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Modified Variational Iteration Method for the Multi-pantograph Equation with Convergence Analysis

[1]Mohsen Alipour, [2]Kobra Karimi, [3]Davood Rostamy

[1]Member of young research club, Islamic Azad University, Sari branch, P.O. Box 48164-194, Sari, Iran.
[2]Member of young research club, Islamic Azad University, Karaj branch, Iran.
[3]Department of Mathematics, Imam Khomeini International University, P.O. Box 34149-16818, Ghazvin, Iran.

Abstract: In this paper, one effective modification of the variational iteration method is applied for finding the solution of the multi-pantograph equation. Moreover, convergence analysis for this method is discussed. Finally, some numerical examples are given to show the effectiveness of the proposed method.

Key words: Pantograph equation, modified variantional iteration method, convergence analysis.

INTRODUCTION

In the mathematical description of a physical process, one generally assumes that the behavior of the process considered depends only on the present state, an assumption which is verified for a large class of dynamical systems. However, there exist situations where this assumption is not satisfied and the use of a "classical" model in systems analysis and their design may lead to poor performance. In such cases, it is better to consider that the systems behavior includes also information on the former state. These systems are called time-delay systems. Many of the processes, both natural and artificial, in biology, medicine, chemistry, physics, engineering and economics involve time delay.

In fact, time delays occur so often, in almost every situation, that to ignore them is to ignore reality. The pantograph equation is(Liu *et al.* 2004).

$$\begin{cases} u'(t) = a(t)u(t) + \sum_{l=1}^{r} b_l(t)u(q_l t) + g(t), & t \in I := (0,T] \\ u(0) = d \end{cases} \tag{1}$$

where g, a, b_l, $l \in \{1, 2, ..., l\}$ are known functions and d, ql are constants , such that $0 < q_l < 1$.

The pantograph equation is a kind of delay equations. Eq. (1) was studied by many authors numerically and analytically.Sezer *et al.* (2008) obtained the approximate solution of the pantograph equation with non-homogenous terms using Taylor polynomials, Keskin *et al.* (2007) applied the differential transform method to obtain approximate solution, Liu *et al.* (2004, 2005, 2006) used the Runge-Kutta method, Muroya *et al.* (2003) used the collocation method to solve the equation numerically and finally Dehghan *et al.* (2009) applied variational iteration method to solve the generalized pantograph equation. This paper is organized as follows. In Section 2, we introduce the modified variational method (MVIM) and apply it for the multi-pantograph equation in (1). In Section 3, convergence analysis of this method for solving Eq (1) is discussed. In Section 4, numerical examples are simulated to show the reasonableness of our theory and demonstrate the high performance of proposed method. Finally, Some conclusions are summarized in the last section.

Modified Variational Iteration Method for Multi-pantograph Equation:

In this section, we first present a brief review of variational iteration method (VIM) (He, 1999, 2000). Then we will propose modification of VIM (Ghorbani *et al.*, 2009).Consider the following differential equation:

Corresponding Author: Mohsen Alipour, Member of young research club, Islamic Azad University, Sari branch, P.O. Box 48164-194, Sari,Iran.
E-mail: m.alipour2323@gmail.com

$$\mathbf{Lu(t) + Ru(t) + Nu(t) = g(t),} \tag{2}$$

where L is the highest order derivative that is assumed to be easily invertible, R is a linear differential operator of order less than L, N is a nonlinear operator and g(t) is an inhomogeneous term. In the variational iteration method, a correctional functional as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\left(Lu_n(s) + R\tilde{u}_n(s) + N\tilde{u}_n(s) - g(s)\right)ds, \quad n = 0,1,2,\cdots \tag{3}$$

is made, where λ is a general Lagrangian multiplier which can be identified optimally via the variantional theory. Obviously the successive approximations , j = 0, 1, . . . can be computed by determining λ. Here, the function is a restricted variation which means $\delta\tilde{u}_n = 0$. In summary, we have the following variational formula for (2),

$$\begin{cases} u_0(t) \quad \text{is arbitrary} \\ u_{n+1}(t) = f(t) - L^{-1}[Ru_n] - L^{-1}[Nu_n] \end{cases} \tag{4}$$

where f is the terms arising from integration the term g and from using the given conditions, all are assumed to be prescribed. The modified form of variational iteration method can be established based on the assumption that the function f(t) can be divided in two parts, namely $f_0(t)$ and $f_1(t)$. According to this assumption $f(t) = f_0(t) + f_1(t)$, and relationship (3) we have:

$$\begin{cases} u_0(t) = f_0(t) \\ u_{n+1}(t) = f(t) - L^{-1}[Ru_n(t)] - L^{-1}[Nu_n(t)], \quad n = 0,1,2,.. \end{cases} \tag{5}$$

assuming the multiplier Lagrange λ, has been identified.

