International Journal of Algebra, Vol. 6, 2012, no. 6, 271 - 275

A Note on Strongly Euclidean Semirings

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Abstract. Theory of ideals in the semiring $\mathbb{Z}_0^+$ was given by P. J. Allen and L. Dale [2] and they proved that $\mathbb{Z}_0^+$ is a Noetherian semiring. Further, characterization of subtractive ideals and prime ideals in the semiring $\mathbb{Z}_0^+$ has been given by V. Gupta and J. N. Chaudhari ([3], [7]). In this paper, we study ideal theory in the semiring ($\mathbb{Z}_0^+$, gcd, lcm) and obtain characterizations of Q-ideals, prime ideals, maximal ideals and primary ideals. Also it is proved that, if R is a strongly Euclidean IS-semiring, then R and $R_{n\times n}$ are principal ideal semirings.

Mathematics Subject Classification: 16Y60, 11A05, 11A41

Keywords: Semiring, subtractive ideal, Q-ideal, prime ideal, semiprime ideal, primary ideal, Noetherian semiring, strongly Euclidean semiring

1. Introduction

A non-empty set R together with two associative binary operations addition and multiplication is called a semiring if i) addition is a commutative operation ii) there exists $0 \in R$ such that $x+0 = x = 0+x$, $x\cdot 0 = 0 = 0\cdot x$ for each $x \in R$ and iii) multiplication distributes over addition both from left and right. The concept of ideal, finitely generated ideal, principal ideal, prime ideal, maximal ideal, semiprime ideal, primary ideal in a commutative semiring with identity 1 can be defined on the similar lines as in commutative rings with identity 1. All semirings are assumed to be semirings with identity element. $\mathbb{Z}_0^+$ ($\mathbb{N}$) will denote the set of all non-negative (positive) integers. An ideal I of a semiring R is called (1) subtractive ideal (= k-ideal) if a, $a+b \in I$, $b \in R$, then $b \in I$. (2) Q-ideal (= partitioning ideal) if there exists a subset Q of R such that

1. $R = \cup\{q + I : q \in Q\}$.
2. if q_1, $q_2 \in Q$, then $(q_1 + I) \cap (q_2 + I) \neq \emptyset \Leftrightarrow q_1 = q_2$.

Lemma 1.1. *(**[1], Lemma 7**) Let I be a Q-ideal of a semiring R. If $x \in R$, then there exists a unique $q \in Q$ such that $x + I \subseteq q + I$.*

Theorem 1.2. *([6], **Theorem 1.4**) An ideal I of a strongly Euclidean semiring R is Q-ideal if and only if I is principal ideal.*

Lemma 1.3. *([3], **Page 648**) A is an ideal of the matrix semiring $R_{n\times n}$ if and only if there exists an ideal I of R such that $A = I_{n\times n}$.*

2. Ideals in the Semiring ($\mathbb{Z}_0^+$, gcd, lcm)

For $a, b \in \mathbb{Z}_0^+$, we define,

1) $a \oplus b = \gcd\{a, b\}$ if $a, b \in \mathbb{N}$;
2) $a \odot b = \text{lcm}\{a, b\}$ if $a, b \in \mathbb{N}$;
3) $a \oplus 0 = a$ and $a \odot 0 = 0$ for all $a \in \mathbb{Z}_0^+$;
4) $ab =$ usual product of a and b;
5) $a^n = aaa...a$(n-times).

Clearly ($\mathbb{Z}_0^+$, $\oplus$, $\odot$) is a commutative semiring with identty element 1. For $a \in (\mathbb{Z}_0^+, \oplus, \odot)$, we denote, $< a > = \{n \odot a : n \in \mathbb{Z}_0^+\}$, the principal ideal generated by a.

