Frequency-Domain Transient Analysis of Multitime Partial Differential Equation Systems

Haotian Liu[1], Fengrui Shi[2], Yuanzhe Wang[1] and Ngai Wong[1]
[1]Department of Electrical and Electronic Engineering
The University of Hong Kong, Hong Kong
Email: htliu, yzwang, nwong@eee.hku.hk
[2]Department of Electrical Engineering
Zhejiang University, Hang Zhou, China
Email: arthur.shifengrui@gmail.com

***Abstract*—Multitime partial differential equations (MPDEs) provide an efficient method to simulate circuits with widely separated rates of inputs. This paper proposes a fast and accurate frequency-domain multitime transient analysis method for MPDE systems, which fills in the gap for the lack of general frequency-domain solver for MPDE systems. A block-pulse function-based multidimensional inverse Laplace transform strategy is adopted. The method can be applied to discrete input systems. Numerical examples then confirm its superior accuracy, under similar efficiency, over time-domain solvers.**

I. Introduction

In the simulation of RF, mixed-signal and some analog circuits, the input excitation might contain several widely separate components, e.g., the switched-capacitor filters, voltage-controlled oscillators, mixers, etc. Time-domain analysis of such (often nonlinear) circuits can be very time consuming, since the time step should be small enough to respond to higher frequency (fast) components, while the overall time span should be long enough to capture the slow-varying components. This results in a huge number of time steps used in the time-domain analysis. To overcome this, multitime partial differential equation (MPDE) [1] is proposed. Specifically, different components of input excitation are separated into different time variables. The response of an MPDE system is in a multidimensional time domain. These time variables are independent and have their own ranges and scales, so the overall time steps can be minimized which provides large speedup in simulation. However, existing numerical MPDE solvers are time-domain or mixed frequency-time domain steady-state solvers [1], [2] subject to periodic boundary conditions, and could not be applied to transient analysis.

On the other hand, block-pulse function (BPF) has been used in numerical multidimensional inversion Laplace transforms (MILT) for decades [3]–[5]. The operational matrices in the block-pulse domain are analogous to certain operations in the Laplace and time domains [5]. However, this method can only be used to deal with closed-form input expressions while in most cases the inputs are arbitrary.

In this paper, we propose a purely frequency-domain method for transient analysis of (possibly nonlinear) MPDE systems. The input signal could be either a function of multiple times or discrete samples. First, a series of multivariable frequency-domain linear time-invariant (LTI) systems is obtained from the original MPDEs via Volterra series. Then, MILT is iteratively applied on each order of frequency response to get the multivariable time-domain response. The accuracy of the algorithm is demonstrated by numerical examples.

II. Background

A. MPDE

For simplicity, we consider a single-input single-output (SISO) system with zero initial value. The differential algebraic equation (DAE) of a time-invariant system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(q\left(x(t)\right)\right) = f\left(x(t)\right) + u(t), \tag{1}$$

where $x(t)$ is a scalar state variable of node voltage (such scalar assumption is made for the ease of presentation, extension to vector state variables can trivially be derived), f and q are the nonlinear functions related to nonlinear conductance and capacitance, u is the current input.

If the input signal contains r widely separate components, the variables often can be expressed by functions of multitime variables [1], [6], [7]. Then, the input and output signals u and x could be represented by $\hat{u}(t_1, \ldots, t_r)$ and $\hat{x}(t_1, \ldots, t_r)$. The original DAE system is re-expressed as an MPDE system:

$$\frac{\partial q(\hat{x})}{\partial t_1} + \cdots + \frac{\partial q(\hat{x})}{\partial t_r} = f(\hat{x}) + \hat{u}(t_1, \ldots, t_r). \tag{2}$$

Once the multitime domain response $\hat{x}(t_1, \ldots, t_r)$ is obtained, the transient result is restored by $x(t) = \hat{x}(t, \ldots, t)$.

