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Properties of the longest-edge n-section refinement scheme for triangular meshes

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ABSTRACT

We prove that the longest-edge n-section of triangles for $n \geqslant 4$ produces a sequence of triangle meshes with minimum interior angle converging to zero. The so called degeneracy property of LE for $n \geqslant 4$ is proved.

The stability condition or non-degeneracy property means that the interior angles of all elements have to be bounded uniformly away from zero. Non-degeneracy is essential, for example, for the approximation properties of finite element spaces and the convergence behavior of multigrid and multilevel algorithms.

Rosenberg and Stenger [1] showed the non-degeneracy property for LE-bisection: if α_0 is the minimum angle of initial given triangle, and α_k is the minimum interior angle in new triangles appeared at iteration k, then $\alpha_k \geqslant \alpha_0/2$. A similar bound has been obtained recently for the LE-trisection: $\alpha_k \geqslant \alpha_0/c$ where $c = \frac{\pi/3}{\arctan(\frac{\sqrt{3}}{11})}$ [2].

Theorem 1. *The iterative application of longest-edge n-section when $n \geqslant 4$ to a given arbitrary triangle $\triangle ABC$ generates a sequence of new triangles in which $\lim_{k\to\infty} \alpha_k = 0$, α_k being the minimum triangle angle in iteration k.*

Proof. It is enough to prove that there exists a sequence $\{\tau_k\}_{k=0}^{\infty}$ such that:

(1) τ_k is the value of the interior angle obtained after kth iteration of the LE n-section of the given triangle $\triangle ABC$.
(2) $\lim_{k\to\infty} \tau_k = 0$.

In fact, for all $k \geqslant 1$ we have $\alpha_k \leqslant \tau_k$, then: $0 \leqslant \lim_{k\to\infty} \alpha_k \leqslant \lim_{k\to\infty} \tau_k = 0$, where, clearly, $\lim_{k\to\infty} \alpha_k = 0$.

We now prove that there exists such a sequence $\{\tau_k\}_{k=0}^{\infty}$. Let $n \geqslant 4$ and $\triangle ABC$ be an arbitrary triangle with sides $|\overline{AB}| \leqslant |\overline{AC}| \leqslant |\overline{BC}|$. We consider a triangle sequence $\{\Delta_k\}_{k=0}^{\infty}$ such that $\Delta_0 = \triangle A_0B_0C_0$, $A_0 = A$, $B_0 = B$, $C_0 = C$. For all $k \geqslant 0$ let $\Delta_{k+1} = \triangle A_{k+1}B_{k+1}C_{k+1}$ where $A_{k+1} \in \overline{B_kC_k}$ such that $|\overline{A_{k+1}C_k}| = \frac{1}{n}|\overline{B_kC_k}|$, $B_{k+1} = C_k$ and $C_{k+1} = A_k$. It should be noted that for all $k \geqslant 1$, $|\overline{A_kB_k}| \leqslant |\overline{A_kC_k}| < |\overline{B_kC_k}|$ and that Δ_k is one of the triangles generated by applying the LE n-section to triangle Δ_{k-1}.

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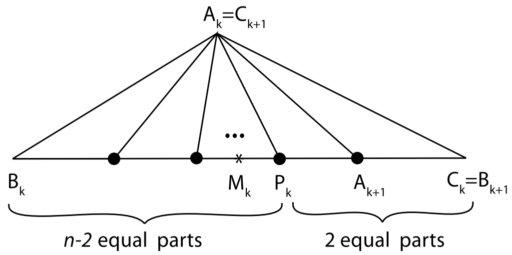
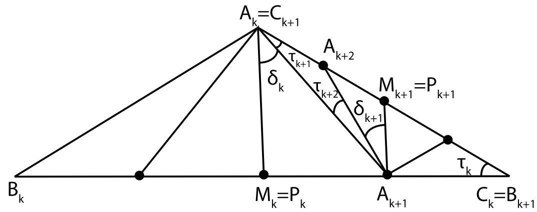
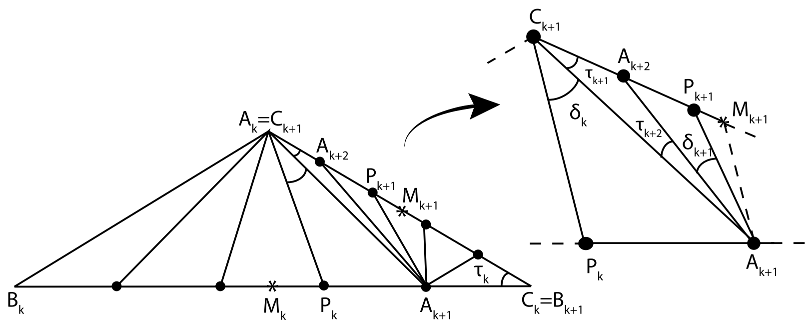


Fig. 1. Scheme for the constructed triangle sequence in the LE n-section.



(a) LE 4-section.



(b) LE 5-section.

Fig. 2. LE n-section ($n = 4, 5$) of triangle $A_kB_kC_k$ and of its descendant $A_{k+1}B_{k+1}C_{k+1}$.

