Int. J. Nonlinear Anal. Appl. 7 (2016) No. 1, 279-287
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2015.147

Regularized fractional derivatives in Colombeau algebra

M. Alimohammady*, F. Fattahi

Department of Mathematics, University of Mazandaran, Babolsar, Iran

(Communicated by A. Ghaffari)

Abstract

The present study aims at indicating the existence and uniqueness result of system in extended colombeau algebra. The Caputo fractional derivative is used for solving the system of ODEs. In addition, Riesz fractional derivative of Colombeau generalized algebra is considered. The purpose of introducing Riesz fractional derivative is regularizing it in Colombeau sense. We also give a solution to a nonlinear heat equation illustrating the application of the theory.

Keywords: Colombeau algebra; Caputo fractional derivative; Riesz fractional derivative.
2010 MSC: Primary 46F30; Secondary 26A33, 34G20.

1. Introduction and preliminaries

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. Moreover fractional processes have been increased many developments in the last decade. For instance, they are suitable for describing the long memory properties of many time series.

Colombeau algebras (usually denoted by the letter $\mathcal{G}$) are differential (quotient) algebras with unit, and were introduced by J. F. Colombeau (cf.[1],[2],[3]) as a nonlinear extension of distribution theory to deal with nonlinearities and singularities in PDE theory. These algebras contain the space of distributions $\mathcal{D}'$ as a subspace with an embedding realized through convolution with a suitable mollifier. Elements of these algebras are classes of nets of smooth functions.

The fractional calculus by application of distributed order PDEs in Colombeau algebra was started by [6]. Furthermore, Caputo and Riemann-Liouville derivatives of a Colombeau generalized process

*Corresponding author
Email addresses: amohsen@umz.ac.ir (M. Alimohammady), f.fattahi@stu.umz.ac.ir (F. Fattahi)

Received: March 2014 *Revised:* May 2015

have been studied in [5]. The reason for introducing fractional derivatives into the Colombeau algebra of generalized functions was the possibility of solving nonlinear problems with singularities and derivatives of arbitrary real order in it.

This article first, considers the Caputo fractional derivatives on the interval $[0,t)$, $t<T$, $T>0$ and then discusses the study of Riesz fractional derivatives on the whole axis $\mathbf{R}$ which is based on the Liouville-Weyl definition of the fractional derivative with higher order [4].

The paper is organized as follows. After the introduction some basic preliminaries such as notation and definitions of the used objects are given. Also the spaces of Colombeau generalized functions are introduced. In addition, existence and uniqueness result in a system of ordinary differential equation is considered. Moreover imbedding the Riesz fractional derivative into extended Colombeau algebra of generalized functions is shown. Finally, the existence-uniqueness result of a nonlinear heat equation is proven. Furthermore an equation driven by the fractional derivative of delta distribution is certified. This means the equation illustrates the application of the theory in a framework of the extended algebra of generalized functions. Besides moderateness and the negligibility for entire and fractional derivatives are clarified.

1.1. Colombeau algebra

First the definitions of some generalized function algebras of Colombeau type are mentioned which are as follows. The elements of Colombeau algebras $\mathcal{G}$ are equivalent classes of regularizations, i.e., sequences of smooth functions satisfying asymptotic conditions in the regularization parameter ϵ. Therefore, for any set X, the family of sequences $(u_\epsilon)_\epsilon \in (0,1]$ of elements of a set X will be denoted by $X^{(0,1]}$; such sequences will also be called nets and simply written as u_ϵ.

Let Ω be an open subset of $\mathbb{R}^d$. The algebra of generalized functions on Ω equals $\mathcal{G}(\Omega)$, is defined $\mathcal{G}(\Omega)=\mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$ where

$$\mathcal{E}_M(\Omega)=\{(u_\epsilon)_\epsilon \in (C^\infty(\Omega))^{(0,1]} \mid \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{N}_0^n\ \exists N \in \mathbb{N} \text{ s.t. } \sup_{x\in K}|\partial^\alpha u_\epsilon(x)| = O(\epsilon^{-N}),\ \epsilon \to 0\},$$

$$\mathcal{N}(\Omega)=\{(u_\epsilon)_\epsilon \in (C^\infty(\Omega))^{(0,1]} \mid \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{N}_0^n\ \forall s \in \mathbb{N} \text{ s.t. } \sup_{x\in K}|\partial^\alpha u_\epsilon(x)| = O(\epsilon^{s}),\ \epsilon \to 0\}.$$

Element of $\mathcal{E}_M(\Omega)$ and $\mathcal{N}(\Omega)$ are called moderate, negligible functions, respectively. Families $(r_\epsilon)_\epsilon$ of complex numbers such as $|r_\epsilon| = O(\epsilon^{-p})$ as $\epsilon \to 0$ for some $p \geq 0$ are called moderate, in which $|r_\epsilon| = O(\epsilon^q)$ for every $q \geq 0$ are termed negligible. The ring $\tilde{\mathbb{R}}$ of Colombeau generalized numbers is obtained by factoring moderate families of complex numbers with respect to negligible families.