B. BPF

For a given time span $[0,\ T)$ and the number of time-steps m, block-pulse functions (BPFs) are defined as [3]

$$\phi_i(t) = \begin{cases} 1, & ih \le t < (i+1)h \\ 0, & \text{otherwise} \end{cases}, \quad i = 0,\ \cdots, m-1 \tag{3}$$

where h is the time interval between steps. A time-domain function $f(t)$ defined in $[0,\ T)$ can be approximated by

$$f(t) = \sum_{i=0}^{m-1} f_i \phi_i(t) = f_{(m)}^T \phi_{(m)}(t), \tag{4}$$

where $f_i = \frac{1}{h}\int_{ih}^{(i+1)h} f(t)dt$, $f_{(m)}$ and $\phi_{(m)}(t)$ stand for $[f_0, f_1, \ldots, f_{m-1}]^T$ and $[\phi_0(t), \phi_1(t), \ldots, \phi_{m-1}(t)]^T$, respectively. From (4), a function $f(t)$ can be represented if its coefficients $f_{(m)}$ is known. The integration of BPFs can be represented by an integral operator matrix:

$$\int_0^t \phi_{(m)}(\tau)d\tau = H_{(m)}\phi_{(m)}(t), \quad t \in [0,\ T), \tag{5}$$

where $H_{(m)} = \frac{h}{2}\left(I + Q_{(m)}\right)\left(I - Q_{(m)}\right)^{-1}$, and $Q_{(m)} \in \mathbb{R}^{m\times m}$ is a nilpotent matrix with all ones on its super-diagonal [3].

Similarly, it can be proven that the differential operational matrix is the inversion of $Q_{(m)}$, namely, $D_{(m)} = \frac{2}{h}\left(I - Q_{(m)}\right)\left(I + Q_{(m)}\right)^{-1}$. Therefore, the derivative of a function $f(t)$ can now be computed by $f^T_{(m)}D_{(m)}\phi_{(m)}(t)$.

C. Numerical Inverse Laplace Transform

To calculate the inverse Laplace transform of $F(s)$, we first decompose it into $F(s) = \tilde{F}(s)\frac{1}{s}$. The inverse Laplace of $\frac{1}{s}$ is the step function $u(t)$, which can be represented as $u(t) = [1,\ \ldots, 1]\phi_{(m)}(t)$. Taking Laplace transform, we have $\frac{1}{s} = [1,\ \ldots, 1]\Phi_{(m)}(s)$, where $\Phi_{(m)}(s)$ is the Laplace transform of $\phi_{(m)}(t)$. Then $F(s)$ is converted to

$$F(s) = [1,\ \ldots,\ 1]\left(\tilde{F}(s)\Phi_{(m)}(s)\right). \tag{6}$$

In (6), $\tilde{F}(s)$ can be interpreted an operator on $\Phi_{(m)}(s)$. The operational matrix can be obtained by replacing s with $D_{(m)}$. For a rational function $F(s)$, it can be calculated by:

1) Take a bilinear transform $s = \frac{2}{h}\frac{1-q}{1+q}$ on $\tilde{F}(s)$.
2) Take a long division up to q^{m-1} term and obtain

$$\tilde{F}(s)|_{s=\frac{2}{h}\frac{1-q}{1+q}} = \tilde{c}_0 + \tilde{c}_1 q + \cdots + \tilde{c}_{m-1}q^{m-1}. \tag{7}$$

3) Replace q by $Q_{(m)}$ and obtain the operator $F_{(m)} = \tilde{c}_0 I + \tilde{c}_1 q Q_{(m)} + \cdots + \tilde{c}_{m-1}Q^{m-1}_{(m)}$.

Therefore, $F(s) = [1,\ 1,\ \ldots,\ 1]F_{(m)}\Phi_{(m)}(s)$ and its time-domain response is $f(t) = f^T_{(m)}\phi_{(m)}(t)$ with

$$f_{(m)} = F^T_{(m)}[1,\ 1,\ \ldots,\ 1]^T = [f_0,\ f_1,\ \ldots,\ f_{m-1}]^T, \tag{8}$$

where $f_i = \sum_{j=0}^{i} \tilde{c}_j$.