Denote by P_k ($k \geqslant 0$) the point within the segment $\overline{B_kC_k}$ such that:

$$\frac{|\overline{B_kP_k}|}{|\overline{P_kC_k}|} = \frac{n-2}{2} \tag{1}$$

and let M_k be the midpoint of segment $\overline{B_kC_k}$. See Fig. 1 for a graphical illustration of point P_k.

Note that $|\overline{P_kA_{k+1}}| = |\overline{A_{k+1}C_k}|$. Moreover, from Eq. (1) and recalling that $n \geqslant 4$ we have:

$$\frac{|\overline{B_kP_k}|}{|\overline{P_kC_k}|} \geqslant 1. \tag{2}$$

From inequality (2) we have $P_{k+1} \in \overline{M_{k+1}C_{k+1}} = \overline{M_{k+1}A_k}$.

On the other hand, it is evident that $A_kP_k \parallel A_{k+1}M_{k+1}$; see Fig. 2(a) and (b) for $n = 4$ and $n = 5$, respectively. Let $\angle P_kA_kA_{k+1} = \delta_k$ and $\angle A_kC_kB_k = \tau_k$. Then, by equality of alternate interior angles between parallels and the sum of consecutive angles:

$$\begin{aligned}\angle P_kA_kA_{k+1} &= \angle A_kA_{k+1}M_{k+1} = \angle P_{k+1}A_{k+1}M_{k+1} + \angle P_{k+1}A_{k+1}A_{k+2} + \angle A_kA_{k+1}A_{k+2}\\ &\geqslant \angle P_{k+1}A_{k+1}A_{k+2} + \angle A_kA_{k+1}A_{k+2}.\end{aligned}$$

This is $\delta_k \geqslant \delta_{k+1} + \tau_{k+2}$, consequently:

$$\tau_{k+2} \leqslant \delta_k - \delta_{k+1}. \tag{3}$$

Note that the equality in (3) holds for $n = 4$; see Fig. 2(a) which illustrates the case of LE quartersection of triangle $A_kB_kC_k$ and of its descendant $A_{k+1}B_{k+1}C_{k+1}$. The case $\tau_{k+2} < \delta_k - \delta_{k+1}$ is attained when $n > 4$ and this situation is depicted in Fig. 2(b) for $n = 5$.

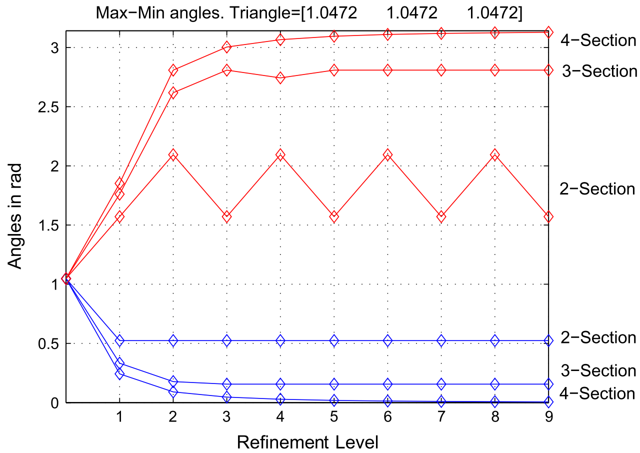


Fig. 3. A simple test: max–min angle evolution in iterative refinement with LE n-section when $n = 2, 3$ and 4.

It can be noted from inequality (3) that $\{\delta_k\}_{k=0}^{\infty}$ is a decreasing sequence. Since this sequence is bounded from below by 0, using the Bolzano–Weierstrass Theorem we conclude that $\{\delta_k\}_{k=0}^{\infty}$ converges and thus $\lim_{k\to\infty}(\delta_k - \delta_{k+1}) = 0$. It follows:

$$0 \leqslant \lim_{k\to\infty} \tau_k = \lim_{k\to\infty} \tau_{k+2} \leqslant \lim_{k\to\infty} (\delta_k - \delta_{k+1}) = 0$$

and then $\{\tau_k\}_{k=0}^{\infty}$ exists and converges to 0 which proves the result of the theorem. □

Finally, in order to show a face-to-face comparison among LE bisection, LE trisection and LE quartersection ($n = 2, 3, 4$), we show in Fig. 3 max–min angles generated in repeated refinements using such triangle partitions and considering an initial triangle with equal interior angles of $\pi/3$ rads (other examples get analogous behavior and are omitted for brevity).

In this example, LE bisection, as expected, exhibits a better tight max–min angle in comparison to LE trisection and quartersection which is in agreement with reported results.

In this paper, we have responded to how good is longest-edge n-section of triangles. Proven results by Rosenberg and Stenger [1], Perdomo et al. [2] and Plaza et al. [3] show that LE bisection and LE trisection exhibit non-degeneracy in iterative application. We show that degeneracy of LE n-section is attained for the so called LE quartersection ($n = 4$). We then find a frontier where LE n-section methods start to degenerate. A matter of similar interest is to study the similarity classes of triangles in the LE n-section for $n \geqslant 4$.

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