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Characterizing Coherence, Correcting Incoherence

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Abstract

Lower previsions defined on a finite set of gambles can be looked at as points in a finite-dimensional real vector space. Within that vector space, the sets of sure loss avoiding and coherent lower previsions form convex polyhedra. We present procedures for obtaining characterizations of these polyhedra in terms of a minimal, finite number of linear constraints. As compared to the previously known procedure, these procedures are more efficient and much more straightforward. Next, we take a look at a procedure for correcting incoherent lower previsions based on pointwise dominance. This procedure can be formulated as a multi-objective linear program, and the availability of the finite characterizations provide an avenue for making these programs computationally feasible.

Keywords. Coherence, avoiding sure loss, linear constraint, polytope, enumeration, projection, multi-objective linear programming, incoherence, dominance.

1 Introduction

In the theory of coherent lower previsions (for an overview, see Walley 1991 or Miranda 2008), its coherence condition takes a central role: it defines which models—lower previsions—are fully rational, meaning that they do not implicitly encode commitments—in terms of buying prices for gambles—that are more demanding than the ones explicitly made. The consequences of this criterion have been extensively studied both in the unconditional and the conditional case, in finite and infinite spaces.

In this paper, we study the coherence criterion for unconditional lower previsions defined on a finite set of gambles, which in turn are essentially defined on a finite possibility space. What can we still add in this restricted setting? Results that make new numerical applications feasible, namely, procedures for obtaining a characterization of coherence in terms of a minimal, finite number of linear constraints that are more efficient than the existing one. These results are presented in Section 4. Note that our procedures give an answer to the question "Which lower previsions are coherent?", and should not be confused with verification procedures, which deal with the question "Is this specific lower prevision coherent?". Of course, the characterization our procedures generate can be used for verification purposes, but this may be reasonable only if many verifications need to be performed

One may wonder what new kinds of applications are possible once we have a minimal linear constraints characterization? In Section 5, we provide one example in a proposal for a method to correct an incoherent lower prevision downward to make it coherent. Similarly to natural extension, this method is formulated in terms of pointwise dominance of lower previsions.

Because of the finitary context of this paper and its aim to be an enabler for numerical applications, it is advantageous to reformulate a number of variants of the coherence criterion and the related criterion of avoiding sure loss in matrix terms; we do this in Section 3.

We make use of polytope theory concepts throughout this paper. We also make use of multi-objective linear programming both for our downward correction method as well as for some of our procedures to obtain the minimal linear constraints characterization. Therefore, we start out with short primers on these topics in Section 2.

2 Primers

2.1 Polytope Theory Essentials

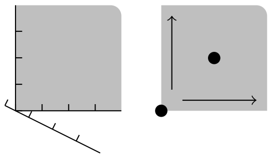
Let us review some concepts and techniques from polytope theory (for more information, see, e.g., Grünbaum 1967, Ziegler 1995, or Fukuda 2004). Any convex polyhedron in a n-dimensional space can be described in two ways:

As an *H-representation* $\{x \in \mathbb{R}^n : Ax \leq b\}$: A set of k linear constraints (inequalities/half-spaces) defined by a matrix A in $\mathbb{R}^{k\times n}$ and a column vector b in $\mathbb{R}^k$; denoted compactly as $[A, b]$, where the comma denotes horizontal concatenation of matrices.

As a *V-representation* $\{x \in \mathbb{R}^n : x = V\mu \wedge \mu \geq 0 \wedge w^\mathsf{T}\mu = 1\}$: A set of ℓ points and rays, defined by a matrix V in $\mathbb{R}^{n\times\ell}$ and a row vector w in $\mathbb{R}^\ell$, with the zero components indicating rays; denoted compactly as $[V;w]$, where the semicolon denotes vertical concatenation of matrices.

The two representations are dual in the sense that $[A^\mathsf{T};b^\mathsf{T}]$ is the V-representation of some polyhedron and $[V^\mathsf{T},w^\mathsf{T}]$ is the H-representation of some—possibly different—polyhedron. This duality is also present in the algorithms of polytope theory.

On the right, we give a simple 2-dimensional polyhedron, in gray, in both a visual H- and V-representation.



H- and V-representations may contain redundant constraints and points or rays, i.e., those that are implied by the other constraints or the other points or rays. Non-redundant extreme points or rays are called vertices and extreme rays. In our illustration, there is one redundant constraint in the H-representation and one redundant point in the V-representation. Let i be the total number of constraints or points and j the non-redundant number; redundancy removal algorithms essentially require solving i linear programming problems of size $n \times j$ (Clarkson 1994).

Moving between the H- and V-representations is done using vertex enumeration algorithms and the dual facet enumeration algorithms. There are enumeration algorithms with a complexity linear in n, k, and ℓ (Avis & Fukuda 1992). Nevertheless, enumeration is inherently highly complex, as ℓ can be exponential in k and vice versa.

Projecting a polyhedron is straightforward in V-representation: project the vertices and then remove the redundant ones. However, in H-representation the best technique depends on the polyhedron's properties: the classical approach, Fourier–Motzkin elimination, is inefficient and on top of that generates a lot of redundant constraints; another approach, block elimination, is inefficient when the number of vertices is high, which is common. The equality set projection approach is claimed to be useful in such cases (Jones et al. 2004), but our input data caused errors in the available code (Kvasnica et al. 2006).

Below, we assume that the output of enumeration and projection algorithms is minimal, i.e., non-redundant.

2.2 Multi-Objective Linear Programming

We here give a brief introduction to multi-objective linear programming (for more information, see, e.g., Ehrgott 2005). We assume familiarity with standard, single objective linear programming (if not, have a quick look at a standard reference such as Bertsimas & Tsitsiklis 1997).

Any *multi-objective linear program* can be put in the following form:

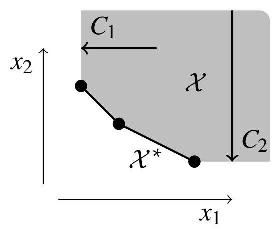
$$\begin{aligned} &\text{maximize} \;\; y = Cx, \\ &\text{subject to} \;\; Ax \leq b \;\text{ and }\; x \geq 0. \end{aligned} \tag{1}$$

In this program, x denotes the n-dimensional real *optimization vector*, y is the m-dimensional *objective vector*, and $Ax \leq b$ is a set of k *linear constraints*; so we assume $C \in \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{k\times n}$, and $b \in \mathbb{R}^k$ as given. Vector inequalities should be read as follows: $x \geq z \Leftrightarrow \min(x-z) \geq 0$ and $x > z \Leftrightarrow x \geq z \wedge x \neq z$. Here, min (max) selects its argument vector's minimum (maximum) component value.