Lemma 2.1. *If $a \in (\mathbb{Z}_0^+, \oplus, \odot)$, then $< a > = \{na : n \in \mathbb{Z}_0^+\}$.*

Proof. We have $na = na \odot a \in < a >$. Thus $\{na : n \in \mathbb{Z}_0^+\} \subseteq < a >$. On the other hand, if $x \in < a >$, then there exists $n \in \mathbb{Z}_0^+$ such that $x = n \odot a = ka$ for some $k \in \mathbb{Z}_0^+$. So $< a > \subseteq \{na : n \in \mathbb{Z}_0^+\}$. □

Lemma 2.2. *Every ideal of ($\mathbb{Z}_0^+$, $\oplus$, $\odot$) is a principal ideal.*

Proof. Let I be a non-zero ideal in ($\mathbb{Z}_0^+$, $\oplus$, $\odot$) and choose least non-zero element $d \in I$. Claim: $I = < d >$. If $a \in I$, then $a \oplus d \in I$ and $a \oplus d = d$ as d is the least non-zero element of I. Now $a = kd$ for some $k \in \mathbb{Z}_0^+$. By Lemma 2.1, $a \in < d >$. Hence $I \subseteq < d >$. On the other hand, for any $n \in \mathbb{Z}_0^+$, $nd = nd \odot d \in I$. By Lemma 2.1, $< d > \subseteq I$. □

Theorem 2.3. *($\mathbb{Z}_0^+$, $\oplus$, $\odot$) is a Noetherian semiring.*

Lemma 2.4. *Every ideal of ($\mathbb{Z}_0^+$, $\oplus$, $\odot$) is subtractive.*

Proof. Let I be an ideal of ($\mathbb{Z}_0^+$, $\oplus$, $\odot$). By Lemma 2.2, $I = < d >$ for some $d \in I$. If a, $a \oplus b \in I = < d >$, then a, $a \oplus b = \gcd\{\text{a, b}\}$ are multiples of d and hence b is a multiple of d. By Lemma 2.1, $b \in < d > = I$. □

Lemma 2.5. *If I is a non-zero proper ideal of ($\mathbb{Z}_0^+$, $\oplus$, $\odot$), then I is not a Q-ideal.*

Proof. Let I be a non-zero proper ideal of ($\mathbb{Z}_0^+$, $\oplus$, $\odot$). By Lemma 2.2, $I = < d >$ for some $d \in \mathbb{Z}_0^+ - \{0, 1\}$. Take $d = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$ where p_1, p_2,..., p_k are pairwise distinct primes and $k, r_i \in \mathbb{N}$. Suppose that I is a Q-ideal. We claim that there exists a unique $q \in Q$ such that $p \in q \oplus I$ for all primes p other than $p_i'^s$. Let p', p'' be any distinct primes other than $p_i'^s$. By Lemma 1.1, there are unique q_1, $q_2 \in Q$ such that $p' \in p' \oplus I \subseteq q_1 \oplus I$ and $p'' \in p'' \oplus I \subseteq q_2 \oplus I$. Since p', p'' are primes other than $p_i'^s$, $1 = p' \oplus d \in q_1 \oplus I$ and $1 = p'' \oplus d \in q_2 \oplus I$.

Hence $(q_1 \oplus I) \cap (q_2 \oplus I) \neq \emptyset$. Since I is a Q-ideal, $q_1 = q_2 = q$ say. Now we have a unique $q \in Q$ such that $p \in q \oplus I$ for all primes p other than $p_i^{'s}$. Clearly $q \geq 1$. By above claim choose a prime $f > q$ such that $f \in q \oplus I$. By Lemma 2.1, $f = q \oplus nd$ for some $n \in \mathbb{Z}_0^+$. So $f \mid q$, a contradiction. Therefore I is not a Q-ideal. □

Theorem 2.6. *$\{0\}$ and $\mathbb{Z}_0^+$ are the only Q-ideals in the semiring $(\mathbb{Z}_0^+, \oplus, \odot)$.*

