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A TIME-STEPPING SCHEME FOR QUASISTATIC MULTIBODY SYSTEMS *

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Abstract

Two new instantaneous-time models for predicting the motion and contact forces of three-dimensional, quasistatic multi-rigid-body systems are developed; one linear and one nonlinear. The nonlinear characteristic is the result of retaining the usual quadratic friction cone in the model. Discrete-time versions of these models provide the first time-stepping methods for such systems. As a first step to understanding their usefulness in simulation and manipulation planning, a theorem defining the equivalence of solutions of a time-stepping method for the nonlinear model and a global optimal solution of a related convex optimization problem is given. In addition, a Proposition giving necessary and sufficient conditions for solution uniqueness of the nonlinear time-stepping method is given. Finally, a simple example is discussed to help develop intuition about quasistatic systems and to solidify the reader's understanding of the theorem and proposition.

1 Introduction

Robots are primarily passive observers and simple electronic companions in the unstructured environments that exist outside factories. This is true despite the fact that, as a society, enormous productivity gains could be accrued by expanding the skills of robots to include manipulation tasks; tasks that cannot be accomplished without making and breaking contact between the robot and physical objects in a controlled fashion. Nearly one million house-hold robots are in use world wide today, but these robots cannot perform manipulation tasks autonomously. Even the highly capable Sony QRIO robot cannot do such tasks, although it can walk and dance on sloping terrain. Currently, robotic dexterous manipulation can only be performed in unstructured environments by tele-operation, and it is well-known that this approach is exceedingly slow and places great demand on the operator. As a result, autonomous grasping controllers are being developed, but are still of limited capability [11].

Manipulation tasks can be partitioned into two classes: dynamic and quasistatic. The former class is by far the broadest and includes high-speed assembly, juggling, and running. However, despite being narrower, the latter class includes a large number of important tasks, such as low-speed assembly, static grasping, walking using tripods of support. The ability to perform tasks in this class motivate the study presented below.

1.1 Background

The field of multibody dynamics has been of interest since DaVinci's work in the 1490's. His interest stemmed from a desire to build better machines. About 250 years later, some basic "laws" of mechanics had been developed by Newton and Coulomb, which allowed one to formulate an instantaneous-time mathematical model of dynamic multi-rigid-body systems. This model is composed of the Newton-Euler equation, Coulomb's friction law, and nonpenetration constraints with unknown con-

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tact forces and body accelerations. In 1895, Painleve was the first to discover that this model does not always admit a solution (this is sometimes referred to as Painleve's paradox) [13]. Existence and uniqueness questions were studied for more general systems after the advent of complementarity theory in the 1960's [5]. In particular, Lötstedt found that when friction is absent, the model can be cast as a linear complementarity problem (LCP) that possesses a property known as "w-uniqueness." The physical interpretation of this property is that the body accelerations are unique, but the contact forces are not [9,10]. Since Lötstedt's work, existence and uniqueness properties have been extended to include limited results for systems with friction [14,20]. Specifically, solution existence can only be guaranteed if the friction coefficients at the contacts are below some threshold value, which unfortunately, is exceedingly difficult to compute and is sensitive to the contact geometry.

Because of the weakness of the existence and uniqueness results, it is not advisable to apply standard time-stepping methods directly to the instantaneous model [7, 8]. A superior approach is to derive a discrete-time model written in terms of the unknown contact impulses and body velocities [1, 17]. The Stewart-Trinkle formulation results in an LCP that incorporates constraint stabilization and is nearly always solvable [17]. Moreover, when a solution exists, it can be found using Lemke's algorithm [5]. If a solution to the Stewart-Trinkle LCP does not exist, one simply drops the constraint stabilization term, yielding the Anitescu-Potra mixed LCP for which a solution always exists and can be found by Lemke's algorithm [1]. One might wonder why a solution always exists for the discrete-time model when the same is not true for the instantaneous-time model. An intuitive explanation is that since the discrete-time model is written in terms of impulses (applied over the current time-step), it implicitly expands the space of possible contact force functions to include infinite impulses. This is consistent with the resolution to Painleve's paradox offered by Mason and Wang [12].

Since time-stepping methods are now reasonably well developed for dynamic rigid body systems [1–3, 16, 17], one might wonder why the focus of this paper is on quasistatic models. The reasons spurs from an interest in the development of planning algorithms. Dynamic systems "live" in state space, which has twice the dimension of configuration space, in which quasistatic systems "live." Secondly, quasistatic systems move slowly, so inertial, Coriolis, and impulsive forces are absent. Finally, in some cases, a quasistatic manipulation plan can serve as a good initial guess for a dynamic plan.

In previous work, Pang *et al.* [15] formulated an instantaneous-time planar quasistatic model as an uncoupled complementarity problem (UCP) and developed a bilinear programming algorithm to solve it. In this paper, the work is extended to three dimensions, a simple time-stepping scheme is derived, and a new uniqueness result is given.