When extending to high dimensions, similarly, if the time-domain response is defined in $t_i \in [0,\ T_i),\ i = 1,\ 2,\ \ldots,\ r$, the procedure can be summarized as:

1) Decompose $F(s_1,\ s_2,\ \ldots,\ s_r)$ into $F(s_1,\ s_2,\ \ldots,\ s_r) = \tilde{F}(s_1,\ s2,\ \ldots,\ s_r)\frac{1}{s_1 s_2\cdots s_r}$.
2) Apply multidimensional bilinear transform on $\tilde{F}(s_1,\ s_2,\ \ldots,\ s_r)$: $s_i = \frac{2}{h_i}\frac{1-q_i}{1+q_i}, \quad h_i = \frac{T_i}{m_i}, \quad i = 1,\ 2,\ \ldots,\ r.$
3) Apply long division on $\tilde{F}(q_1,\ q_2,\ \ldots,\ q_r)$ to obtain the coefficients of the expansion:

$$\tilde{F}(q_1,\ q_2,\ \ldots,\ q_r) = \sum_{i_1=0}^{m_1-1}\cdots\sum_{i_r=0}^{m_r-1}\tilde{c}_{i_1,\ \ldots,\ i_r}q_1^{i_1}\cdots q_r^{i_r}. \tag{9}$$

4) Finally, the multitime domain response is calculated by

$$f_{i_1,\ \ldots,\ i_r} = \sum_{j_1=0}^{i_1}\cdots\sum_{j_r=0}^{i_r}\tilde{c}_{j_1,\ \ldots,\ j_r}. \tag{10}$$

In [3], the input of a system cannot be arbitrary. The exact expression of frequency-domain response $F(s)$ must be given. Therefore, the BPF-based method in [3] cannot be used to solve systems with arbitrary inputs.

A direct method is proposed in Section IV-A for systems with arbitrary numerical inputs. Thus, the BPF-based inverse Laplace transform can deal with not only exact expressions of $F(s)$, but also systems with arbitrary inputs.

III. Frequency-domain Representation of MPDE Systems

The SISO MPDE system (1) can be expanded around its bias point x_0 by Talyor series [8],

$$\frac{\partial}{\partial t}(C_1\hat{x} + C_2\hat{x}^2 + \cdots) - (G_1\hat{x} + G_2\hat{x}^2 + \cdots) = \hat{u}, \tag{11}$$

where $\frac{\partial}{\partial t}$ denotes the operator $\left(\frac{\partial}{\partial t_1} + \cdots + \frac{\partial}{\partial t_r}\right)$, $G_i,\ C_i \in \mathbb{R}$ are the ith order nonlinear conductance and capacitance defined by $G_i = \frac{1}{i!}\frac{\partial^i f}{\partial x^i}\Big|_{x=x_0}$, $C_i = \frac{1}{i!}\frac{\partial^i q}{\partial x^i}\Big|_{x=x_0}$.

On the other hand, according to Volterra series theory, the response $\hat{x}$ can be represented by the sum of the responses at each order

$$\hat{x}(t_1,\ldots,t_r) = \sum_{n=1}^{\infty} x_n(t_1,\ldots,t_r), \tag{12}$$

where x_i denotes the ith-order response of the MPDE system. Each order of input/output relationship can be obtained by the Variational Equation Approach [9, Chapter 3]. For instance, the first three orders of response in (11) are given by

$$\frac{\partial}{\partial t}(C_1x_1) - G_1x_1 = \hat{u}, \tag{13}$$

$$\frac{\partial}{\partial t}(C_1x_2) - G_1x_2 = -\frac{\partial}{\partial t}(C_2x_1^2) + G_2x_1^2, \tag{14}$$

$$\frac{\partial}{\partial t}(C_1x_3) - G_1x_3 = -\frac{\partial}{\partial t}(C_3x_1^3 + 2C_2x_1x_2) + G_3x_1^3 + 2G_2x_1x_2. \tag{15}$$

Equations (13)-(15) can be regarded as LTI systems by assuming the right hand side of the equations are the inputs. Thus the responses are the solutions to the recursive LTI systems. The right hand side of each equation can be defined as the equivalent input, say, $\hat{u}_i$, which can be calculated with $x_1,\ldots,x_{i-1}$ and $\hat{u}$. Then, the responses become

$$\frac{\partial}{\partial t}(C_1x_i) - G_1x_i = \hat{u}_i, \quad i = 1,\ 2,\ \ldots \tag{16}$$

Applying multidimensional Laplace transform on both sides of (16) will give the multivariable frequency-domain representation of each order,

$$X_i(s_1,\ldots,s_m) = \frac{\mathfrak{L}_{(r)}\{\hat{u}_i\}}{C_1(s_1 + \ \cdots + s_r) - G_1}, i = 1,\ 2,\ \ldots, \tag{17}$$

where X_i is the frequency-domain response of x_i, $\mathfrak{L}_{(r)}\{\cdot\}$ denotes the r-dimension Laplace transform operator.