The definition of extended Colombeau algebras of generalized functions on open subset of Ω is in a sense of extension of the entire derivatives to the fractional ones. Let $\mathcal{E}^e(\Omega)$ be an algebra of all nets $(u_\epsilon)_{\epsilon>0}$ of real valued smooth functions $u_\epsilon \in C^\infty(\Omega)$. Suppose that

$$\mathcal{E}^e_M(\Omega)=\{(u_\epsilon)_\epsilon \in \mathcal{E}^e(\Omega) \mid \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{R}_+ \cup \{0\}, \exists N \geq 0 \text{ s.t. } \sup_{x\in K}|D^\alpha u_\epsilon(x)| = O(\epsilon^{-N})\ \ \epsilon \to 0\},$$

$$\mathcal{N}^e(\Omega)=\{(u_\epsilon)_\epsilon \in \mathcal{E}^e(\Omega) \mid \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{R}_+ \cup \{0\}, \forall s \geq 0 \text{ s.t. } \sup_{x\in K}|D^\alpha u_\epsilon(x)| = O(\epsilon^{s})\ \ \epsilon \to 0\},$$

where $D^\alpha u_\epsilon(x)$ is the Caputo fractional derivative. The extended Colombeau algebra of generalized functions is the set $\mathcal{G}^e(\Omega)=\mathcal{E}^e_M(\Omega)/\mathcal{N}^e(\Omega)$.

The algebra $\mathcal{G}_\tau(\mathbb{R}^d)$ of tempered generalized functions was introduced by J. F. Colombeau in (cf. [3]) in order to develop a theory of Fourier transform in algebras of generalized functions. We have

$$\mathcal{O}_M(\mathbb{R}^d)=\{f \in C^\infty(\mathbb{R}^d) \mid \forall \alpha \in \mathbb{N}_0^d\ \exists N \in \mathbb{N} : \sup_{x\in\mathbb{R}^d}\langle x\rangle^{-N}|D^\alpha u_\epsilon(x)| < \infty\}.$$

The Colombeau algebra of tempered generalized functions is $\mathcal{G}^e_\tau(\mathbb{R}^d) = \mathcal{E}^e_\tau(\mathbb{R}^d)/\mathcal{N}^e_\tau(\mathbb{R}^d)$. where

$$\mathcal{E}^e_\tau(\mathbb{R}^d) = \{(u_\epsilon)_\epsilon \in \mathcal{O}_M(\mathbb{R}^d)^{(0,1]} |\ \forall\alpha \in \mathbb{R}_+ \ \cup\ \{0\}, \exists N \geq 0 : \sup_{x\in\mathbb{R}^d} \langle x\rangle^{-N} |D^\alpha u_\epsilon(x)| = O(\epsilon^{-N})\},$$

$$\mathcal{N}^e_\tau(\mathbb{R}^d) = \{(u_\epsilon)_\epsilon \in \mathcal{O}_M(\mathbb{R}^d)^{(0,1]} |\ \forall\alpha \in \mathbb{R}_+ \ \cup\ \{0\}, \exists N \geq 0\ \forall s \geq 0 : \sup_{x\in\mathbb{R}^d} \langle x\rangle^{-N} |D^\alpha u_\epsilon(x)| = O(\epsilon^{s})\ \}.$$

Here, D^α, $m-1 \leq \alpha < m$, $m \in \mathbb{N}$, is the Caputo fractional derivative. Imbedding $S'(\mathbb{R})$ into $\mathcal{G}^e_\tau(\mathbb{R})$ for the entire derivatives is given by the map $i : \nu \to [(\nu * \varphi_\epsilon)_{\epsilon>0}]$, where

$$\varphi_\epsilon(x) = \frac{1}{\epsilon}\varphi(\frac{x}{\epsilon}),\ \varphi(x) \in C^\infty_0(\mathbb{R}), \varphi(x) \geq 0, \int \varphi(x)dx = 1, \int x^\alpha \varphi(x)dx = 0, \forall\alpha \in \mathbb{N}, |\alpha| > 0.$$

The new definition of extended Colombeau algebra is based on the ratio of spatial variable x. Moreover for a fractional derivative in the Riesz sense is used. An interval $\Omega = (-\infty, \infty)$, and for PDEs the derivative (w.r.) to spatial variable x in the domain $\Omega = ((0, T] \times \mathbb{R})$ is considered.