2 Instantaneous-time models

Let $q \in \Re^{n_q}$ be the configuration of a system of rigid bodies, $\nu \in \Re^{n_\nu}$ be the generalized velocity, and $f(q,t) \in \Re^{n_\nu}$ represent the applied external generalized force, with t being time. Further, let $\{\lambda_{in} \geq 0\}_{i=1}^{n_c}$ be the nonnegative normal force at the i^{th} contact point, and λ_{it} and λ_{io} be the corresponding orthogonal friction force components. Since a quasistatic system must satisfy equilibrium at all times, the equilibrium equation is needed. It can be written as:

$$0 = W_{\mathrm{n}}(q)\lambda_{\mathrm{n}} + W_{\mathrm{t}}(q)\lambda_{\mathrm{t}} + W_{\mathrm{o}}(q)\lambda_{\mathrm{o}} + f(q,t) \tag{1}$$

where $\lambda_{\mathrm{n}},\ \lambda_{\mathrm{t}},\ \lambda_{\mathrm{o}} \in \Re^{n_c}$ are the vectors of normal and friction force components of the contacts (also called wrench intensities), $W_{\mathrm{n}},\ W_{\mathrm{t}},\ W_{\mathrm{o}} \in \Re^{(n_c \times n_\nu)}$, are matrices whose columns are unit wrenches of the contact normals, and orthogonal tangent plane directions.

The system must also obey a nonpenetration constraint at each contact and a complementarity relationship between the normal component of contact force and the distance function $\psi_{in}(q,t)$ between the contacting bodies. The complementarity constraint is:

$$0 \leq \lambda_{\mathrm{n}} \ \perp\ \psi_{\mathrm{n}}(q,t) \geq 0 \tag{2}$$

where $\psi_{\mathrm{n}}(q,t) \in \Re^{n_\nu}$ is the vector of distance functions with i^{th} element given by $\psi_{in}(q,t)$, the symbol $\perp$ implies perpendicularity (*i.e.,* $\lambda_{\mathrm{n}} \cdot \psi_{\mathrm{n}} = 0$). The physical interpretation of equation (2) is that a force may act at contact i only if the distance between the bodies is zero.

The force at each contact is assumed to obey Coulomb's friction law, which states that the contact force must lie within a cone during rolling contact and must lie on the boundary of the cone in the direction that dissipates the most energy during sliding. Since sliding is a function of body velocities, the following kinematic relationship will be needed:

$$\dot{q} = G(q)\nu \tag{3}$$

where G depends on the specific orientation parameterization used for three-dimensional systems and is the identity matrix for planar systems. Equation (3) provides a connection between the distance functions and the matrix W_{n} as follows: $W_{\mathrm{n}}^T = \frac{\partial \psi_{\mathrm{n}}}{\partial q} G$. Note that one can define analogous (local) tangential displacement functions ψ_{t} and ψ_{o} with elements ψ_{it} and ψ_{io} for which the following hold: $W_{\mathrm{t}}^T = \frac{\partial \psi_{\mathrm{t}}}{\partial q} G$ and $W_{\mathrm{o}}^T = \frac{\partial \psi_{\mathrm{o}}}{\partial q} G$.

As noted above, when the contact is sliding, the contact force must be one from among all those in the cone that maximizes energy dissipation. For $\lambda_{in} \geq 0$, let $\mathcal{F}_i(\mu_i, \lambda_{in})$ denote the friction cone at contact i:

$$\mathcal{F}_i(\mu_i, \lambda_{\text{in}}) = \{(\lambda_{\text{it}}, \lambda_{\text{io}}):\ \mu_i^2\lambda_{\text{in}}^2 - \lambda_{\text{it}}^2 - \lambda_{\text{io}}^2 \geq 0\} \quad (4)$$

where μ_i is the coefficient of friction acting at contact i.

Next, define orthogonal sliding velocity components v_{it} and v_{io}. The vectors of sliding velocities for all the contacts are: $v_{\text{t}} = W_{\text{t}}^T\nu + \frac{\partial\psi_{\text{t}}}{\partial t}$ and $v_{\text{o}} = W_{\text{o}}^T\nu + \frac{\partial\psi_{\text{o}}}{\partial t}$ with i^{th} elements $v_{\text{it}} = W_{\text{it}}^T\nu + \frac{\partial\psi_{\text{it}}}{\partial t}$ and $v_{\text{io}} = W_{\text{io}}^T\nu + \frac{\partial\psi_{\text{io}}}{\partial t}$, respectively. Then Coulomb's law at contact i may be written as follows:

$$(\lambda_{it}, \lambda_{io}) \in \underset{(\lambda_{it}, \lambda_{io}) \in \mathcal{F}_i}{\arg\max} \left(-\lambda_{it}v_{\text{it}} - \lambda_{io}v_{\text{io}}\right), \quad (5)$$