Therefore, the representation of each order of frequency-domain response is obtained. Here, $\mathfrak{L}_{(r)}\{\hat{u}_1\} = \mathfrak{L}_{(r)}\{\hat{u}\}$ can be in function of $s_1,\ldots,s_r$. For $i > 1$, $\mathfrak{L}_{(r)}\{\hat{u}_i\}$ are all in the form of a "pseudo Laplace transform" on discrete numerical input of $\hat{u}_i$.

IV. Multidimensional Inverse Laplace Transform

In this section, a method of numerical MILT of (17) will be given. First, the inverse Laplace transform with numerical input will be introduced. Then, the procedure of multivariable frequency-domain MPDE solver is summarized.

A. Inverse Laplace Transform with Numerical Input

In one-dimensional inverse Laplace transform, the frequency response $F(s)$ is split into product of $\tilde{F}(s)$ and $\frac{1}{s}$. In (6), $\tilde{F}(s)$ is interpreted as operator on $\Phi_{(m)}(s)$ with step function input. Thus, the results (8) can be regarded as the convolution of the "discrete" step function $[1,\ 1,\ \ldots,\ 1]$ and the "discrete" LTI system $\tilde{F}(s)$ which is represented by the coefficients $\tilde{c}_i,\ i = 1,\ \ldots,\ m$.

In other words, the step function is an artificial "input" to the artificial "system" $\tilde{F}(s)$. If the discrete numerical input $u = [u_0,\ u_1,\ \ldots,\ u_{m-1}]^T$ is given, there is no need to apply a bilinear transform and a long division on the artificial "system" $\tilde{F}(s)$. These steps can be applied on the real system $F(s)$ to obtained a new series of coefficients, say, $c_i,\ i = 1,\ \ldots,\ m$. Then the results (8) can be rewritten as

$$f_{(m)} = F_{(m)}^T [u_0,\ u_1,\ \ldots,\ u_{m-1}]^T = [f_0,\ f_1,\ \ldots,\ f_{m-1}]^T, \tag{18}$$

where $f_i = \sum_{j=0}^{i} u_{i-j} c_j$.

The conclusion can be generalized to MILT as well. If we take the multivariate bilinear transform and long division on the original system $F(s_1,\ s_2,\ \ldots,\ s_r)$ to obtained a new series of coefficients $c_{j_1,\ \ldots,\ j_r}$, $j_i = 1,\ \ldots,\ m_i$; $i = 1,\ \ldots,\ r$, the results (10) will become

$$f_{i_1,\ \ldots,\ i_r} = \sum_{j_1=0}^{i_1} \cdots \sum_{j_r=0}^{i_r} u_{i_1-j_1,\ \ldots,i_r-j_r} c_{j_1,\ \ldots,j_r}, \tag{19}$$

where $u_{j_1,\ \ldots,\ j_r}$ is the multidimensional discrete numerical input. Consequently, the LTI systems (17) with either closed-form or discrete inputs can be solved by MILT.

B. Frequency-Domain MPDE Solver Algorithm

The procedure of our multidimensional frequency-domain MPDE solver is summarized in Algorithm 1.

Algorithm 1: Frequency-domain MPDE Solver

Input: MPDE system
Output: Multitime domain response $f_{i_1,\ \ldots,\ i_r}$
1: Expand the system by Volterra series and truncate to kth order; {usually $k \leq 3$}
2: **for** $j = 1$ to k **do**
3: Calculate closed-form representation or discrete numerical values of input $\hat{u}_j$;
4: Apply bilinear transform and polynomial long division to obtain the coefficients for frequency-domain response X_j;
5: Use the coefficients and discrete input values to obtain the jth-order time-domain response $x_{i_1,\ \ldots,\ i_r}^{(j)}$;
6: **end for**
7: $f_{i_1,\ \ldots,\ i_r} = \sum_{j=1}^{k} x_{i_1,\ \ldots,\ i_r}^{(j)}$

The most expensive step in the algorithm is step 5, wherein multidimensional convolutions (19) are calculated. The computational complexity of (19) is $O(m_1^2 m_2^2\ \cdots m_r^2)$, where m_i is the number of time-steps observed in the ith time scale.