The Colombeau algebra generalized functions is the set $\mathcal{G}^e_{L^\infty}(\Omega) = \mathcal{E}^e_{M,L^\infty}(\Omega)/\mathcal{N}^e_{L^\infty}(\Omega)$ where

$$\mathcal{E}^e_{M,L^\infty}(\Omega) = \{(u_\epsilon)_\epsilon \in \mathcal{E}^e(\Omega) |\ \forall\alpha \in \mathbb{R}_+ \ \cup\ \{0\}, \exists N \geq 0,\ s.t\ \|D^\alpha u_\epsilon(x)\|_{L^\infty(\Omega)} = O(\epsilon^{-N})\ as\ \epsilon \to 0\},$$

$$\mathcal{N}^e_{L^\infty}(\Omega) = \{(u_\epsilon)_\epsilon \in \mathcal{E}^e(\Omega) |\ \forall\alpha \in \mathbb{R}_+ \ \cup\ \{0\}, \forall s \geq 0,\ s.t\ \|D^\alpha u_\epsilon(x)\|_{L^\infty(\Omega)} = O(\epsilon^{s})\ as\ \epsilon \to 0\}.$$

Imbedding the fractional derivatives (w.r.) to the spatial variable is given by the convolution of the Riesz derivative with the delta sequence:

$$i_{frac} : \nu \to [\tilde{D}^\alpha(\nu_\epsilon)_{\epsilon>0}] = [D^\alpha(\nu_\epsilon * \varphi_\epsilon(x))_{\epsilon>0}],$$

where ν_ϵ represents the entire derivative.

1.2. Fractional derivatives

Let f_ϵ be a representative of a Colombeau generalized function. The fractional derivatives of order $\alpha > 0$ in the Caputo sense on the interval $[0, t)$ is defined by:

$$D^\alpha_t f_\epsilon(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m_\epsilon(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m \\ \frac{d^m}{dt^m} f_\epsilon(t), & \alpha = m \end{cases},$$

for $m \in \mathbb{N}$ and $\epsilon \in (0, 1]$. For imbedding of the fractional derivatives of distributions into $\mathcal{G}^e([0, T))$, $T > 0$ we use the convolution with mollifier $\varphi_\epsilon(t) = \frac{1}{\epsilon}\varphi(\frac{t}{\epsilon})$.

Let ν be the distribution. Then $\nu_\epsilon = \nu * \varphi_\epsilon(t)$, according to the classical Colombeau theory for entire derivatives. Consider $\tilde{D}^\alpha\nu_\epsilon$, $\alpha \in \mathbb{R}_+ \cup \{0\}$. For $0 < \alpha < 1$ we have:

$$\begin{aligned}\tilde{D}^\alpha\nu_\epsilon = D^\alpha\nu_\epsilon * \varphi_\epsilon(t) &= \frac{1}{\Gamma(1-\alpha)}\Big(\int_0^t \frac{\dot{\nu}_\epsilon(\tau)}{(t-\tau)^\alpha} d\tau\Big) * (\varphi_\epsilon(t)) \\ &\leqslant \frac{1}{\Gamma(1-\alpha)} \sup_{t\in[0,T)} \Big| \int_0^t \frac{\dot{\nu}_\epsilon(\tau)}{(t-\tau)^\alpha} d\tau \Big| \cdot \|\varphi_\epsilon(t)\|_{L^1} \\ &\leqslant \frac{C}{\Gamma(1-\alpha)} \sup_{t\in[0,T)} |\dot{\nu}_\epsilon(t)| \frac{T^{1-\alpha}}{1-\alpha} \leqslant C_{T,\alpha}\epsilon^{-N}, \quad \exists N > 0,\ \alpha \in \mathbb{R}_+ \cup \{0\}.\end{aligned}$$

In a similar way, for higher fractional derivatives we apply the semigroup property of fractional differentiation.

2. Existence-uniqueness result with Caputo fractional derivative

It was proved in [6] the existence-uniqueness result for the system $\dot{x}_\epsilon(t) = F_\epsilon(t, x_\epsilon(t)) + D^\alpha \delta_\epsilon(t)$ for the Caputo fractional derivative. Here, existence and uniqueness result for the system of ODEs

$$\dot{x}_\epsilon(t) = F_\epsilon(t, x_\epsilon(t)) + H_\epsilon(t), \tag{2.1}$$

for the fractional derivative in extended Colombeau algebra of generalized functions are proved.