which has a useful equivalent formulation [20]:

$$0 = \mu_i\lambda_{\text{in}}(W_{\text{it}}^T\nu + \frac{\partial\psi_{it}}{\partial t}) + \lambda_{it}\sigma_i \quad (6)$$

$$0 = \mu_i\lambda_{\text{in}}(W_{\text{io}}^T\nu + \frac{\partial\psi_{io}}{\partial t}) + \lambda_{io}\sigma_i \quad (7)$$

$$0 \leq \sigma_i \perp \mu_i^2\lambda_{\text{in}}^2 - \lambda_{\text{it}}^2 - \lambda_{\text{io}}^2 \geq 0 \quad (8)$$

where σ_i is a Lagrange multiplier arising from the conversion of the maximum dissipation condition from its "argmax" form into the inequality form given above. Note that at a solution of these conditions, $\sigma_i = \sqrt{v_{\text{it}}^2 + v_{\text{io}}^2}$, which is the magnitude of the slip rate at contact i. One should also observe that for all slip velocities such that at least one of v_{it} and v_{io} is nonzero, then equations (6) and (7) uniquely determine the direction of the tangential force vector $(\lambda_{it}, \lambda io)$ and equation (8) uniquely determines its length.

Compactly, Coulomb's law for all contacts is:

$$0 = (U\lambda_{\text{n}}) \circ (W_{\text{t}}^T\nu + \frac{\partial\psi_{\text{t}}}{\partial t}) + \lambda_{\text{t}} \circ \sigma \quad (9)$$

$$0 = (U\lambda_{\text{n}}) \circ (W_{\text{o}}^T\nu + \frac{\partial\psi_{\text{o}}}{\partial t}) + \lambda_{\text{o}} \circ \sigma \quad (10)$$

$$0 \leq \sigma \perp (U\lambda_{\text{n}}) \circ (U\lambda_{\text{n}}) - \lambda_{\text{t}} \circ \lambda_{\text{t}} - \lambda_{\text{o}} \circ \lambda_{\text{o}} \geq 0 \quad (11)$$

where U is the diagonal matrix with i^{th} diagonal element equal to μ_i and $\circ$ connotes the Hadamard product.

Some of the above equations are nonlinear in the unknowns (forces, configuration, and velocity), so their direct use in a time-stepping scheme would require the solution of mixed nonlinear complementarity problems (mixed NCPs). In order to obtain a scheme based on LCPs, a piecewise linear approximation of the quadratic friction cone with nonnegative force variables is needed (see Figure 1). Let n_{d} friction force direction vectors d_j be chosen such that they positively span the space of possible friction forces, and let $\{(\lambda_{i\text{f}})_j\}_{j=1}^{n_{\text{d}}}$ be the friction force components in those directions. Also, let $\{(\psi_{i\text{f}}(q,t))_j\}_{j=1}^{n_{\text{d}}}$ be the corresponding (local) tangential displacement function. Then the equilibrium equation can be approximated as:

$$0 = W_{\text{n}}(q)\lambda_{\text{n}} + W_{\text{f}}(q)\lambda_{\text{f}} + f(q,t) \quad (12)$$

where $\lambda_{\text{f}} \in \Re^{n_{\text{c}}n_{\text{d}}}$ has n_{c} elements $\lambda_{i\text{f}} \in \Re^{n_{\text{d}}}$ with elements $(\lambda_{i\text{f}})_j$, the vector $\psi_{\text{f}} \in \Re^{n_{\text{c}}n_{\text{d}}}$ is defined analogously, and $W_{\text{f}}^T = \frac{\partial\psi_{\text{f}}}{\partial q}G$.

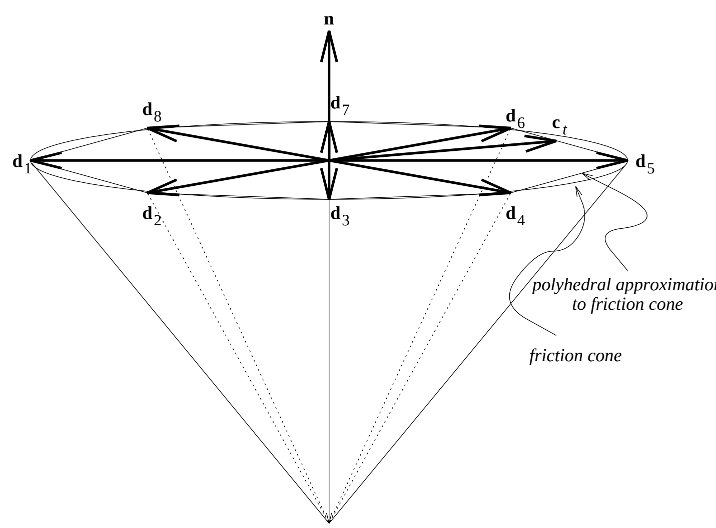

Figure 1. Friction cone approximated by an eight-sided pyramid defined by friction direction vectors d_j.