Since the time-domain methods for MPDE systems are all steady-state solvers, we apply the Multivariate Finite Difference Time Domain (MFDTD) method [1] on the time-domain transient analysis of MPDE systems to compare the results against our frequency-domain solver. The Newton-Raphson method is used in roots finding for nonlinear equations. The numerical results are shown in Section V.

V. Numerical Examples

A. Example 1: A Comparator Circuit

This example is modified from the example in [1]. A two-tone input passes a comparator followed by an RC filter, as shown in Fig. 1.

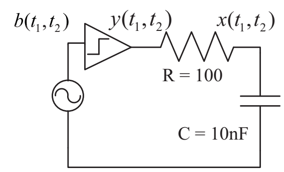

Fig. 1. A comparator circuit with RC.

Its input is a two-tone quasi-periodic signal defined by

$$b(t_1,\ t_2) = \sin(\frac{2\pi}{T_1} t_1) \sin(\frac{2\pi}{T_2} t_2),\ T_1 = 0.1ms,\ T_2 = 0.01ms. \tag{20}$$

The current output of the comparator is described by

$$y(t_1,\ t_2) = \text{comp}\left(b(t_1,\ t_2)\right),\ \text{comp}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}.$$

The example can be regarded as a multitime LTI system with nonlinear input. Of course, the response can be calculated with numerical discrete inputs by a 2-D convolution in (19), it also can be solved by closed-form representation. The time-domain representation of $y(t_1,\ t_2)$ is

$$y(t_1,\ t_2) = \text{comp}\left(\sin(\frac{2\pi}{T_1} t_1) \sin(\frac{2\pi}{T_2} t_2)\right).$$

Take 2-D Laplace transform of $y(t_1,\ t_2)$, its frequency-domain representation $Y(s_1,\ s_2)$ reads

$$Y(s_1,\ s_2) = \frac{1 + e^{-(T_1 s_1 + T_2 s_2)/2}}{s_1 s_2 (1 + e^{-T_1 s_1/2})(1 + e^{-T_2 s_2/2})}.$$

Therefore, the frequency-domain voltage response $X(s_1,\ s_2)$ can be obtained by (17):

$$X(s_1,\ s_2) = \frac{Y(s_1,\ s_2)}{C(s_1 + s_2) + 1/R}.$$

By applying our direct coefficient method, the coefficients of $q_1^{i_1} q_2^{i_2}$ can be solved. Finally, the time-domain response of the system is obtained.

The time-domain response in the time span $[0,\ T_1) \times [0,\ T_2)$ is shown in Fig. 2, with time-steps $m_1 = 200,\ m_2 = 100$ in the time scales. The exact response of the circuit can be found along the $t_1 = t_2$ axis. The exact solution in the time interval $[0,\ T_1)$ is displayed in Fig. 3, together with the results of

MFDTD method with different time-steps. The reference line in Fig. 3 is the original system solved by ODE solver. Further benchmarks are given in Table I.

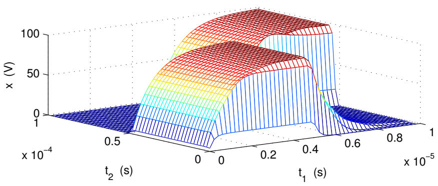

Fig. 2. Multitime domain response of Example 1.

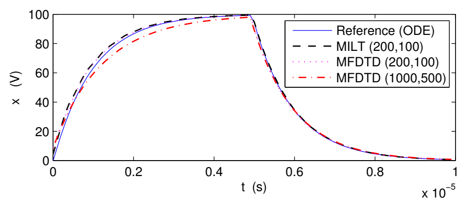

Fig. 3. Time-domain response of Example 1.

B. Example 2: A Nonlinear Capacitor RC Filter

This example is a RC low-pass filter with nonlinear capacitor. The parameters are defined by

$$R = 100\Omega, \quad C = 1 + 0.128V_C + 0.068V_C^2\mu\text{F},$$

where R and C are the resistance and capacitance of the filter, V_C is the voltage applied on the capacitor. The excitation is the same quasi-periodic signal as in (20), $m_1 = m_2 = 100$ time steps in both directions.