For $0 < \alpha < 1$ we have

$$D^\alpha H_\epsilon(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{H^1_\epsilon(t)}{(t-\tau)^\alpha} d\tau = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}.$$

This derivative is divergent at the point $t = 0$. The regularization of the fractional derivative with a delta sequence is used. The regularization of Heaviside distribution is

$$H(x, \epsilon) = \frac{1}{\epsilon} \int_{-\infty}^x \varphi(\frac{\eta}{\epsilon}) d\eta = \int_{-\infty}^x \varphi_\epsilon(\eta) d\eta.$$

Hence, we consider of regularized equation.

Lemma 2.1. *Assume that $F \in (\mathcal{G}_\tau(\mathbb{R}^{n+1}))^n$ and $|\nabla_x F| \leq |\ln \epsilon|$. For the given $x_0 \in \tilde{\mathbb{R}}^n$, the system*

$$\dot{x}_\epsilon(t) = F_\epsilon(t, x_\epsilon(t)) + H_\epsilon(t) \tag{2.2}$$

has a unique solution $x \in (\mathcal{G}^e(\mathbb{R}))^n$ in extended Colombeau algebra of a generalized functions.

Proof . Utilize the regularized fractional operator both of (2.2). Consider fractional derivative $\tilde{D}^\alpha$, $0 < \alpha < 1$, it follows that

$$\tilde{D}^\alpha \dot{x}_\epsilon(t) = \tilde{D}^\alpha F_\epsilon(t, x_\epsilon(t)) + \tilde{D}^\alpha H_\epsilon(t). \tag{2.3}$$

Instead of the system (2.3), we consider the regularized equation

$$\tilde{D}^\alpha \dot{x}_\epsilon(t) = \tilde{D}^\alpha F_\epsilon(t, x_\epsilon(t)) + \tilde{D}^\alpha \int_{-\infty}^t \varphi_\epsilon(y) dy * \varphi_\epsilon(t)), \quad x_\epsilon(t_0) = x_{0\epsilon}. \tag{2.4}$$

The integral form of this system is given by:

$$|\tilde{D}^\alpha x_\epsilon(t)| \leq |\ \tilde{D}^\alpha x_{0\epsilon}| + \int_0^t |\tilde{D}^\alpha F_\epsilon(0, h)| dh + \int_0^t |\int_0^1 (\nabla_x F_\epsilon(h, \sigma x_\epsilon(h)) d\sigma || \tilde{D}^\alpha x_\epsilon(h)| dh$$

$$+ \int_0^t |(D^\alpha \int_{-\infty}^\tau \varphi_\epsilon(y) dy * \varphi_\epsilon(\tau))| d\tau. \tag{2.5}$$

For three expressions of the above estimate by using the Gronwall inequality we obtain

$$|\tilde{D}^\alpha x_\epsilon(t)| \leq (C\epsilon^{-N} + C\epsilon^{-N}) \exp(-T \ln \epsilon) \leq C\epsilon^{-N}, \qquad \exists N > 0,\ t \in \mathbb{R}_+ \cup \{0\},\ t < T,\ T > 0.$$

Here, we prove the moderateness of the last expression in sup-norm, so for $0 < \alpha < 1$, the following result is obtained:

$$\begin{aligned}
\int_0^t (D^\alpha \int_{-\infty}^\tau \varphi_\epsilon(y)dy * \varphi_\epsilon(\tau))d\tau &\leq C \int_0^t \sup_{\tau\in[0,t)} |D^\alpha \int_{-\infty}^\tau \varphi_\epsilon(y)dy|d\tau \\
&\leq \frac{1}{\Gamma(1-\alpha)} \int_0^t \sup_{\tau\in[0,t)} \int_0^\tau |\frac{(\int_{-\infty}^s \varphi_\epsilon(y)dy)'}{(\tau-s)^\alpha} ds|d\tau \\
&\leq \int_0^t (\int_0^\tau \frac{\varphi_\epsilon(s)}{(\tau-s)^\alpha} ds)d\tau \leq C \int_0^t (\int_0^{\frac{\tau}{\epsilon}} \frac{\phi(h)}{(\tau-\epsilon h)^\alpha} dh)d\tau \\
&\leq C \int_0^t \sup_{h\in(0,\frac{t}{\epsilon})} |\phi(h)| \int_0^{\frac{\tau}{\epsilon}} (\frac{dh}{(\tau-\epsilon h)^\alpha})d\tau \\
&\leq \frac{C_{\alpha,\phi}}{\epsilon} \int_0^t \frac{\tau^{-\alpha+1}}{-\alpha+1} d\tau \leq C_{\alpha,\phi}\epsilon^{-1}t^{2-\alpha} \leq C_{\alpha,\phi}\epsilon^{-1}T^{2-\alpha}, \quad t < T,\ T > 0.
\end{aligned}$$