The approximate friction cone can be represented as:

$$\overline{\mathcal{F}}_i(\mu_i, \lambda_{\text{in}}) = \{\lambda_{i\text{f}} \mid \mu_i\lambda_{\text{in}} - e^T\lambda_{i\text{f}} \geq 0,\ \lambda_{i\text{f}} \geq 0\} \quad (13)$$

where $e \in \Re^{n_{\text{d}}}$ is a column vector of ones. Let $v_{i\text{f}} = [(v_{i\text{f}})_1 \ldots (v_{i\text{f}})_{n_{\text{d}}}]^T = \frac{\partial\psi_{i\text{f}}}{\partial q}G\nu = W_{i\text{f}}^T\nu$ be the vector of components of the sliding velocity at contact i in the friction directions. The approximate version of the dissipation condition becomes:

$$\lambda_{i\text{f}} \in \underset{\lambda_{i\text{f}} \in \overline{\mathcal{F}}_i}{\arg\max} \left(-\lambda_{i\text{f}}^T W_{i\text{f}}^T\nu\right). \quad (14)$$

Reusing the slack variable σ_i (with slightly different meaning now), a useful equivalent mixed LCP formulation of the maximum dissipation condition for the approximate friction cone is:

$$0 \leq \lambda_{i\mathrm{f}} \perp W_{i\mathrm{f}}^T \nu + e\sigma_i + \frac{\partial \psi_{i\mathrm{f}}}{\partial t} \geq 0 \tag{15}$$

$$0 \leq \sigma_i \perp \mu_i \lambda_{i\mathrm{n}} - e^T \lambda_{i\mathrm{f}} \geq 0, \tag{16}$$

where now σ_i *approximates* the sliding speed at contact i. Notice that for all generic cases in which the sliding speed is nonzero, the friction force will lie along one of the direction vectors with magnitude equal to $\mu_i \lambda_{i\mathrm{n}}$, so in generic cases the "arg max" function has a unique solution, but the friction force does not directly oppose the sliding direction. However, nongeneric cases occur when the two minimum values of the product $W_{i\mathrm{f}}^T \nu + \frac{\partial \psi_{i\mathrm{f}}}{\partial t}$ are equal. In this case, there exist sets of sliding velocities over a range of directions which all yield the same energy dissipation rate. Thus, while linearization of the friction cone has clear benefits, it also opens the potential for solution nonuniqueness.

Maximum dissipation for all contacts can be written compactly as:

$$0 \leq \lambda_{\mathrm{f}} \perp W_{\mathrm{f}}^T \nu + E\sigma + \frac{\partial \psi_{\mathrm{f}}}{\partial t} \geq 0 \tag{17}$$

$$0 \leq \sigma \perp U\lambda_{\mathrm{n}} - E^T \lambda_{\mathrm{f}} \geq 0 \tag{18}$$

where E is the block diagonal matrix with i^{th} block on the main diagonal given by e.

To summarize, there are two models of interest which differ only in their descriptions of the friction cone.

Model-IQC (quadratic cones): equations (1-3,9-11).
Model-ILC (linear cones): equations (2,3,12,17,18).

3 Discrete-time models

A desirable outcome for any time-stepping scheme is that its solution at the end of each time step of the discrete-time model equals the (continuous) solution of the instantaneous-time model at the same time. Typically however, computational efficiency and/or convergence issues force one to design a scheme that does not exactly meet this desire. To prepare for the design of a time-stepper that solves a linear problem for each time step, the quadratic friction cone was approximated by a piecewise linear cone. In the following, two time-stepping schemes will be presented. The unknowns for both are the configuration vector, contact forces, and sliding speeds at the end of the time step.

Let t^ℓ and denote the time at which one has a solution and let $t^{\ell+1} = t^\ell + h$ denote the time at which one would like an estimate of the solution (the term h is the called the step size). To eliminate ν, $\dot{q}$ can be approximated using a backward Euler formula as follows:

$$\Delta q = q^{\ell+1} - q^\ell = G(q)\nu^{\ell+1} h \tag{19}$$

where $q^\ell = q(t^\ell)$. Note that since Δq is in the range of G (see equation (3)), the following useful identity holds: $\Delta q = GG^T \Delta q$.

3.1 A mildly nonlinear model: Model-DQC

After substituting equation (19) into **Model-IQC**, and replacing all occurrences of the variables $(q,\, \lambda_{\mathrm{n}},\, \lambda_{\mathrm{t}},\, \lambda_{\mathrm{o}},\, \sigma)$ with their values at the end of the time step, $(q^{\ell+1},\, \lambda_{\mathrm{n}}^{\ell+1},\, \lambda_{\mathrm{t}}^{\ell+1},\, \lambda_{\mathrm{o}}^{\ell+1},\, \sigma^{\ell+1})$, all model equations are nonlinear in the unknowns.