Finally, with using this method for solving Eq. (1). Let us, assume Lu = $\frac{du}{dt}$, Ru = −a(t)u(t) and Nu $\sum_{l=1}^{r} b_l(t)u(q_l t)$, therefore we have:

$$\begin{cases} u_0 = f_0 \\ u_{n+1}(t) = f(t) - \int_0^t (-a(s)\,u_n(s) - \sum_{I=1}^{r} b_l(s)\,u_n(q_l s))\,ds \; t \in I := [0,T], n = 0,1,.... \end{cases} \tag{6}$$

In the next section, we discuss convergence analysis sequence $\{u_n(t)\}$ in (6).

Convergence Analysis of Mvim for Multi-pantograph Equation:

In the following we show that sequence $\{u_n(t)\}$ in (6) is convergent to exact solution of Eq. (1).

Theorem 3.1:

Let $\bar{\Omega}$ = [0, T] and $u(t) \in C^1(\bar{\Omega}) \cap L_2(\bar{\Omega})$ the exact solution of (1) and $u_n(t) \in C^1(\bar{\Omega}) \cap L_2(\bar{\Omega})$ be the obtained solutions of the sequence defined by (6). If $E_n(t) = u_n(t) - u(t)$, a(t) and $b_l \in L_2(\bar{\Omega})$, $l \in \{1, 2, . . . , r\}$, the nth sequence of (6) converges to the solution(1).

Proof:

According to (6), note that

$$u_{n+1}(t) = f(t) - \int_0^t -a(s)\ u_n(s) - \sum_{l=1}^{r} b_l(s)\ u_n(\ q_l s)\, ds \tag{7}$$

and we know that the following equation is satisfied for the exact solution

$$u(t) = f(t) - \int_0^t -a(s)u(s) - \sum_{l=1}^{r} b_l(s)u(q_l s)\, ds. \tag{8}$$

Therefore we continue

$$E_{n+1}(t) = \int_0^t a(s)E_n(s) + \sum_{l=1}^{r} b_l(s)E_n(q_l s)\, ds. \tag{9}$$

Taking 2-norm of both sides of (9), we have

$$\|E_{n+1}(t)\|_2 \le \int_0^t (\|\ a(s)\|_2\ \|E_n(s)\|_2 + \sum_{l=1}^{r} \|b_l(s)\|_2\ \|E_n(q_l s)\|_2)\, ds$$

$$\|E_{n+1}(t)\|_2 \le M_1 \int_0^t \|E_n\ (s)\|_2\, ds + \sum_{l=1}^{r} M_2 \int_0^t \|E_n(q_l s)\|_2\, ds$$

$$= M_1 \int_0^t \|E_n(s)\|_2\, ds + \sum_{l=1}^{r} \frac{M_2}{q_l} \int_0^{q_l t} \|E_n(s)\|_2\, ds$$

$$\le M_1 \int_0^t \|E_n(s)\|_2\, ds + \sum_{l=1}^{r} \frac{M_2}{q_l} \int_0^t \|E_n(s)\|_2\, ds$$

$$= M \int_0^t \|E_n(s)\|_2\, ds$$

where $M_1 = \max_{t\in(0,T]} \|a(t)\|_2, M_2 = \max_{t\in(0,T]} \|b_l(t)\|_2,\ \forall l \in \{1,\dots,r\}. M_3 = \sum_{l=1}^{r} \frac{M_2}{q_l}$ and $M=M_1+M_2$

Now we put n = 0

$$\|E1(t)\|_2 \le M \int_0^t \|E_0\ (s)\|_2\, ds \le M\|E_0(\xi)\|_2 \int_0^t ds = Mt\|E_0(\xi)\|_2$$

where, $\|E_0(\xi)\|_2 \max_{s\in[0,T]} \|E_0(s)\|_2$. Also for n = 1, 2, ..., k we have:

$$\|E_2(t)\|_2 \le M \int_0^t \|E_1\ (s)\|_2\, ds \le M^2\ \|E_0(\xi)\ \|_2 \frac{t^2}{2!}$$

$$\|E_K(t)\|_2 \le M\int_0^t \|E_{k-1}(s)\|_2\, ds \le \frac{(Mt)^K}{K!}\|E_0(\xi)\|_2 \le \frac{(MT)^k}{k!}\|E_0(\xi)\|_2$$

Therefore, we can conclude

$$\frac{(MT)^k}{k!} \longrightarrow 0$$

as $k \to \infty$, and this completes the proof.

In the next section, we give three numerical examples to show the reasonableness of our theory and demonstrate the high performance of proposed method.