Thus,

$$\int_0^t (D^\alpha \int_{-\infty}^\tau \varphi_\epsilon(y)dy * \varphi_\epsilon(\tau))d\tau \leq C_{\alpha,\phi}\epsilon^{-1}T^{2-\alpha} \leq C\epsilon^{-1}, \quad T > 0.$$

Then, equation (2.5) leads to,

$$|\ \tilde{D}^\alpha x_\epsilon(t)| \leq C\epsilon^{-N},\ \exists N > 0,\ t \in \mathbb{R}_+ \cup \{0\}.$$

Apply $\tilde{D}^\beta$ where $0 < \beta < 1$ in equation (2.4), we have:

$$\begin{aligned}
|\tilde{D}^\beta(\tilde{D}^\alpha \dot{x}_\epsilon(t))| \leq |\tilde{D}^\beta(\tilde{D}^\alpha F_\epsilon(t,0)(t))| + |\ln\epsilon||\tilde{D}^\beta(\tilde{D}^\alpha x_\epsilon(t))| \\
+ |\tilde{D}^\beta(D^\alpha \int_{-\infty}^t \varphi_\epsilon(y)dy * \varphi_\epsilon(t))|.
\end{aligned} \tag{2.6}$$

Now, we are investigating the above final term whose other terms are similar the previous steps

$$\begin{aligned}
|(\tilde{D}^\beta(D^\alpha \int_{-\infty}^t \varphi_\epsilon(y)dy*\varphi_\epsilon(t))) &= \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{(D^\alpha \int_{-\infty}^\tau \varphi_\epsilon * \varphi_\epsilon)'_\tau d\tau}{(t-\tau)^\beta} * \varphi_\epsilon(t) \\
&= \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{(D^\alpha \int_{-\infty}^\tau \varphi_\epsilon * \varphi_\epsilon')_\tau d\tau}{(t-\tau)^\beta} * \varphi_\epsilon(t) \\
&\leq \frac{1}{\Gamma(1-\beta)\Gamma(1-\alpha)} \frac{C}{\epsilon^2(1-\alpha)} \int_0^t \frac{\tau^{1-\alpha}}{(t-\tau)^\beta}\, d\tau * \varphi_\epsilon(t) \\
&\leq C_{T,\alpha,\beta,\phi}\, \epsilon^{-2-\beta}.
\end{aligned}$$

Therefore,

$$|(\tilde{D}^\beta(D^\alpha \int_{-\infty}^t \varphi_\epsilon(y)dy * \varphi_\epsilon(t)))| \leq C_{T,\alpha,\beta,\phi}\, \epsilon^{-2-\beta}.$$

By the Gronwall inequality, the estimate (2.6) becomes

$$|\tilde{D}^\beta(\tilde{D}^\alpha x_\epsilon(t))| \leq C\epsilon^{-N}\exp(-T\ln\epsilon) \leq C\epsilon^{-N}, \quad \exists N > 0,\ t \in \mathbb{R}_+ \cup \{0\}.$$

For further fractional derivatives, when $m-1<\beta<m$, $\forall m\in\mathbb{N}$, we repeat the same procedure.

Let us see the uniqueness. We have two different solutions to the regularized equation (2.3), whose difference is denoted by ω_ϵ satisfying the inequality

$$|D^\beta \dot{\omega}_\epsilon(t)| \leqslant |ln\epsilon||D^\beta \omega_\epsilon(t)|.$$

Integration on $[0,t)$, $t<T$, $T>0$, yields

$$|D^\beta \omega_\epsilon(t)| \leq |D^\beta \omega_{0\epsilon}| + |ln\epsilon| \int_0^t |D^\beta \omega_\epsilon(\tau)| d\tau.$$

By the Gronwall inequality, since $|D^\beta \omega_{0\epsilon}|=0$,

$$|D^\beta \omega_\epsilon(t)| \leq 0, \quad 0<\beta<1,\ t\in\mathbb{R}_+\cup\{0\}.$$

$$|D^\beta \omega_\epsilon(t)| \leq \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\dot{\omega}_\epsilon(\tau)}{(t-\tau)^\beta} d\tau \leqslant \sup_{\tau\in[0,T)} |\dot{\omega}_\epsilon(t)| C_T \approx 0.$$

It follows that,

$$|\dot{\omega}_\epsilon(\tau)| \approx 0,\ |\omega_\epsilon(t)| \approx |\omega_\epsilon(0)| = 0.$$

We apply the same manner for the uniqueness when $m-1<\beta<m$, $\forall m\in\mathbb{N}$. □