To remove some of the nonlinearities from the time-stepper, let W_{n}, W_{t}, W_{o}, G, and f be evaluated at $(t_\ell,\, q^\ell)$. In addition, let the distance function vector be approximated by the linear terms in its Taylor series expansion:

$$0 \leq \lambda_{\mathrm{n}}^{\ell+1} \;\perp\; W_{\mathrm{n}}^T G^T q^{\ell+1} + b_{\mathrm{n}} \geq 0 \tag{20}$$

where $b_{\mathrm{n}} = \psi_{\mathrm{n}}^\ell + \frac{\partial \psi_{\mathrm{n}}^\ell}{\partial t} h - W_{\mathrm{n}}^T G^T q^\ell$. Now the only remaining nonlinearities are the quadratic terms in Coulomb's law. The result is a mildly nonlinear discrete-time model, **Model-DQC**. For each time step, the NCP composed of equations (1,20-23) must be solved:

$$0 = (U\lambda_{\mathrm{n}}) \circ (W_{\mathrm{t}}^T G^T q + b_{\mathrm{t}}) + \lambda_{\mathrm{t}} \circ \sigma h \tag{21}$$

$$0 = (U\lambda_{\mathrm{n}}) \circ (W_{\mathrm{o}}^T G^T q + b_{\mathrm{o}}) + \lambda_{\mathrm{o}} \circ \sigma h \tag{22}$$

$$0 \leq \sigma \perp (U\lambda_{\mathrm{n}}) \circ (U\lambda_{\mathrm{n}}) - \lambda_{\mathrm{t}} \circ \lambda_{\mathrm{t}} - \lambda_{\mathrm{o}} \circ \lambda_{\mathrm{o}} \geq 0 \tag{23}$$

where the variables q, λ_{n}, λ_{t}, λ_{o}, and σ appearing in equations (21-23) are to be evaluated at time $t^{\ell+1}$, $b_{\mathrm{t}} = \frac{\partial \psi_{\mathrm{t}}^\ell}{\partial t} h - W_{\mathrm{t}}^T G^T q^\ell$ and $b_{\mathrm{o}} = \frac{\partial \psi_{\mathrm{o}}^\ell}{\partial t} h - W_{\mathrm{o}}^T G^T q^\ell$.

Summary of **Model-DQC**:
For each time step, solve the mixed NCP of size $n_{\mathrm{q}} + 4n_{\mathrm{c}}$ defined by equations (1,20-23).

3.2 A linear model: Model-DLC

The other discrete-time model of interest, **Model-DLC** can be derived from **Model-ILC** by the same procedure. The result is a mixed LCP defined as follows:

$$\begin{pmatrix} 0 \\ \rho_{\mathrm{n}}^{\ell+1} \\ \rho_{\mathrm{f}}^{\ell+1} \\ s^{\ell+1} \end{pmatrix} = B \begin{pmatrix} q^{\ell+1} \\ \lambda_{\mathrm{n}}^{\ell+1} \\ \lambda_{\mathrm{f}}^{\ell+1} \\ \sigma^{\ell+1} \end{pmatrix} + \; b \tag{24}$$

$$0 \leq \begin{pmatrix} \rho_{\rm n}^{\ell+1} \\ \rho_{\rm f}^{\ell+1} \\ s^{\ell+1} \end{pmatrix} \perp \begin{pmatrix} \lambda_{\rm n}^{\ell+1} \\ \lambda_{\rm f}^{\ell+1} \\ \sigma^{\ell+1} \end{pmatrix} \geq 0 \tag{25}$$

where

$$B = \begin{pmatrix} 0 & W_{\rm n} & W_{\rm f} & 0 \\ W_{\rm n}^T G^T & 0 & 0 & 0 \\ W_{\rm f}^T G^T & 0 & 0 & E \\ 0 & U & -E^T & 0 \end{pmatrix}, \quad b = \begin{pmatrix} f \\ b_{\rm n} \\ b_{\rm f} \\ 0 \end{pmatrix}, \tag{26}$$

$b_{\rm n}$ is defined as above, and $b_{\rm f} = \frac{\partial \psi_{\rm f}^{\ell}}{\partial t} h - W_{\rm f}^T G^T q^{\ell}$.

Summary of **Model-DLC**:
For each time step, solve the mixed LCP of size $n_{\rm q} + (2 + n_{\rm d}) n_{\rm c}$ defined by equations (24,25).

4 Solution Uniqueness

The theorem presented here is the first known solution uniqueness result for general quasistatic multibody systems with dry friction. It applies only to the discrete-time models, **Model-DQC** and **Model-DLC**.[1] Because of space limitations, the results are presented without proof, but these will be available in [4].