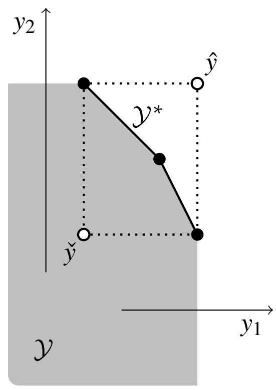
Whereas in single objective linear programming, with $m = 1$, all optimization vectors x are completely ordered by the single objective, whenever $m > 1$, they are only partially ordered through the standard ordering of the objective vectors. Consequently, whereas in single objective linear programming all optimal solutions are equivalent from the objective value point of view, in multi-objective linear programming there are in general multiple sets of incomparable 'Pareto' optimal (or 'efficient')—i.e. *C-undominated*—solutions.

The sets of feasible optimization and objective vectors are $\mathcal{X} := \{x \in \mathbb{R}^n : Ax \leq b \wedge x \geq 0\}$ and $\mathcal{Y} := \{Cx : x \in \mathcal{X}\}$, respectively. Furthermore, $\mathcal{X}^* := \{x \in \mathcal{X} : (\forall z \in \mathcal{X} : Cx \not< Cz)\}$ is the set of C-undominated solutions, and so $\mathcal{Y}^* := \{Cx : x \in \mathcal{X}^*\}$ is the set of undominated objective vectors. The sets of extreme points of the sets of undominated solutions and objectives are $\operatorname{ext}\mathcal{X}^*$ and $\operatorname{ext}\mathcal{Y}^*$, respectively.

Let us give a simple graphical illustration (with $n = m = 2$) below right to clarify the concepts just introduced. The sets $\mathcal{X}$ and $\mathcal{Y}$ are shaded gray. The sets $\mathcal{X}^*$ and $\mathcal{Y}^*$ are shown as black lines. The members of $\operatorname{ext}\mathcal{X}^*$ and $\operatorname{ext}\mathcal{Y}^*$ are shown as black dots. The vectors C_1 and C_2—rows of C—that point towards higher objective vector component values are drawn free: only their direction and magnitude matter.



In the picture of the objective vector space, we have included the so-called *ideal point* $\hat{y}$ and *nadir point* $\check{y}$, the upper and lower envelopes of $\mathcal{Y}^*$, respectively. They provide bounds on the values attained by the undominated objective vector components.



The main computational tasks are, in non-decreasing order of complexity:

M1. Finding the ideal point $\hat{y}$, which can be done by solving a linear program maximizing each of the components of y separately.

M2. Finding the nadir point $\check{y}$ (for algorithms, see Ehrgott & Tenfelde-Podehl 2003 and Alves & Costa 2009).

M3. Finding the extreme points $\operatorname{ext}\mathcal{Y}^*$ and the whole set $\mathcal{Y}^*$ of undominated objective vectors (for algorithms, see Benson 1998 and Ehrgott et al. 2012; these are relatively efficient only if m is small compared to n).

M4. Finding the extreme points $\operatorname{ext}\mathcal{X}^*$ of the set of optimal optimization vectors (for algorithms, MOLP simplex solvers, see, e.g., Evans & Steuer 1973, Strijbosch et al. 1991, or Ehrgott 2005, Sec. 7).

M5. Finding the whole set $\mathcal{X}^*$ of optimal optimization vectors (for algorithms, based on post-processing the MOLP simplex solver output, see, e.g., Yu & Zeleny 1975 or Isermann 1977).

3 Matrix Formulations of Avoiding Sure Loss and Coherence

Consider a finite possibility space Ω and a finite set of gambles $\mathcal{K} \subset \mathbb{R}^{\Omega}$ on this possibility space. The elements of $\mathcal{K}$ can be looked at as vectors; we group them as columns in a gamble matrix $K \in \mathbb{R}^{\Omega\times\mathcal{K}}$. We use the same notation for scalars and constant vectors; the identity matrix is denoted $\mathbb{I}$; there will be no ambiguity in this paper because we leave their size implicit. The columns of $K^{\top}$ are the degenerate previsions, so $\{K^{\top}\mu : \mu \geq 0 \wedge 1^{\top}\mu = 1\}$ is the set of linear previsions. Any lower prevision $\underline{P}$ defined on $\mathcal{K}$ can be looked at as a column vector in $\mathbb{R}^{\mathcal{K}}$. Similarly, min and max can also thought of at as column vectors in $\mathbb{R}^{\mathcal{K}}$.

A lower prevision $\underline{P}$ on $\mathcal{K}$ is said to *avoid sure loss* (cf., e.g., Walley 1991, §2.4) if and only if

$$\forall \lambda \geq 0 : \ \underline{P}^{\top}\lambda \leq \max(K\lambda), \tag{2}$$

or, based on dominance by a linear prevision (cf. Walley 1991, §3.3.3(a)), if

$$\exists \mu \geq 0 : \ \underline{P} \leq K^{\top}\mu \ \wedge \ 1^{\top}\mu = 1, \tag{3}$$

or, by introducing slack variables, if

$$\exists \mu, \nu \geq 0 : \ \underline{P} = K^{\top}\mu - \mathbb{I}\nu \ \wedge \ 1^{\top}\mu = 1. \tag{4}$$

This last form shows that the set of all sure loss avoiding lower previsions is a convex polyhedron by providing a V-representation

$$\begin{bmatrix} V \\ w \end{bmatrix} := \begin{bmatrix} K^{\top} & -\mathbb{I} \\ 1^{\top} & 0^{\top} \end{bmatrix}. \tag{5}$$

Now, let $\mathcal{S}$ denote the set of matrices obtained from the identity matrix by changing at most one 1 to -1. Then a lower prevision $\underline{P}$ on $\mathcal{K}$ is called *coherent* (cf., e.g., Walley 1991, §2.5) if and only if

$$\forall S \in \mathcal{S} : \forall \lambda \geq 0 : \ \underline{P}^{\top}S\lambda \leq \max(KS\lambda), \tag{6}$$

or, by formal analogy to Equation (3) and because $S^{\top} = S$, if

$$\forall S \in \mathcal{S} : \exists \mu_S \geq 0 : \ S\underline{P} \leq SK^{\top}\mu_S \ \wedge \ 1^{\top}\mu_S = 1, \tag{7}$$

or, by introducing slack variables and because $S^{-1} = S$, if

$$\forall S \in \mathcal{S} : \exists \mu_S, \nu_S \geq 0 : \ \underline{P} = K^{\top}\mu_S - S\nu_S \ \wedge \ 1^{\top}\mu_S = 1. \tag{8}$$

This last form shows that the set of all coherent lower previsions is an intersection of $|\mathcal{K}| + 1$ convex polyhedra with V-representations

$$\begin{bmatrix} V_S \\ w_S \end{bmatrix} := \begin{bmatrix} K^{\top} & -S \\ 1^{\top} & 0^{\top} \end{bmatrix}, \tag{9}$$

and therefore is a convex polyhedron. Furthermore, coherence implies that $\min \leq \underline{P} \leq \max$ (cf. Walley 1991, §2.6.1(a)), so the set of coherent lower previsions is a bounded polyhedron, i.e., a *polytope*.