Numerical Examples:

Example 4.1P:

We consider the following pantograph equation

$$\begin{cases} \acute{u}(t) = -\frac{5}{6}\, u(t) + 4u\left(\frac{t}{2}\right) + 9u\left(\frac{t}{3}\right) + t^2 - 1 \qquad 0 < t \le 1 \\ u(0) = 1 \end{cases}$$

According to (1) $a(t) = -\frac{5}{6}, b_1(t) = +4, b_2(t) = 9, q_1 = \frac{1}{2}, q_2 = \frac{1}{3}$ and

$g(t) = t^2 - 1$. Therefore we can write the following statement:

$$\int_0^t u'(t)dt = \int_0^t \left(-\frac{5}{6}u(x) + 4u(\frac{x}{2}) + 9u(\frac{x}{3}) + x^2 - 1\right)dx,$$

so we have:

$$u(x) = \frac{t^3}{3} - t + 1 + \int_0^t \left(-\frac{5}{6}\, u(x) + 4u\left(\frac{x}{2}\right) + 9(\frac{x}{3})\right)dx.$$

$$\begin{cases} f(t) = \frac{t^3}{3} - t + 1 \\ f_0(t) = -t \\ u_0(t) = f_0(t) \\ u_{n+1}(t) = f(t) + \int_0^t -\frac{5}{6}u_n(x) + 4u_n(\frac{x}{2}) + 9u_n(\frac{x}{3})dx \qquad n = 0,1,2, ... \end{cases}$$

Hence, we have:

$$u_0 = \frac{t^3}{3}$$

$$u_1 = \frac{t^3}{3} - t + 1 - \frac{25}{12}t^2$$

$$u_2 = \frac{-103}{216}t^3 + \frac{67}{6}t + 1 - \frac{25}{12}t^2$$

$$u_3 = \frac{-103}{216}t^3 + \frac{67}{6}t + 1 + \frac{1675}{72}t^2$$

$$u_4 = \frac{12157}{1296}t^3 + \frac{1675}{72}t^2 + \frac{67}{6}t + 1.$$

This problem has the exact solution $u^*(t) = \frac{12157}{1296}t^3 + \frac{1675}{72}t^2 + \frac{67}{6}t + 1$ in this example, we observe that Theorem 3.1 guarantees that u_n (t) converges to exact solution. We obtained the exact solution by four iterations.

Example 4.2:

We consider the following system of pantograph equation

$$\begin{cases} \overrightarrow{u'(t)} = B(t)\overrightarrow{u(qt)} + \overrightarrow{g(t)}, \\ \overrightarrow{u(0)} = [0,1]^T, \end{cases}$$

where $\overrightarrow{u(t)} = [u^1(t), u^2(t)]^T$ and we choose

$$B(t) = \begin{bmatrix} \sin(t) & \cos(t) \\ -\cos(t) & \sin(t) \end{bmatrix}, q = \frac{1}{2}$$

And

$$g(t) = \begin{bmatrix} \cos(t) - \cos\left(\frac{t}{2}\right) \\ -\sin(t) - \sin\left(\frac{t}{2}\right) \end{bmatrix}$$

Therefore, in this problem, we deal with the following equation

$$\begin{cases} u^{1'}(t) = \sin(t)u^1(\frac{t}{2}) + \cos(t)u^2(\frac{t}{2}) + \cos(t) - \cos(\frac{t}{2}) \\ u^{2'}(t) = -\cos(t)u^1(\frac{t}{2}) + \sin(t)u^2(\frac{t}{2}) - \sin(t) - \sin(\frac{t}{2}). \end{cases}$$

The exact solution of this problem is $\overrightarrow{uu(t)}^* = [\sin(t), \cos(t)]^T$. Now, by using (5) we have

$$\begin{cases} u^1(t) = L^{-1}[\sin(t)u^1(\frac{t}{2}) + \cos(t)u^2(\frac{t}{2}) + L^{-1}(\cos(t) - \cos(\frac{t}{2})) + u^1(0) \\ u^2(t) = L^{-1}[-\cos(t)u^1(\frac{t}{2})\sin(t)u^2(\frac{t}{2}) + L^{-1}(-\sin(t) - \sin(\frac{t}{2}))u^2(0). \end{cases}$$

Therefore, we have

$$\begin{cases} f^1(t) = \sin(t) - 2\sin\left(\frac{t}{2}\right) \\ f^2(t) = \cos(t) + 2\cos(\frac{t}{2}) - 2. \end{cases}$$

Let us start to the following initial approximation

$$\begin{cases} u_0^1(t) = f_0^1 = \sin(t) \\ u_0^2(t) = f_0^2 = \cos(t). \end{cases}$$

Therefore, we obtain the next iterations by (6) as follows

$$\begin{cases} u_1^1(t) = \sin(t) - 2\sin(\frac{t}{2}) + 2\sin(\frac{t}{2}) = \sin(t) \\ u_1^2(t) = \cos(t) + 2\cos(\frac{t}{2}) - 2\cos(\frac{t}{2}) = \cos(t). \end{cases}$$

If we carry out the sequence, the following calculation are obtained

$$\begin{cases} u_n^1(t) = \sin(t), & n = 2,3,\ldots \\ u_n^2(t) = \cos(t), & n = 2,3,\ldots. \end{cases}$$

Note that, we get to the exact solution by one iteration.