Before stating the result, the friction force components can be written as the following functions of the normal force component and the relative tangential displacement components $\Delta_{it} = W_{it}^T G^T q^{\ell+1} + b_{it}$ and $\Delta_{io} = W_{io}^T G^T q^{\ell+1} + b_{io}$:

$$\lambda_{it} = -\mu_i \, \lambda_{in} \, \frac{\Delta_{it}}{\sqrt{\Delta_{it}^2 + \Delta_{io}^2}} \tag{27}$$

$$\lambda_{io} = -\mu_i \, \lambda_{in} \, \frac{\Delta_{io}}{\sqrt{\Delta_{it}^2 + \Delta_{io}^2}} \tag{28}$$

where when $\Delta_{it} = \Delta_{io} = 0$, the fractions appearing in equations (27) and (28) are both equal to 0/0, and are taken to be a suitable pair of scalars (α, β) such that $\alpha^2 + \beta^2 \leq 1$.

[1] A result for continuous-time frictionless quasistatic systems was given in [18]

For given $\{\mu_i \lambda_{in}\}_{i=1}^{n_c}$, consider the following convex, non-differentiable optimization problem in the variable $q^{\ell+1}$:

$$\left.\begin{array}{l} \min \; -f^T G^T q^{\ell+1} + \displaystyle\sum_{i=1}^{n_c} \mu_i \, \lambda_{in} \sqrt{\Delta_{it}^2 + \Delta_{io}^2} \\ \text{s.t:} \;\; W_n^T G^T q^{\ell+1} + b_n \, \geq \, 0 \end{array}\right\} \tag{29}$$

where recall that Δ_{it} and Δ_{io} are functions of $q^{\ell+1}$. The physical interpretation of this problem is that the displacement of the system is one that avoids penetration while minimizing the work done against external and frictional forces. In other words, the system is "lazy" and so moves no more than it absolutely must.

The following result describes the precise connection between the above optimization problem (29) and the discrete-time model **Model-DQC**.

Theorem 1. *If* $(q^{\ell+1}, \lambda_n, \lambda_t, \lambda_o)$ *solves* **Model-DQC** *then* $q^{\ell+1}$ *is a globally optimal solution to (29) corresponding to* λ_n*. Conversely, if* $q^{\ell+1}$ *is a globally optimal solution to (29) for a given* λ_n *and if* λ_n *is equal to an optimal Karush-Kuhn-Tucker (KKT) multiplier of the constraint in (29), then defining* (λ_t, λ_o) *by (27) and (28), the tuple* $(q^{\ell+1}, \lambda_n, \lambda_t, \lambda_o)$ *solves* **Model-DQC**.

A question relevant to the design of fixed-point time stepping schemes is whether or not the convex optimization problem (29) has a unique solution, for fixed $\{\mu_i \lambda_{in}\}_{i=1}^{n_c}$. Let $(q^{\ell+1}, \lambda_n, \lambda_t, \lambda_o)$, solve **Model-DQC**. Denote by dq a small change in $q^{\ell+1}$, and define the index sets:

$$\mathcal{I} \equiv \{\, i \, : \, \psi_{\rm in} \, = \, 0 \, < \, \lambda_{\rm in} \,\} \tag{30}$$

$$\mathcal{J} \equiv \{\, i \, : \, \psi_{\rm in} \, = \, 0 \, = \, \lambda_{\rm in} \,\}. \tag{31}$$

Proposition 1. *Corresponding to the solution* $(q^{\ell+1}, \lambda_n, \lambda_t, \lambda_o)$ *of* **Model-DQC***,* $q^{\ell+1}$ *is the unique solution of (29) if and only if the following implication holds:*

$$\left.\begin{array}{l} W_{\rm in}^T G^T dq \, \geq \, 0, \, i \, \in \, \mathcal{I} \cup \mathcal{J} \\ W_{\rm it}^T G^T dq \, = \, 0, \, i \, \in \, \mathcal{I} \\ W_{\rm io}^T G^T dq \, = \, 0, \, i \, \in \, \mathcal{I} \\ f^T G^T dq \, \geq \, 0 \end{array}\right\} \Rightarrow \; dq \, = \, 0. \tag{32}$$

Next, consider an alternative model where the quadratic friction cone at each contact i is replaced by a four-sided linearized cone (as suggested in [20]):

$$\overline{\mathcal{F}}_i(\mu_i, \lambda_{\rm in}) = \{\, (\, \lambda_{it}, \lambda_{io} \,) : \, \max(|\lambda_{it}|, |\lambda_{\rm io}|) \, \leq \, \mu_i \, \lambda_{in} \,\}. \tag{33}$$

In this case, instead of (27) and (28), we have

$$\lambda_{it} = -\mu_i \lambda_{in} \frac{\Delta_{it}}{|\Delta_{it}|} \tag{34}$$

$$\lambda_{io} = -\mu_i \lambda_{in} \frac{\Delta_{io}}{|\Delta_{io}|}. \tag{35}$$

Moreover, a result similar to Theorem 1 holds with the optimization problem (29) replaced by the following linear program:

$$\left.\begin{array}{l} \min \; -f^T G^T q^{\ell+1} + \sum_{i=1}^{n_c} \mu_i \, \lambda_{in} \, (\,|\,\Delta_{it}\,| + |\,\Delta_{io}\,|) \\ \text{s.t:} \;\; W_n^T G^T q^{\ell+1} + b_n \, \geq \, 0 \end{array}\right\} \tag{36}$$

where again recall that Δ_{it} and Δ_{io} are functions of $q^{\ell+1}$.