3. Imbedding the Riesz fractional differentiation into extended Colombeau algebra of generalized functions

Let $f_\epsilon(x)$ represents a Colombeau generalized function $f(x)\in\mathcal{G}^e(\mathbb{R})$. The Riesz fractional derivative for $0\leq\alpha<1$ is defined by:

$$D^\alpha f_\epsilon(x) = \Gamma(1+\alpha)\frac{\cos(\frac{\alpha\pi}{2})}{\pi} \int_0^\infty \frac{f_\epsilon(x+\xi)-f_\epsilon(x-\xi)}{\xi^{\alpha+1}} d\xi.$$

We use the regularization for $0\leq\alpha<1$,

$$\tilde{D}^\alpha f_\epsilon(x) = \Gamma(1+\alpha)\frac{\cos(\frac{\alpha\pi}{2})}{\pi} \int_0^\infty \int_0^\infty (f_\epsilon(x+\xi)-f_\epsilon(x-\xi))\xi^{-1-\alpha}\varphi_\epsilon(\xi-t) dt d\xi.$$

We indicate that $|\tilde{D}^\alpha f_\epsilon(x) - D^\alpha f_\epsilon(x)| \approx 0$.

$$\sup_{x\in\mathbb{R}} |\tilde{D}^\alpha f_\epsilon(x) - D^\alpha f_\epsilon(x)| = \Gamma(1+\alpha)\frac{\cos(\frac{\alpha\pi}{2})}{\pi}$$

$$\times \sup_{x\in\mathbb{R}} \Big(\int_0^\infty |(f_\epsilon(x+\xi)-f_\epsilon(x-\xi))\xi^{-1-\alpha}| d\xi * |\varphi_\epsilon(x)-\delta(x)| \Big) \longrightarrow 0$$

as $\epsilon\longrightarrow 0$. Since $\lim_{\epsilon\longrightarrow 0}|\varphi_\epsilon(x)-\delta(x)|\longrightarrow 0$, then $\tilde{D}^\alpha f_\epsilon(x)\approx D^\alpha f_\epsilon(x)$. Using the fact that $\varphi_\epsilon(x)$ has the compact support on $[0,x]$, and defines $\forall x, g_x(\xi) = f_\epsilon(x+\xi)-f_\epsilon(x-\xi)$ where $g_x(\xi)$ has the compact support on $[0,x]$, so by Holder inequalities have the following calculations:

$$\sup_{x\in\mathbb{R}} |\tilde{D}^\alpha f_\epsilon(x)| \leq \Gamma(1+\alpha)\frac{\cos(\frac{\alpha\pi}{2})}{\pi} \int_0^\infty (f_\epsilon(x+\xi)-f_\epsilon(x-\xi))\xi^{-1-\alpha} * \varphi_\epsilon(\xi) d\xi$$

$$= \Gamma(1+\alpha)\frac{\cos(\frac{\alpha\pi}{2})}{\pi}\int_0^\infty (f_\epsilon(x+\xi)-f_\epsilon(x-\xi))\int_0^\infty (\xi-h)^{-1-\alpha}\varphi_\epsilon(h)dhd\xi$$
$$\le C\int_0^\infty (f_\epsilon(x+\xi)-f_\epsilon(x-\xi))\int_0^\infty (\xi-\epsilon p)^{-1-\alpha}\phi(p)dpd\xi$$
$$\le C\int_0^\infty (f_\epsilon(x+\xi)-f_\epsilon(x-\xi))\sup_{p\in[0,x]}\phi(p)\int_0^x (\xi-\epsilon p)^{-1-\alpha}dpd\xi$$
$$\le C\int_0^\infty (f_\epsilon(x+\xi)-f_\epsilon(x-\xi))\sup_{p\in[0,x]}\phi(p)\frac{1}{\epsilon}\int_{\xi-\epsilon x}^{\xi}(k)^{-1-\alpha}dkd\xi$$
$$\le C\sup_{x\in\mathbb{R}}(f_\epsilon(x+\xi)-f_\epsilon(x-\xi))\sup_{p\in[0,x]}\phi(p)\int_0^x\frac{1}{\epsilon}\int_{\xi-\epsilon x}^{\xi}(k)^{-1-\alpha}dkd\xi$$
$$\le C\sup_{x\in\mathbb{R}}(f_\epsilon(x+\xi)-f_\epsilon(x-\xi))\sup_{p\in[0,x]}\phi(p)\int_0^x\frac{1}{\epsilon}\frac{1}{-\alpha}((\xi)^{-\alpha}-(\xi-\epsilon x)^{-\alpha})d\xi$$
$$\le C\sup_{x\in\mathbb{R}}(f_\epsilon(x+\xi)-f_\epsilon(x-\xi))\sup_{p\in[0,x]}\phi(p)\frac{1}{\epsilon^2}\frac{1}{-\alpha}((\xi)^{-\alpha+1}-(\xi-\epsilon x)^{-\alpha+1})\Big|_0^x$$
$$\le C\sup_{x\in\mathbb{R}}(f_\epsilon(x+\xi)-f_\epsilon(x-\xi))\sup_{p\in[0,x]}\phi(p)\frac{1}{\epsilon^2}C_\alpha\epsilon^{-\alpha+1}X^{-\alpha+1}$$
$$\le C\sup(f_\epsilon(x+\xi)-f_\epsilon(x-\xi))C_{\alpha,\phi}\epsilon^{-\alpha+1}X^{-\alpha+1}\le C_{\alpha,\phi}\epsilon^{-N}X^{-\alpha+1},\ 0\le\alpha<1,$$