We will later on in this paper use the Lower Envelope Theorem (see, e.g., Walley 1991, §2.6.3):

Theorem. The lower envelope $\underline{P}$ of a subset $\mathcal{Q}$ of the coherent lower previsions on a set of gambles $\mathcal{K}$ is coherent. (So $\underline{P}f := \inf_{\underline{Q}\in\mathcal{Q}} \underline{Q}f$ for each gamble f in $\mathcal{K}$.) ◁

We give a proof based on Equation (7)—a version of the coherence criterion a shallow search of ours left unencountered in the literature:

Proof. By coherence of the $\underline{Q}$ in $\mathcal{Q}$, we have a vector $\mu_{\underline{Q},S}$ such that $S\underline{Q} \leq SK^{\top}\mu_{\underline{Q},S}$ for each S in $\mathcal{S}$. By the lower envelope definition, for $S := \mathbb{I}$, we have $\underline{P} \leq \underline{Q} \leq K^{\top}\mu_{\underline{Q},\mathbb{I}}$ for any $\underline{Q}$ in $\mathcal{Q}$. For other S, let g_S denote the gamble corresponding to the -1 diagonal component in S. Let $\underline{Q}_S$ be a coherent lower prevision from $\mathcal{Q}$ such that $\underline{P}g_S = \underline{Q}_S g_S$. Then $S\underline{P} \leq S\underline{Q}_S \leq SK^{\top}\mu_{\underline{Q}_S,S}$. □

In the literature on verification procedures—which are typically formulated in the more general conditional context—, there is a clear separation between algorithms based on criteria formulations of the type of Equations (2) and (6) (cf. Walley et al. 2004), and those of the type of Equations (3)–(4) and (7)–(8) (see, e.g., Vicig 1996 and Biazzo & Gilio 2000). This separation is also present in the characterization procedures we present; the latter type leads to the procedures in Section 4.1, the former to those in Section 4.2.

4 Computing Constraints Efficiently

Building on earlier work with lower probabilities (Walley 1991, App. A; Quaeghebeur & De Cooman 2008; Quaeghebeur 2009), we presented a procedure for obtaining characterizations of the polytope of coherent lower previsions in terms of a minimal, finite number of linear constraints (Quaeghebeur 2010). However, the procedure is such that a relatively large number of redundant constraints are generated, which at a later step need to be removed—a computationally demanding task. Moreover, the procedure and its derivation is somewhat involved.

It is possible to derive procedures in a more direct way. Some of these more direct procedures turn out to be computationally more efficient as well, resulting in running times that are up to an order of magnitude shorter.

What are our concrete goals? We wish to find minimal H-representations for the set of all lower previsions $\underline{P}$

A. that avoid sure loss ($[\Lambda_A, \alpha_A]$),
B. that avoid sure loss and for which $\underline{P} \geq \min$ ($[\Lambda_B, \alpha_B]$),
C. that are coherent ($[\Lambda_C, \alpha_C]$).

So for each goal, we want to obtain a block matrix $[\Lambda, \alpha]$ that stands for the linear constraints $\Lambda\underline{P} \leq \alpha$.

These goals are formulated based on experimental results from earlier work (Quaeghebeur & De Cooman 2008; Quaeghebeur 2009, 2010): For coherence, we observed that the V-representations have a much larger size than the H-representations, and to such a degree that it currently seems impractical to generate and use them. We observed that avoiding sure loss with lower bound constraints leads to a smaller H-representation than plain avoiding sure loss. As the lower bound constraints are uncontroversial in most contexts, it may be useful to use this combination as a 'lighter' proxy for plain avoiding sure loss.

Below, we first discuss the direct procedures and follow this up with a look at improved versions of our earlier, involved approach. We close the section with a short discussion of our numerical experiments.

4.1 Straightforward Procedures

The straightforward procedures for Goal A go as follows:

A1. Apply a facet enumeration algorithm to the V-representation of the polyhedron of lower previsions that avoid sure loss in Equation (5) to obtain $[\Lambda_A, \alpha_A]$.

A2. As can be seen from Equation (3), we know an H-representation for pairs $[\underline{P}; \mu_{\mathbb{I}}]$ of which the $\underline{P}$-components are lower previsions that avoid sure loss:

$$\begin{bmatrix} A_{\mathbb{I},\underline{P}} & A_{\mathbb{I},\mu_{\mathbb{I}}} & b_0 \end{bmatrix} \coloneqq \begin{bmatrix} \mathbb{I} & -K^\top & 0 \\ & -\mathbb{I} & 1 \\ & 1^\top & 1 \\ & -1^\top & -1 \end{bmatrix}. \tag{10}$$

Project this H-representation onto the $\underline{P}$-part to obtain $[\Lambda_A, \alpha_A]$.

The straightforward procedures for Goal B build on those for Goal A:

B1. Start from the resulting H-representation of Procedure A1 and add the lower bound constraints to it, i.e., the block row $[-\mathbb{I}, -\min(K)^\top]$, where the minimum is taken column-wise. Because some constraints may have become redundant because of this, perform redundancy removal to obtain $[\Lambda_B, \alpha_B]$.

B2. Idem as Procedure B1, but now starting from the H-representation resulting from Procedure A2.

The straightforward procedures for Goal C are based on the similarities of the underlying problem with that of Goal A:

C1. Recall that the polytope of coherent lower previsions is the intersection of $|\mathcal{S}| = |\mathcal{K}| + 1$ polyhedra, one for each value of S. So apply a facet enumeration algorithm to the V-representation as given in Equation (9) for each S to obtain the corresponding H-representations $[A_S, b_S]$. An H-representation of the intersection polyhedron of polyhedra given as H-representations is the vertical concatenation of these matrices. (Intersection of polyhedra in V-representation, or mixed representations is not straightforward.) Perform redundancy removal on this concatenation H-representation to obtain $[\Lambda_C, \alpha_C]$.