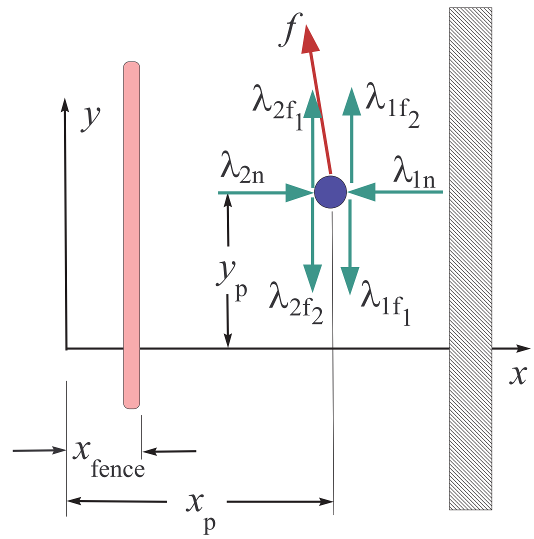

Figure 2. Schematic of fence-particle system.

5 Example: fence-particle problem

Consider the problem of manipulating a particle (shown as a finite disc) of mass m initially at rest on a horizontal plane (the (x, y)-plane in Figure 2). The configuration of this system is $q = [x_\text{p}\; y_\text{p}\; z_\text{p}]^T$, where z_p is the height of the particle above the plane (of the page). The wall on the right is parallel to the (y, z)-plane (perpendicular to the plane of the page) and of infinite extent. The fence is parallel to the wall, of infinite extent, and can translate in the x- and y-directions, but cannot translate in the z-direction or rotate.[2] The vector of noncontact and noninertial forces $f = [0\; 0\; -mg]^T$ is the gravitational force which acts in the negative z-direction.

The three nonpenetration constraints, $\psi_\text{n}(q, t) = [\psi_{1\text{n}}(q)\;\; \psi_{2\text{n}}(q, t)\;\; \psi_{3\text{n}}(q)]^T$ are written as:

$$\psi_{1n} = 1 - x_\text{p} \geq 0 \tag{37}$$

$$\psi_{2n} = x_\text{p} - x_\text{fence}(t) \geq 0 \tag{38}$$

$$\psi_{3n} = z_\text{p} \geq 0. \tag{39}$$

The corresponding lagrange multipliers are the normal components of the contact forces, $\lambda_\text{n}(q, t) = [\lambda_{1\text{n}}\;\; \lambda_{2\text{n}}\;\; \lambda_{3\text{n}}]^T$. Even though as shown, the particle is not in contact with the fence or wall on the right, the components of the corresponding contact forces are shown.[3] The possible contact force components between the particle and the (x, y)-plane are not shown.

[2] The latter constraint is to simplify the problem by making the particle remain within the (x, y)-plane.

[3] Since translation in the z-direction is not possible in this problem, friction forces can act only in the plane of motion of the particle. This is why there are only two friction force directions for contacts 1 and 2.

In this example, solution uniqueness will be explored using the friction laws discussed above acting at the contact between the particle and the (y, z)-plane. An interesting observation, is that for dynamic systems, the absence of friction guarantees solution existence and uniqueness of the predicted motion (not necessarily uniqueness of the contact forces) and the inclusion of friction leads to motion nonuniqueness. In the quasistatic system studied here, the reverse is true. For the case of linearized friction, the quadratic cone will be approximated by a four-sided friction pyramid (see Figure 3). The various friction direction

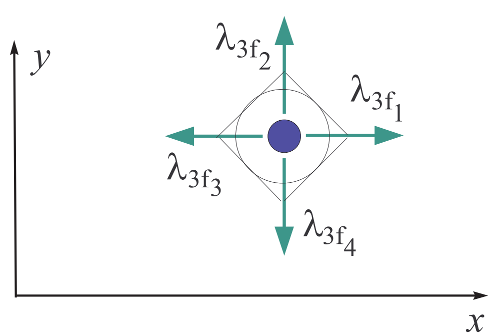

Figure 3. Friction direction vectors between the particle and the (x, y)-plane.