since $x<X, X>0$ and $f_\epsilon(x)$ is of the moderate class. Hence,

$$\sup_{x\in\mathbb{R}}|\tilde{D}^\alpha f_\epsilon(x)|\le C_{\alpha,\phi}\epsilon^{-N}X^{-\alpha+1},\quad 0\le\alpha<1.$$

In order to prove moderateness for higher derivatives a similar calculation is applied.

3.1. Application of the Reaiz fractional derivative of the spatial variable

We investigate the existence and uniqueness result for a nonlinear parabolic heat equation and an equation driven by the fractional derivative of the delta distribution in the extended algebra of generalized functions.

We consider the problem

$$\partial_t u(t,x)=\Delta u(t,x)+g(u(t,x)),\quad t>0,\ x\in\mathbb{R}^d,$$

where $g(u)\in L^\infty_{loc}([0,T),\mathbb{R}^n)$. Without loss of generality we set $u_{0\epsilon}(x)=\delta_\epsilon(x)$ with the following regularization

$$u_{0\epsilon}(x)=|\ln\epsilon|^{an}\phi(x.|\ln\epsilon|),\quad a>0.$$

We use the following regularization for $g(u)$

$$\|\nabla g_\epsilon(u_\epsilon)\|_{L^\infty}\le(|\ln\epsilon|)^b,\quad 0<b<1,$$

where $\phi(x)\in C_0^\infty(\mathbb{R}^n),\quad \phi(x)\ge 0,\quad \int\phi(x)dx=1.$

Theorem 3.1. *Regularized equation to heat equation*

$$\partial_t u_\epsilon(t,x)=\Delta u_\epsilon(t,x)+g_\epsilon(u_\epsilon(t,x)),\quad u_{0\epsilon}(x)=\delta_\epsilon(x),\tag{3.1}$$

has a unique solution in the space $\mathcal{G}^e([0,T)\times\mathbb{R}^n)$.

Proof . The integral form of equation (3.1)

$$u_\epsilon(t,x) = E_{n\epsilon}(t,x) * u_{0\epsilon}(x) + \int_0^t \int_{\mathbb{R}^n} E_{n\epsilon}(t-\tau, x-y) g_\epsilon(u_\epsilon(\tau,x)) dy d\tau, \quad t \in [0,T],\ x \in \mathbb{R}^n,$$

where E_n is the heat kernel. By the Young's inequality,

$$\|u_\epsilon(t,\cdot)\|_{L^\infty} \leq \|E_{n\epsilon}(t, x-\cdot)\|_{L^1} \|u_{0\epsilon}\|_{L^\infty} + \int_0^t \|E_{n\epsilon}(t-\tau, x-\cdot)\|_{L^1} \|\nabla g_\epsilon(\theta u_\epsilon)\|_{L^\infty} \|u_\epsilon(\tau,\cdot)\|_{L^\infty} d\tau,$$

$$\|u_\epsilon(t,\cdot)\|_{L^\infty} \leq C|\ln\epsilon|^{an} + \int_0^t C(|\ln\epsilon|)^b \|u_\epsilon(\tau,\cdot)\|_{L^\infty} d\tau.$$

By Gronwall inequality

$$\|u_\epsilon(t,\cdot)\|_{L^\infty} \leq C|\ln\epsilon|^{an} \exp(CT(|\ln\epsilon|)^b) \leq C\epsilon^{-N}, \quad \exists N > 0, \qquad x \in \mathbb{R}^n,\ t \in [0,T],\ \epsilon \in (0,1].$$