C2. As can be seen from Equation (7), for each S we also know an H-representation for pairs $[\underline{P}; \mu_S]$ of which the $\underline{P}$-component belongs to the polyhedron corresponding to S already mentioned in Procedure C1:

$$\begin{bmatrix} A_{S,\underline{P}} & A_{S,\mu_S} & b_0 \end{bmatrix} \coloneqq \begin{bmatrix} S & -SK^\top & 0 \\ & -\mathbb{I} & 1 \\ & 1^\top & 1 \\ & -1^\top & -1 \end{bmatrix}. \tag{11}$$

Project this H-representation onto the $\underline{P}$-part to obtain the H-representation $[A_S, b_S]$ already encountered in Procedure C1, the remainder of which is to be followed here as well.

C3. Equation (7) also shows that we can actually create a single H-representation for pairs $[\underline{P}; \mu]$ of which the $\underline{P}$-components are coherent lower previsions:

$$\begin{bmatrix} A_{\underline{P}} & A_\mu & b \end{bmatrix} \coloneqq \begin{bmatrix} A_{\mathbb{I},\underline{P}} & A_{\mathbb{I},\mu_{\mathbb{I}}} & & b_0 \\ \vdots & & \ddots & \vdots \\ A_{S_g,\underline{P}} & & A_{S_g,\mu_{S_g}} & b_0 \\ \vdots & & \ddots & \vdots \end{bmatrix}, \tag{12}$$

where $S_g \in \mathcal{S}$, with g in $\mathcal{K}$, has negative diagonal g-component. Projecting this H-representation onto the $\underline{P}$-part again gives us $[\Lambda_C, \alpha_C]$. Because of the block diagonal structure of the set of columns to be removed by projection, this procedure is essentially identical to Procedure C2 from the computational point of view.

Comparing the two main procedure types, enumeration-based (A1, B1, C1) and projection-based (A2, B2, C2, C3), our numerical experiments showed that the enumeration-based ones were faster by at least an order of magnitude. It is not yet clear whether this is inherent or whether this is due to the fact that the enumeration implementation used (the double description method of Fukuda & Prodon 1996) is efficient, and the facet projection implementations used (Fourier–Motzkin and block elimination) are not.

4.2 A More Involved Type of Procedure

All of the procedures in the previous section were based on Equations (3)–(4) and (7)–(8). In these expressions, $\underline{P}$ appears free, i.e., without being multiplied by a variable vector such as λ, this in contrast to the other expressions characterizing avoiding sure loss and coherence, Equations (2) and (6). This allowed us to consider $\underline{P}$ as variable as well, directly leading to the straightforward procedures.

In our earlier work (Quaeghebeur 2010), we created a procedure starting from the expressions with bound $\underline{P}$. It is, by the standard set by the best performing of the straightforward procedures, inefficient. However, it is possible to create bound-$\underline{P}$-based procedures that are relatively efficient; we present the ones we found here, as the techniques used might be useful in other contexts as well.

We first make an assumption, namely that all gambles are non-constant and non-negative with zero minimum, or NNZM. In Appendix 4.3 immediately following this section we show that for coherent lower previsions this assumption is non-limiting and how to move between general gamble sets and NNZM gamble sets. The assumption is, however, limiting for lower previsions that only avoid sure loss. Note that $\underline{P} \geq \min$ becomes $\underline{P} \geq 0$ for an NNZM set of gambles $\mathcal{K}$; i.e., positivity constraints.

We do not develop procedures for Goal A here and move straight to Goal B, which because of the limiting nature of the NNZM assumption must be seen as preparation for the procedures for Goal C:

B3. We can rewrite Equation (2) as

$$\forall \gamma \in \mathbb{R} : \forall \lambda \geq 0 : \ \max(K\lambda) = \gamma \ \Rightarrow \ \underline{P}^{\top}\lambda \leq \gamma, \tag{13}$$

which, because $\mathcal{K}$ is NNZM, can be normalized to

$$\forall \lambda \geq 0 : \ \max(K\lambda) = 1 \ \Rightarrow \ \underline{P}^{\top}\lambda \leq 1. \tag{14}$$

Now, again because $\mathcal{K}$ is NNZM, $K\lambda$ is pointwise strictly increasing in λ. So we know that the feasible set $\{\lambda \geq 0 : K\lambda \leq 1\}$ is bounded and that apart from 0, all its vertices satisfy $\max(K\lambda) = 1$. So in our procedure, we first vertex enumerate

$$\begin{bmatrix} A & b \end{bmatrix} \coloneqq \begin{bmatrix} K & 1 \\ -\mathbb{I} & 0 \end{bmatrix}, \tag{15}$$

and then use this V-representation $[V; w]$ for the λ's to construct an H-representation $[V^{\top}, w^{\top}]$ for lower previsions. Add positivity constraints $[-\mathbb{I}, 0]$; then after redundancy removal we obtain $[\Lambda_{\mathrm{B}}, \alpha_{\mathrm{B}}]$.

B4. Because we assume $\mathcal{K}$ is NNZM, $\underline{P} \geq 0$, so we know that all pointwise dominated vertices of the feasible set $\{\lambda \geq 0 : K\lambda \leq 1\}$ encountered in Procedure B3 result in redundant constraints (cf. the implicand in Equation (14)). So we can use the MOLP

$$\begin{aligned} &\text{maximize } \ \lambda, \\ &\text{subject to } \ K\lambda \leq 1 \ \text{ and } \ \lambda \geq 0, \end{aligned} \tag{16}$$

to select only the undominated vertices. Gather them as columns in a matrix $\hat{V}$ and construct the H-representation $[\hat{V}^{\top}, 1]$ to replace $[V^{\top}, w^{\top}]$ of Procedure B3.

B5. Because $K\lambda$ is pointwise strictly increasing in λ, we can replace the MOLP (16) by

$$\begin{aligned} &\text{maximize } \ K\lambda, \\ &\text{subject to } \ K\lambda \leq 1 \ \text{ and } \ \lambda \geq 0. \end{aligned} \tag{17}$$

We are now ready to present the procedures for Goal C, which strongly parallel those for Goal B:

C4. We can rewrite Equation (6) as

$$\begin{aligned} &\forall S \in \mathcal{S} : \forall \lambda \geq 0 : \forall \gamma \in \mathbb{R} : \\ &\qquad \max(KS\lambda) = \gamma \ \Rightarrow \ \underline{P}^{\top}S\lambda \leq \gamma, \end{aligned} \tag{18}$$

which, because $\mathcal{K}$ is NNZM and only a single column of KS is non-positive, but with zero maximum, can be normalized and rewritten as

$$\begin{aligned} &\forall S \in \mathcal{S} : \forall \kappa \in \mathbb{R}^{\mathcal{K}} : \\ &S\kappa \geq 0 \Rightarrow \begin{cases} \max(K\kappa) = 1 \ \Rightarrow \ \underline{P}^{\top}\kappa \leq 1, \\ \max(K\kappa) = 0 \ \Rightarrow \ \underline{P}^{\top}\kappa \leq 0. \end{cases} \end{aligned} \tag{19}$$