vectors at the three potential contacts imply the following definitions of the local tangential displacement functions:

$$\begin{aligned}
(\psi_{1\mathrm{f}})_1 &= -y_{\mathrm{p}} && (40)\\
(\psi_{1\mathrm{f}})_2 &= y_{\mathrm{p}} && (41)\\
(\psi_{2\mathrm{f}})_1 &= y_{\mathrm{p}} - y_{\mathrm{fence}}(t) && (42)\\
(\psi_{2\mathrm{f}})_2 &= -y_{\mathrm{p}} + y_{\mathrm{fence}}(t) && (43)\\
\psi_{3\mathrm{t}} = (\psi_{3\mathrm{f}})_1 &= x_{\mathrm{p}} && (44)\\
\psi_{3\mathrm{o}} = (\psi_{3\mathrm{f}})_2 &= y_{\mathrm{p}} && (45)\\
(\psi_{3\mathrm{f}})_3 &= -x_{\mathrm{p}} && (46)\\
(\psi_{3\mathrm{f}})_4 &= -y_{\mathrm{p}}. && (47)
\end{aligned}$$

where $y_{\mathrm{fence}}(t)$ is the vertical position of the fence.

The various submatrices appearing in the matrix B are:

$$W_{\mathrm{n}} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad U = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} \tag{48}$$

$$W_{\mathrm{f}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{49}$$

$$E = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \quad G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{50}$$

Other matrices for the nonlinear problem are

$$W_{\mathrm{t}} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad W_{\mathrm{o}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{51}$$

The time-dependent functions needed to define the vectors b_{n}, b_{t}, b_{o}, b_{f} were chosen as:

$$x_{\mathrm{fence}}(t) = 0.5 + 0.4 sin(t) \tag{52}$$
$$y_{\mathrm{fence}}(t) = t \tag{53}$$

With these choices, the fence translates in the y-direction while oscillating in the x-direction without ever hitting the wall.

5.1 Results

Various values of the problem data were chosen to illustrate the theorem and proposition given in section 4. One common aspect of these problems is that the only forces that can act in the z-direction are the gravitational force and the normal component of the contact force between the particle and the (x, y)-plane. This implies that $\lambda_{3\mathrm{n}} = mg > 0$ and $\psi_{3\mathrm{n}} = 0$.

5.1.1 Results: no friction With no friction, **Model-DQC** becomes a mixed LCP and is equivalent to **Model-DLC**. A detailed analysis of the example without friction emphasizes why friction may be needed in some cases for solution uniqueness in quasistatic systems. Looking back at Proposition 1, the 2$^{\mathrm{nd}}$ and 3$^{\mathrm{rd}}$ rows of implication (32) are vacuous in the absence of friction. The removal of these equalities from the implication is what allows the construction of $dq \neq 0$ still satisfying the two remaining inequalities of the implication. The Proposition requires us to look at a solution and then consider variations, dq from $q^{\ell+1}$. Assume a solution of the mixed LCP with contact between the particle and the (x, y)-plane, but not with the wall or fence. In this case, W_{n} is given below and, as above, $f = [0, 0, -mg]^T$:

$$W_{\mathrm{n}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{54}$$

Let $dq = [dx \;\; dy \;\; dz]^T$. Then the inequalities of the implication yield $dz = 0$, but dx and dy are unconstrained. Therefore, there exist $dq \neq 0$ satisfying the inequalities, and the implication fails. Thus the solution of $q^{\ell+1}$ is not unique. In this particular case, the set of $q^{\ell+1}$ solving the frictionless quasistatic model are all those for which the particle remains in contact with the (x, y)-plane, and between the wall and fence. This conclusion was observed in practice. Specifically, the solution obtained was dependent on the initial guess used in the *PATH* solver [6].

5.1.2 Results: Model-DQC with friction Consider a solution for the system when the particle is not touching the fence or wall and the quadratic friction law is in effect at the contact with the (x, y)-plane. In this case, the matrices W_{t} and W_{o} are given as follows:

$$W_{\mathrm{t}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad W_{\mathrm{o}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{55}$$

and W_{n} and f are as in the frictionless case.

From the frictionless case, it is known that the first and last rows of the implication constrain the z-component of dq to 0.

Since the "t" and "o" directions span the contact tangent plane, the equations in the second and third rows of the implication imply that the x- and y-components of dq are also zero. Thus the implication always holds, so all $q^{\ell+1}$ obtained during time-stepping will be unique.

When the particle is in contact with the fence, the matrices $W_{\rm t}$ and $W_{\rm o}$ gain rows, but the conclusion does not change - the motion of the particle is unique.

6 Summary

Two instantaneous-time models of three-dimensional quasistatic multibody systems with Coulomb friction have been presented along with two corresponding discrete-time models. The discrete-time models take the form of complementarity problems for which led to the first known uniqueness results for such systems. A simple example was used to highlight a somewhat unexpected finding. In particular, dynamic multibody systems have unique accelerations when the friction coefficients are small enough. Whereas, for some quasistatic systems, the absence of friction can lead to nonunique system motions.

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