Proof. Let I be an ideal of $(\mathbb{Z}_0^+, \oplus, \odot)$. If $I = \{0\}$, then clearly I is a Q-ideal of $(\mathbb{Z}_0^+, \oplus, \odot)$ with $Q = \mathbb{Z}_0^+$. If $I = \mathbb{Z}_0^+$, then I is a Q-ideal of $(\mathbb{Z}_0^+, \oplus, \odot)$ with $Q = \{0\}$. □

Theorem 2.7. *I is a non-zero prime ideal in $(\mathbb{Z}_0^+, \oplus, \odot)$ if and only if $I = < p^r >$ for some prime p and $r \geq 1$.*

Proof. Let I be a non-zero prime ideal in $(\mathbb{Z}_0^+, \oplus, \odot)$. By Lemma 2.2, $I = < d >$ where $d = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$ where p_1, p_2,..., p_k are pairwise distinct primes and $r_i \in \mathbb{N}$. If $k \geq 2$, then $p_1^{r_1} \odot (p_2^{r_2} ... p_k^{r_k}) = p_1^{r_1} p_2^{r_2} ... p_k^{r_k} \in I$ but $p_1^{r_1} \notin I$ and $p_2^{r_2} ... p_k^{r_k} \notin I$, a contradiction to I is a prime ideal. Hence $k = 1$. Now $d = p_1^{r_1}$. Conversely, let $I = < p^r >$ for some prime p and $r \geq 1$ and let $a \odot b \in I = < p^r >$. By Lemma 2.1, $p^r \mid lcm\{a, b\}$ implies $p^r \mid a$ or $p^r \mid b$. Again by Lemma 2.1, $a \in I$ or $b \in I$ and hence I is a prime ideal of $(\mathbb{Z}_0^+, \oplus, \odot)$. □

Theorem 2.8. *I is a non-zero maximal ideal in $(\mathbb{Z}_0^+, \oplus, \odot)$ if and only if $I = < p >$ for some prime p.*

Proof. Let I be a non-zero maximal ideal in $(\mathbb{Z}_0^+, \oplus, \odot)$. By Lemma 2.2, $I = < d >$ for some $d \in \mathbb{Z}_0^+$. If d is not prime, then $d = pq$ for some $1 < p < d$ and $1 < q < d$. But then $I = < d > \subsetneq < p > \subsetneq \mathbb{Z}_0^+$, a contradiction to I is a maximal ideal. Hence d is a prime number. Conversely, suppose that $I = < p >$ for some prime p. Let J be any ideal of $\mathbb{Z}_0^+$ such that $I \subseteq J \subsetneq \mathbb{Z}_0^+$. By Lemma 2.2, $J = < d >$ for some $d > 1$. Since $< p > = I \subseteq J = < d >$, $d = p$. Hence I is a maximal ideal. □

Theorem 2.9. *Every ideal of the semiring $(\mathbb{Z}_0^+, \oplus, \odot)$ is semiprime.*

Proof. Let I be a non-zero ideal in $(\mathbb{Z}_0^+, \oplus, \odot)$ and $a \odot a \in I$. But then $a \in I$. □

Theorem 2.10. *A non-zero ideal I of the semiring $(\mathbb{Z}_0^+, \oplus, \odot)$ is primary if and only if it is a prime ideal.*

Proof. Let I be a primary ideal of $(\mathbb{Z}_0^+, \oplus, \odot)$ and $a \odot b \in I$. Therefore $a \in I$ or $b \odot b \odot b \odot ... \odot b \in I$ i.e. $a \in I$ or $b \in I$. Hence I is a prime ideal. Converse is trivial. □

3. Strongly Euclidean Semirings

Definition 3.1. *A semiring R is called IS-semiring if every ideal of R is subtractive.*

Example 3.2. By Lemma 2.4, the semiring $(\mathbb{Z}_0^+, \oplus, \odot)$ is a IS-semiring. By Proposition ([3], Proposition 2.19), the semiring $(\mathbb{Z}_0^+, +, \cdot)$ is not IS-semiring. □