Now, again because $\mathcal{K}$ is NNZM, $K\kappa$ is pointwise monotone strictly increasing in κ. So we know that the set $\{S\kappa \geq 0 : K\kappa \leq 1\}$ is bounded and that apart from 0, all its vertices satisfy $\max(K\kappa) = 1$. We also know that the set $\{0 \leq S\kappa \leq 1 : K\kappa \leq 0\}$ is bounded and that all its vertices satisfy $\max(K\kappa) = 0$. So the procedure consists in, for every S in $\mathcal{S}$, vertex enumerating

$$\begin{bmatrix} A_{S,0} & b_{S,0} \end{bmatrix} \coloneqq \begin{bmatrix} K & 0 \\ -S & 0 \\ S & 1 \end{bmatrix}, \quad \begin{bmatrix} A_{S,1} & b_{S,1} \end{bmatrix} \coloneqq \begin{bmatrix} K & 1 \\ -S & 0 \end{bmatrix}; \tag{20}$$

then use the resulting V-representations $[V_{S,1}; w_{S,1}]$ and $[V_{S,0}; w_{S,0}]$ to construct the H-representations $[V^{\top}{}_{S,1}, w^{\top}{}_{S,1}]$ and $[V^{\top}{}_{S,0}, 0]$. Vertically concatenate these H-representations for every S to obtain an H-representation for the set of coherent lower previsions on $\mathcal{K}$ and apply redundancy removal to obtain $[\Lambda_{\mathrm{C}}, \alpha_{\mathrm{C}}]$.

Entirely analogously to what was done in Procedures B4 and B5, we can use MOLPs to generate undominated vertex versions of $[V_{S,\gamma}; w_{S,\gamma}]$ for all S in $\mathcal{S}$ and γ in $\{0, 1\}$:

C5. The κ-variant:

$$\begin{aligned} &\text{maximize } \ \kappa, \\ &\text{subject to } \ K\kappa \leq \gamma, \ S\kappa \geq 0 \ \text{ and, if } \gamma = 0, \ S\kappa \leq 1. \end{aligned} \tag{21}$$

C6. The $K\kappa$-variant:

$$\begin{aligned} &\text{maximize } \ K\kappa, \\ &\text{subject to } \ K\kappa \leq \gamma, \ S\kappa \geq 0 \ \text{ and, if } \gamma = 0, \ S\kappa \leq 1. \end{aligned} \tag{22}$$

In principle, the MOLP-based procedures (B4, B5, C5, and C6) should be more efficient than the vertex enumeration ones (B3, C4), as for both the same polytope needs to be mapped, but for the MOLPs only in part, which also results in less redundant constraints to be removed later on. In our numerical experiments, the vertex enumeration variant turned out to be quite efficient: the number of redundant constraints it produces is about the same as the number of non-redundant ones; for our earlier procedure, this quickly grew beyond a difference of an order of magnitude. However, the results for Procedures B4 and C5 were not as good: the M3-solver at our disposal (Löhne 2012) could not deal in reasonable time with sets of gambles that the enumeration-based procedures digested almost instantly (its author explained that it was not designed for large objective vectors). Procedures B5 and C6 could not be tested due to an apparent lack of publicly available M4-solvers.

4.3 Appendix: the NNZM Assumption & Coherence

Given a general set of gambles $\mathcal{K}$, let $\bar{\mathcal{K}}$ be the subset of constant gambles and $\check{\mathcal{K}}$ the subset of non-constant gambles. Let $\bar{b}$ be the vector with the values of the constant gambles and $\hat{\mathcal{K}}$ an NNZM set of gambles associated with $\check{\mathcal{K}}$. The restrictions of a lower prevision $\underline{P}$ on $\bar{\mathcal{K}} \cup \check{\mathcal{K}} \cup \hat{\mathcal{K}}$ to these sets are $\underline{\bar{P}}$, $\underline{\check{P}}$, and $\underline{\hat{P}}$. (Properties of coherent lower previsions used here can be found in Walley 1991, §2.6.1(b),(c).)

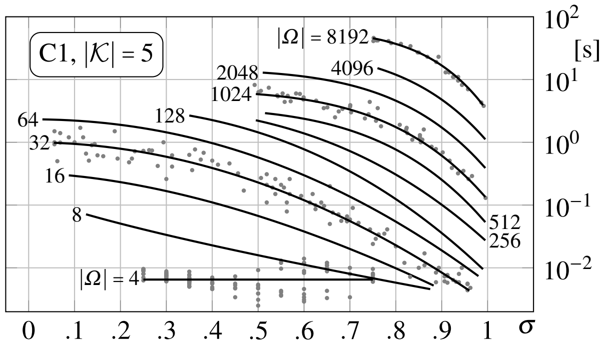
If $\underline{P}$ is coherent, we know that $\underline{P}\beta = \beta$ for any constant gamble β and so the constraints are $\underline{\bar{P}} = \bar{b}$. For any other gamble f in $\mathcal{K}$ we have the linking constraint $\underline{\check{P}}f - \underline{\hat{P}}(f - \min f) = \min f$. Fix $\hat{\mathcal{K}} := \{f - \min f : f \in \check{\mathcal{K}}\}$; this set is NNZM. Let $\hat{A}\underline{\hat{P}} \le \hat{b}$ be the constraints for the polytope of coherent lower previsions $\underline{\hat{P}}$ on $\hat{\mathcal{K}}$, then, using the linking constraints, the corresponding constraints for $\underline{\check{P}}$ on $\check{\mathcal{K}}$ are $\hat{A}\underline{\check{P}} \le \hat{b} + \hat{A}\min$. So the full H-representation of the set of coherent lower previsions $[\underline{\bar{P}};\underline{\check{P}}]$ on $\mathcal{K}$ is

$$\begin{bmatrix} A_{\mathcal{K}} & b_{\mathcal{K}} \end{bmatrix} := \begin{bmatrix} \mathbb{I} & & \bar{b} \\ -\mathbb{I} & & -\bar{b} \\ & \hat{A} & \hat{b} + \hat{A}\min \end{bmatrix}. \tag{23}$$

4.4 Quantitative Results of Numerical Experiments

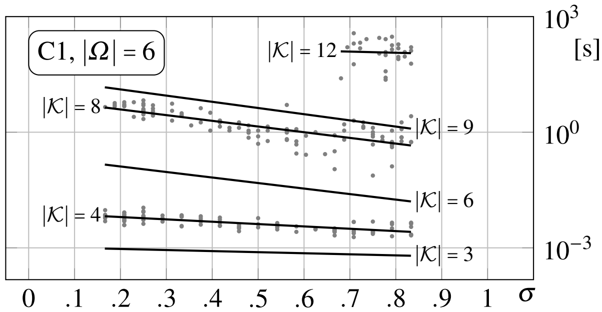
Above, we have already mentioned some qualitative evaluations and comparisons of the different procedures. Here we present more quantitative results. Our CPU-bound numerical (floating point) experiments were run on an Intel i7-2620M processor. (The Python scripts we developed are publicly available: Quaeghebeur, *pycohconstraints*.)