Example 4.3:

We consider the following pantograph equation

$$\begin{cases} u'(t) = -u(t) + \frac{1}{10}u(\frac{2t}{10}) - \frac{1}{10}e^{-\frac{2t}{10}} & 0 < t \le 0.5, \\ u(0) = 1 \end{cases}$$

where we choose $a(t) = -1$, $q = \frac{2}{10}$` $b_1(t) = \frac{1}{10}$ and $g(t) = \frac{-1}{10}e^{-\frac{-2t}{10}}$. This problem has the exact solution $u(t) = e^{-t}$. Since $f = \frac{1}{2} + \frac{1}{2}e^{\frac{1}{5}t} = \frac{1}{2} + \frac{1}{2}(1 - \frac{1}{5}t + \frac{1}{50}t^2 + \ldots)$, let us start to the initial approximation $u_0(t) = f_0(t) = 1 - \frac{1}{10}t + \frac{1}{100}t^2$ and use iteration formula (6). We can obtain the other iterations as follows.

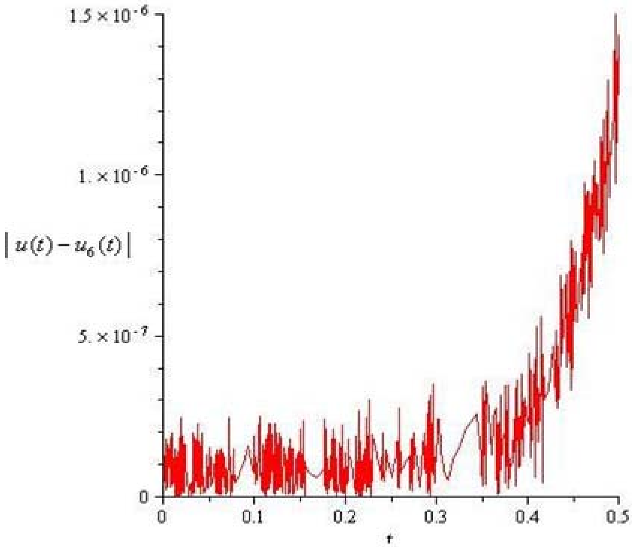

Fig. 1: Error function $|u(t)-u_6(t)|$ for example 4.3 in the interval $0 \leq t \leq 0.5$.

$$u_1(t) = \frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{5}t + \frac{t^2}{50} + \ldots\right) - \frac{9}{10}t + \frac{49}{1000}t^2 - \frac{83}{25000}t^3 = 1 - t + O(t^2),$$

$$u_2(t) - \frac{3}{4} + 3\left(-\frac{1}{5}t + \frac{1}{5}t^2 + \ldots\right) + \frac{9}{20}t + \frac{441}{10000}t^2 - \frac{4067}{2500000}t^3$$

$$+\frac{103667}{125000000}t^4 - \frac{5}{4}\left(1 - \frac{1}{25}t + \frac{t^2}{1250} + \ldots\right) = 1 - \frac{t^2}{2!} + O(t^3),$$

$$u_3(t) - \frac{69}{8} + \frac{31}{2}\left(-\frac{1}{5}t + \frac{1}{50}t^2 + \ldots\right) + \frac{27}{40}t + \frac{441}{2000}t^2$$

$$-\frac{36603}{250000}t^3 + \frac{5079683}{125000000}t^4 - \frac{647815083}{390625000000}t^5 - \frac{155}{4}\left(1 - \frac{1}{25}t + \frac{t}{1250}t^2 + \ldots\right)$$

$$+\frac{125}{8}\left(1 - \frac{1}{125}t + \frac{t^2}{31250} + \ldots\right)$$

$$= 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} O(t^4),$$

In this example, Theorem 3.1 guarantees that $u_n(t)$ converges to exact solution, i.e. $u(t) = \lim_{n\to\infty} u_n(t) = e^{-t}$. In Fig. 1, we see behavior of the error function $|u(t) - u_6(t)|$.

5 Conclusions:

In this work, we used the modified variational iteration method (MVIM) for solving the multi-pantograph equation and compared our results with the exact solution in order to demonstrate the validity and applicability of the method. This method can give very good approximations by means of a few iterations for most cases. Moreover, we proved the convergence of MVIM for multi-pantograph equation. All of the computations have done by the Maple software.

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