Definition 3.3. *A semiring R is called principal ideal semiring (PIS) if every ideal of R is principal ideal.*

Definition 3.4. *A commutative semiring R is called strongly Euclidean if there exists a function $d : R - \{0\} \to \mathbb{Z}_0^+$ such that (1) $d(ab) \geq d(a)$ for all $a, b \in R - \{0\}$ and (2) if $a, b \in R$ with $b \neq 0$, then there exist unique $q, r \in R$ such that $a = bq + r$ where either $r = 0$ or $d(r) < d(b)$.*

Theorem 3.5. *Every strongly Euclidean IS-semiring is a principal ideal semiring.*

Proof. Let R be a strongly Euclidean IS-semiring with function d and I an ideal of R, $I \neq 0$. Let $A = \{d(a) \in \mathbb{Z}_0^+ : a \in I - \{0\}\}$. Then A has the least element say $d(a)$. We claim that $I = < a >$. Let $x \in I$. Then there exist unique $q, r \in R$ such that $x = aq + r$ where $r = 0$ or $d(r) < d(a)$. If $r \neq 0$, then $r \in I$, since I is a subtractive ideal. As $d(a)$ is the least element of A, $d(a) \leq d(r)$, a contradiction. Hence $r = 0$. Now $x = aq \in < a >$. Thus $I \subseteq < a >$. But $< a > \subseteq I$. So $I = < a >$. Hence R is a principal ideal semiring. □

Converse of the Theorem 3.5 is not true.

Example 3.6. By Lemma 2.2, $R = (\mathbb{Z}_0^+, \oplus, \odot)$ is a PIS. If R is strongly Euclidean semiring, then by Theorem 1.2, every principal ideal of R is a Q-ideal, a contradiction to Lemma 2.5. Hence R is not strongly Euclidean semiring. □

Example 3.7. The semiring $(\mathbb{Z}_0^+, +, \cdot)$ is a strongly Euclidean semiring but not PIS. □

Example 3.8. The semiring $R = (\mathbb{Z}_0^+ \cup \{\infty\}, max, min)$ is IS-semiring. By Theorem ([5], Theorem 5), $I = \mathbb{Z}_0^+$ is not a principal ideal and hence R is not a PIS. So by Theorem 3.5, R is not a strongly Euclidean semiring. □

Theorem 3.9. *If R is a strongly Euclidean IS-semiring, then $R_{n\times n}$ is a PIS.*

Proof. Let R be a strongly Euclidean IS-semiring and A be any ideal of $R_{n\times n}$. By Lemma 1.3, $A = I_{n\times n}$ for some ideal I of R. By Theorem 3.5, R is PIS. So I is a principal ideal say $I = < a >$. We claim that $A = < B >$ where $B = \begin{bmatrix} a & 0 & ... & 0 \\ 0 & a & ... & 0 \\ .. & .. & ... & .. \\ 0 & 0 & ... & a \end{bmatrix}$. Let $X = \begin{bmatrix} x_{11} & x_{12} & ... & x_{1n} \\ x_{21} & x_{22} & ... & x_{2n} \\ .. & .. & ... & .. \\ x_{n1} & x_{n2} & ... & x_{nn} \end{bmatrix} \in A = I_{n\times n}$. Therefore $x_{ij} \in I$ $= < a >$. So $x_{ij} = t_{ij}a$ where $t_{ij} \in R$ for all i, j. Take $T = \begin{bmatrix} t_{11} & t_{12} & ... & t_{1n} \\ t_{21} & t_{22} & ... & t_{2n} \\ .. & .. & ... & .. \\ t_{n1} & t_{n2} & ... & t_{nn} \end{bmatrix}$

$\in R_{n\times n}$. Then $X = TB \in < B >$. Thus $A \subseteq < B >$. Other inclusion is trivial. Hence $A = < B >$. Thus $R_{n\times n}$ is a PIS. □

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Received: November, 2011