Our experiments showed that the *sparsity* σ, i.e., the fraction of zero components in the gamble matrix K, has an important influence on the running times of our procedures. The graph below indicates that the running time of Procedure C1 decreases exponentially as a function of the sparsity. The approximate equidistance of the curves of doubling possibility space cardinality $|\Omega|$ indicates that the running time increases approximately linearly as a function of $|\Omega|$. The curves are least-squares fits to the data points obtained from randomly generated NNZM gamble sets with values taken from $\{0,\dots,9\}$. To give an idea of the variance, we have also plotted the data points for $|\Omega|$ in $\{4, 32, 1024, 8192\}$ as gray dots.



The same gamble sets were also processed using Procedure C4; the running times were typically 1.5 times, but sometimes 4 times longer. The other procedures were orders of magnitude too slow for reliable testing.

In the graph below, the approximate equidistance of the lines for $|\mathcal{K}|$ in $\{3,6,9\}$ and for $|\mathcal{K}|$ in $\{4,8,12\}$, respectively, indicates that the running time of Procedure C1 increases (at least) exponentially as a function of $|\mathcal{K}|$. Again to give an idea of the variance, we have plotted the data points for $|\mathcal{K}|$ in $\{4,8,12\}$ as gray dots.



5 Correcting Incoherent Lower Previsions

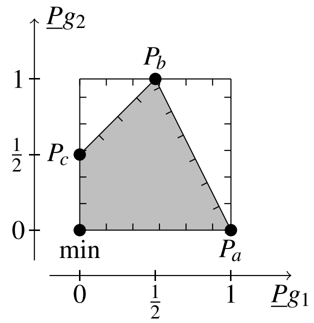
Now that we have procedures for obtaining minimal linear constraint characterizations for lower previsions that avoid sure loss or are coherent, we are ready to look at what lies beyond: sure loss and other forms of incoherence.

Automatic methods for learning lower previsions from data ideally produce coherent lower prevision, but some may not—possibly for good reasons. Also, when eliciting lower previsions from experts—but not in imprecise probability theory—, it is not reasonable to expect the result to be coherent or perhaps even avoid sure loss. For incoherent, but sure loss avoiding lower previsions, we can apply natural extension to perform a pointwise upward correction that makes explicit all implicit commitments. This is appropriate when the user of the automatic method or the elicitee provide informed consent. Otherwise a conservative, downward correction may be more acceptable.

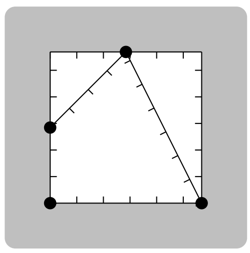
Downward changes of a lower prevision imply a reduction in both explicit and implicit commitments. When it is not possible to decide on the changes with input from the user or the elicitee, automatic downward correction methods are an option, after informed consent. We here propose one such automatic downward correction method.

5.1 Forms of Incoherence

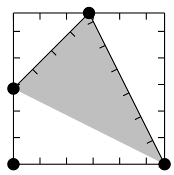
Let us briefly give a categorization of the possible forms of incoherence. To this end, consider a two-gamble example on a possibility space $\{a,b,c\}$: consider the set $\mathcal{K} := \{g_1, g_2\}$, with $g_1 := [1; 1/2; 0]$ and $g_2 := [0; 1; 1/2]$. Using a procedures from Section 4, we have obtained the constraints, drawn using bestubbled lines, delimiting the shaded convex polytope of coherent lower previsions. Its vertices have been named: the vacuous lower prevision min and for every atom ω in $\{a,b,c\}$ the degenerate prevision $P_\omega := [g_1\omega; g_2\omega]$, the columns of $K^\top$.



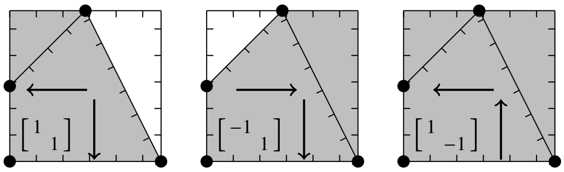
We recalled at the end of Section 3 that coherent lower previsions $\underline{P}$ are bounded, i.e., that $\min \le \underline{P} \le \max$. Our first category of incoherent previsions are those that are out of bounds. On the right, we shaded the magnitude-wise smallest part of this unbounded region in gray.



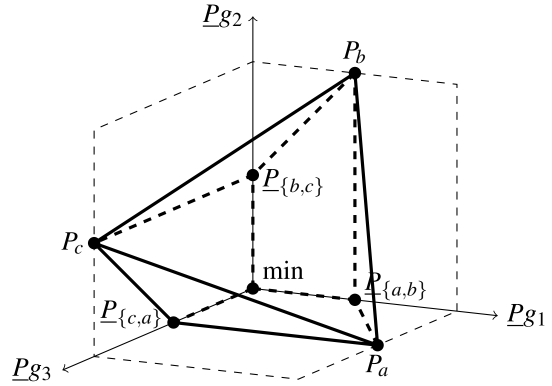
Equations (3)–(4) and (7)–(8) showed us that the convex set of linear previsions can take a central role in both the definitions of avoiding sure loss and coherence. For our example, it is in gray on the right.



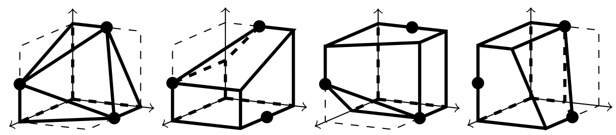
More concretely, Equation (8) made it clear that the polytope of coherent lower previsions is an intersection of polyhedra corresponding to avoiding sure S-loss—i.e., S-dominance by a linear prevision—, one for each S in $\mathcal{S}$. Below, we show, in gray, the part of these polyhedra within bounds, accompanied by their respective S-matrix and the extreme rays of the dominance cone it implies. With each S there corresponds a set whose members incur sure S-loss. The set of incoherent lower previsions is their union.