We expand the system to the 3rd-order Volterra representation. Again, the results in the time interval $[0,\ T_1)$ are displayed in Fig. 4.

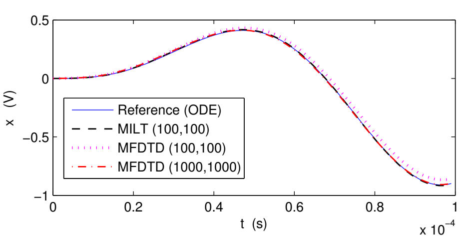

Fig. 4. Time-domain response of Example 2.

The comparison of the MFDTD method and the MILT is listed in Table I. Both examples show that, though the time-domain transient solver is faster than the frequency-domain solver with the same time-steps, the result of MILT is more accurate. When increasing the number of time-steps in MFDTD, it will consume more CPU times to get the solutions with similar accuracy. In Example 2, though only first three orders of Volterra series are preserved, the accuracy of the result is even better than the result of MFDTD method with much more time-steps.

TABLE I
COMPARISON OF MILT AND MFDTD METHODS
(INTEL I5 750@2.67GHZ, 4GB, WINDOWS 7)

		Time-steps	CPU times (s)	Relative error (%)
Example 1	MILT	(200, 100)	2.60	2.98
	MFDTD 1	(200, 100)	0.16	4.52
	MFDTD 2	(1000, 500)	4.14	3.13
Example 2	MILT	(100, 100)	1.95	0.88
	MFDTD 1	(100, 100)	0.16	7.32
	MFDTD 2	(1000, 1000)	13.2	1.45

VI. CONCLUSION

An MILT-based algorithm has been proposed for the multi-dimensional transient analysis of MPDE systems. This method can deal with nonlinear systems with either closed-form or discrete numerical inputs. Numerical examples have demonstrated the proposed algorithm produces fast and accurate transient solutions for both linear and nonlinear MPDE systems.

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REFERENCES

[1] J. Roychowdhury, "Analyzing circuits with widely separated time scales using numerical pde methods," *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, vol. 48, no. 5, pp. 578 –594, May 2001.

[2] O. Narayan and J. Roychowdhury, "Analyzing oscillators using multitime pdes," *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, vol. 50, no. 7, pp. 894 – 903, july 2003.

[3] C. Hwang, T.-Y. Guo, and Y.-P. Shih, "Numerical inversion of multidimensional laplace transforms via block-pulse functions," *Control Theory and Applications, IEE Proceedings D*, vol. 130, no. 5, pp. 250 –254, 1983.

[4] L. S. Shieh, R. E. Yates, and J. M. Navarro, "Solving inverse laplace transform, linear and nonlinear state equations using block-pulse functions," *Computers and Electrical Engineering*, vol. 6, no. 1, pp. 3 – 17, 1979.

[5] P. Sannuti, "Analysis and synthesis of dynamic systems via block-pulse functions," *Electrical Engineers, Proceedings of the Institution of*, vol. 124, no. 6, pp. 569 –571, 1977.

[6] E. Ngoya and R. Larcheveque, "Envelop transient analysis: a new method for the transient and steady state analysis of microwave communication circuits and systems," in *Microwave Symposium Digest, 1996., IEEE MTT-S International*, vol. 3, Jun. 1996, pp. 1365 –1368 vol.3.

[7] H. G. Brachtendorf, G. Welsch, R. Laur, and A. Bunse-Gerstner, "Numerical steady state analysis of electronic circuits driven by multi-tone signals," *Electrical Engineering (Archiv fur Elektrotechnik)*, vol. 79, pp. 103–112, 1996, 10.1007/BF01232919.

[8] P. Li and L. Pileggi, "Compact reduced-order modeling of weakly nonlinear analog and rf circuits," *Computer-Aided Design of Integrated Circuits and Systems, IEEE Transactions on*, vol. 24, no. 2, pp. 184 – 203, 2005.

[9] W. J. Rugh, *Nonlinear System Theory: The Volterra/Wiener Approach*. JOHNS HOPKINS UNIV. PRESS., 1981.