To get a feel for what constellations can occur when faced with larger sets of gambles, we extend our two-gamble example with a gamble $g_3 := [1/2; 0; 1]$. Below, we give the polytope of coherent lower previsions. It is bounded by the cuboid defined by the min and max points. Its edges in the coordinate planes are shown using thin dashed lines. The new vertices can be characterized for g in $\{g_1, g_2, g_3\}$ by $\underline{P}_A g := \min_{\omega \in A} g\omega$. The range of values attained by the vertex lower previsions is $\{0, 1/2, 1\}$.



Below, we furthermore give the $|\mathcal{S}| = |\mathcal{K}| + 1 = 4$ sets of lower previsions that avoid sure S-loss.



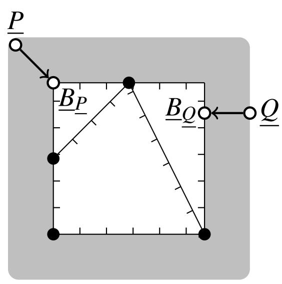
This illustration shows that some lower previsions within bounds may incur sure S-loss for all S; max, for example.

5.2 Bringing Lower Previsions Within Bounds

Correcting a lower prevision $\underline{P}$ that is out of bounds to one that is within bounds is trivial: We replace it by the pointwise closest such lower prevision $\underline{B}_{\underline{P}}$, so for every gamble f in $\mathcal{K}$ we have

$$\underline{B}_{\underline{P}} f := \begin{cases} \min f & \underline{P}f \le \min f, \\ \max f & \underline{P}f \ge \max f, \\ \underline{P}f & \text{otherwise.} \end{cases} \tag{24}$$

This correction method may produce both downward and upward pointwise changes.



From now on we assume that all lower previsions are within bounds.

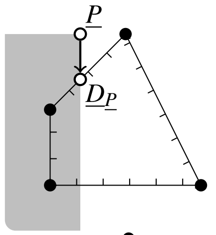
5.3 Maximal Dominated Coherent Lower Previsions

Our proposal for the downward correction of an incoherent lower prevision $\underline{P}$ is the lower envelope of the maximal coherent lower previsions dominated by $\underline{P}$. In other words, it is the nadir point $\underline{D}_P$ of the MOLP (cf. Section 2.2):

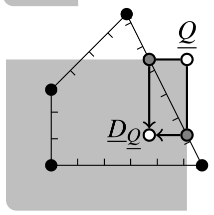
$$\begin{aligned}&\text{maximize } \underline{Q},\\ &\text{subject to } \Lambda_C \underline{Q} \leq \alpha_C \text{ and } \underline{Q} \leq \underline{P}.\end{aligned} \tag{25}$$

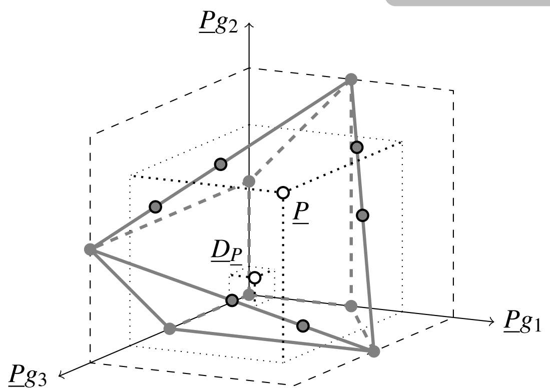
This proposal is essentially the same as the specific so-called *prudential correction* $\overline{P}_H$ mentioned by Pelessoni & Vicig (2003, §3.4). They generalize the interval-probability concept *F-Hülle* (see Weichselberger 2001, 342ff. and 375ff.; translated as *F-cover* in Weichselberger 2000). However, they only aim to apply this correction when sure loss is avoided; we make no such restriction.

On the right, the method is illustrated for two incoherent lower previsions that are within bounds; extreme maximal dominated coherent lower previsions are shown as gray-filled dots.



We should not conclude from these illustrations that the extreme maximal coherent lower previsions dominated by the given incoherent lower prevision can always be reached by reducing single components; a graphical counterexample is given below.





The lower prevision $\underline{D}_P$ satisfies the necessary requirements:

i. It is a downward correction as a lower envelope of lower previsions dominated by $\underline{P}$.
ii. It is coherent by the Lower Envelope Theorem.

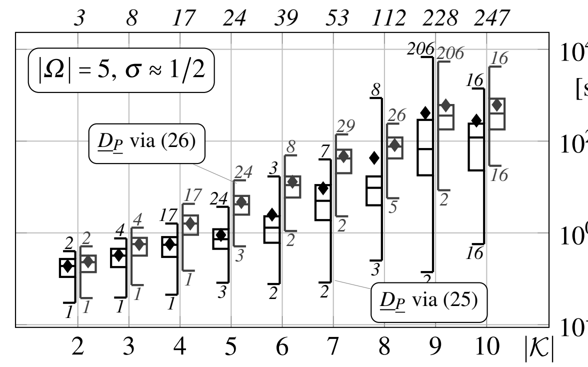
Furthermore, as a nadir point it has a number of further desirable properties:

iii. The correction it embodies is neutral in the sense that no tradeoff between corrections for the different components of $\underline{P}$ is made; this makes it especially suited for unguided corrections.
iv. It is the maximal such neutral correction—the vacuous lower prevision min is another—and therefore preserves as much of the commitments expressed by $\underline{P}$ as possible.
v. The set of coherent lower previsions dominated by an incoherent lower prevision $\underline{P}$ is non-decreasing with pointwise increasing $\underline{P}$. So the more incoherent a lower prevision, the more imprecise its correction.

It is actually not necessary to calculate $[\Lambda_C, \alpha_C]$ in order to find $\underline{D}_P$, because we have a full constraint based characterization of coherence with the H-representation (12). So an alternative to the MOLP (25) is the following MOLP:

$$\begin{aligned}&\text{maximize } \underline{Q},\\ &\text{subject to } A_{\underline{Q}}\underline{Q} + A_\mu \mu \leq b \text{ and } \underline{Q} \leq \underline{P},\end{aligned} \tag{26}$$

where we use the notation of Equation (12). (Weichselberger 2001, 468ff, also proposes an as of yet untested algorithm that is essentially based on a representation such as the one given by Equation (12).) This problem has $(|\mathcal{K}|+1)\cdot|\Omega|$ more variables than the MOLP (25), which has $|\mathcal{K}|$ variables. It has $(|\mathcal{K}|+1)\cdot(|\mathcal{K}|+|\Omega|+2)$ constraints, whereas the MOLP (25) typically has of the order of $3\cdot|\mathcal{K}|$ constraints. This results in a greater average running time for the nadir computation using the alternative MOLP, even if we take the setup time—calculating $[\Lambda_C, \alpha_C]$ (cf. Section 4.4) versus generating $[A_{\underline{Q}}, A_\mu, b]$ (about 10^{-3}s)—into account. This can be seen in the graphical summary of the results of our numerical experiments, which we are going to describe next. (The Octave/Matlab scripts we developed are publicly available: Quaeghebeur, *mcohconstraints*.)



In this experiment, for each value of $|\mathcal{K}|$ in $\{2, \ldots, 10\}$, we generated about 10 NNZM gamble sets $\mathcal{K}$—as in Section 4.4—with sparsity σ fixed at approximately $1/2$, on a possibility space Ω with $|\Omega| = 5$. Next, we calculated the corresponding $[\Lambda_C, \alpha_C]$—using Procedure C1—and generated the corresponding $[A_{\underline{Q}}, A_\mu, b]$. Finally, for each $\mathcal{K}$, we generated about 10 incoherent lower previsions within bounds to correct. This we did using both the MOLP (25) and the MOLP (26), resulting in about 100 computation time samples per $|\mathcal{K}|$ for each of both approaches. Each of these sample sets is summarized using a box plot indicating minimum, lower quartile, median, upper quartile, and maximum; its arithmetic mean is indicated with a lozenge. Black left-leaning box plots are used for the results obtained with the MOLP (25); darkgray right-leaning ones for those obtained with the MOLP (26).

With the M3-solver we used (Löhne 2012), average computation time seems to increase exponentially as a function of $|\mathcal{K}|$. Surprisingly, the number of extreme maximal dominated coherent lower previsions is not a major factor. This is illustrated by the number of these extreme points found for the minimum and maximum computation times—put in italics near the respective box plot whiskers—and the maximum number of extreme points in the sample—listed in italics at the top edge of the plot axis.

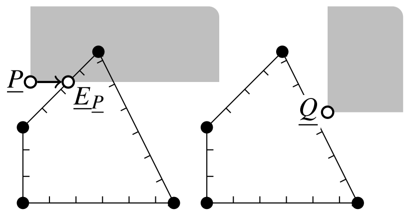
The M3-solver does compute all these extreme points, so we suspect that it is highly inefficient for the task at hand. Therefore we believe substantial efficiency gains can be achieved by switching to an M4-solver, which we expect to be *output sensitive*, i.e., to depend on the number of extreme points. Nadir point calculation algorithms that do not need to calculate all these extreme points (e.g., Alves & Costa 2009) should provide a further increase in efficiency. Because elicited lower previsions can be expected to generally be closer to coherent than our randomly generated ones, we also expect them to generally dominate less extreme points and thus, because of output sensitivity, be faster to correct. We already observed this phenomenon for randomly generated sure loss avoiding lower previsions.

5.4 Least Dominating Coherent Lower Prevision

For completeness's sake, let us also have a look at upward correction using the MOLP approach. Given an incoherent lower prevision $\underline{P}$, we consider the set of minimal pointwise dominating coherent lower previsions; this is the solution to the following MOLP:

$$\begin{aligned}&\text{minimize } \underline{E}_P,\\ &\text{subject to } \Lambda_C\underline{E}_P \leq \alpha_C \text{ and } \underline{E}_P \geq \underline{P}.\end{aligned} \tag{27}$$

Because of the Lower Envelope Theorem, there is only one such $\underline{E}_P$, so we may replace this vector objective by the scalar objective $\sum_{g\in\mathcal{K}} \underline{E}_P g$, reducing the problem to a plain LP. This coherent lower prevision $\underline{E}_P$ is the one least dominating $\underline{P}$, to wit, its natural extension (cf. Walley 1991, §3.1). This plain LP method for obtaining it is illustrated on the right.



Again, we can use the H-representation (12) to formulate an alternative to the MOLP (27):

$$\begin{aligned}&\text{minimize } \underline{E}_P,\\ &\text{subject to } A_{\underline{E}_P}\underline{E}_P + A_\mu\mu \leq b \text{ and } \underline{E}_P \geq \underline{P}.\end{aligned} \tag{28}$$

Thanks to the block structure of the constraint matrix, it is straightforward to deduce some well-known facts:

i. It is necessary that $\underline{P}$ avoids sure loss for a solution $\underline{E}_P$ to exist (cf. right-hand side illustration above).
ii. For each gamble g in $\mathcal{K}$, we can calculate the corresponding natural extension component $\underline{E}_P g$ separately as $\max\{g^\top\mu : \underline{P} \leq K^\top\mu \wedge \mu \geq 0 \wedge 1^\top\mu = 1\}$.

These facts raise the currently still open question of whether there exist specific classes of incoherent lower previsions $\underline{P}$ for which the calculation of $\underline{D}_P$ can be simplified, e.g., to separate calculations for each component.

6 Conclusions

We hope that you are now convinced of the fact that the availability of a finite, minimal linear constraints characterization of coherence opens doors for many new numerical applications dealing with the set of coherent lower previsions. In our application, downward correction of incoherent lower previsions, we saw that it proved useful to keep the running time of the inherently computationally complex implementation of our proposed method a bit in check. We determined that currently, sets of up to 5 gambles can be dealt with sufficiently fast even for interactive applications. In a domain where complex systems are often decomposed into smaller ones linked in some network structure, this is not overly restrictive.

We also hope that this paper has kindled your interest in the application of multi-objective linear programming to imprecise probability problems. We believe that beyond the two applications of them presented in this paper, there are bound to be more in our research field because of the common underlying assumption that incomparability should be modeled, not avoided.

There are some unfinished strands in this paper:

i. Testing an efficient projection implementation (cf. Kvasnica et al. 2006).
ii. Finding and testing a MOLP simplex solver (cf. M4) and a nadir computation algorithm (cf. M2).
iii. Theoretically investigate whether $\underline{D}_P$ can be calculated more efficiently if $\underline{P}$ satisfies some additional conditions beyond being within bounds.

We hope these are picked up by us, or others, in the future.

Acknowledgments

Thanks to the reviewers for pointing out very relevant literature. Thanks to J. De Bock for providing me with extracts from Weichselberger's 2001 tome and Th. Augustin for helping me interpret them correctly. Thanks to G. de Cooman and N. Huntley for sharing their thoughts about the downward correction idea. Thanks to A. P. Pedersen for pointers to potentially relevant belief revision texts, which, I am sorry to say, I could not yet follow up on.

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