For the first derivative

$$\partial_x u_\epsilon(t,x) = \int_{\mathbb{R}^n} E_{n\epsilon}(t,y) \partial_x u_{0\epsilon}(x-y) dy + \int_0^t \int_{\mathbb{R}^n} \frac{(t-\tau)^{\frac{1}{2}} \partial_x E_{n\epsilon}(t-\tau, x-y)}{(t-\tau)^{\frac{1}{2}}} \nabla g_\epsilon(\theta u_\epsilon) u_\epsilon(\tau,y) dy d\tau,$$

$$\|\partial_x u_\epsilon(t,\cdot)\|_{L^\infty} \leq C|\ln\epsilon|^{an+1} + C\int_0^t (|\ln\epsilon|)^b \|u_\epsilon(\tau,\cdot)\|_{L^\infty} d\tau.$$

Using the moderateness of $u_\epsilon(t,x)$,

$$\|\partial_x u_\epsilon(t,\cdot)\|_{L^\infty} \leq C|\ln\epsilon|^{an+1} + (CT(|\ln\epsilon|)^b)\epsilon^{-N} \leq C\epsilon^{-N}, \quad \exists N > 0,\ x \in \mathbb{R}^n,\ t \in [0,T],\ \epsilon \in (0,1].$$

For higher entire derivatives, the procedure is repeated. Take the Riesz fractional derivative for $0 \leq \alpha < 1$,

$$\widetilde{(D)}^\alpha u_\epsilon(x,t) = \int_{\mathbb{R}^n} E_{n\epsilon}(t, x-y) \widetilde{(D)}^\alpha u_{0\epsilon}(y) dy$$
$$+ \int_0^t \int_{\mathbb{R}^n} E_{n\epsilon}(t-\tau, x-y) \nabla g_\epsilon(\theta u_\epsilon) \widetilde{(D)}^\alpha u_\epsilon(\tau,y) dy d\tau.$$

Using the moderateness of $u_\epsilon(t,x)$,

$$\|\widetilde{(D)}^\alpha u_\epsilon(t,\cdot))\|_{L^\infty} \leq \|E_{n\epsilon}(t, x-y)\|_{L^1} \|\widetilde{(D)}^\alpha u_{0\epsilon}(y)\|_{L^\infty}$$
$$+ \int_0^t \|E_{n\epsilon}(t-\tau, x-\cdot)\|_{L^1} \|\nabla g_\epsilon(\theta u_\epsilon)\|_{L^\infty} \|\widetilde{(D)}^\alpha u_\epsilon(\tau,\cdot)\|_{L^\infty} d\tau,$$

$$\|\widetilde{(D)}^\alpha u_\epsilon(x,t))\|_{L^\infty} \leq CX^{1-\alpha}|\ln\epsilon|^{an} + \int_0^t C(|\ln\epsilon|)^b \|\widetilde{(D)}^\alpha u_\epsilon(\tau,\cdot)\|_{L^\infty} d\tau.$$

By Gronwall inequality

$$\|\widetilde{(D)}^\alpha u_\epsilon(x,t))\|_{L^\infty} \leq CX^{1-\alpha}|\ln\epsilon|^{an} \exp(CT|\ln\epsilon|^b).$$

$$\|\widetilde{(D)}^\alpha u_\epsilon(x,t))\|_{L^\infty} \leq C\epsilon^{-N}, \qquad \exists N > 0,\ x \in \mathbb{R}^n,\ t \in [0,T],\ \ 0 \leq \alpha < 1,\ \ \epsilon \in (0,1].$$

It follows moderateness for the Riesz fractional derivative in the space $\mathcal{G}^e([0,T)\times\mathbb{R}^n)$.

For uniqueness suppose that $L_\epsilon(x,t) = u_{1\epsilon}(x,t) - u_{2\epsilon}(x,t)$ are two different solutions which make difference for equation (3.1)

$$\partial_t L_\epsilon(t,x) = \Delta L_\epsilon(t,x) + k_\epsilon(t,x)L_\epsilon(t,x) + N_\epsilon(t,x),$$

$$(L_\epsilon(x,0))_\epsilon = (N_{0\epsilon}(x)) \in (\mathcal{N}_{c^1,L^\infty}(\mathbb{R}^n)),$$

where $N_\epsilon(x,t) \in \mathcal{N}_{c^1,L^\infty}(\mathbb{R}^n\times[0,T)), \quad \|k_\epsilon(\tau,x)\|_{L^\infty} \leq C(|\ln\epsilon|)^b \quad 0<b<1.$

$$\begin{aligned}\|\widetilde{(D)}^\alpha L_\epsilon(t,\cdot))\|_{L^\infty} \leq{}& \|E_{n\epsilon}(t,x-\cdot)\|_{L^1}\|\widetilde{(D)}^\alpha N_{0\epsilon}(\cdot)\|_{L^\infty}\\ &+\int_0^t \|E_{n\epsilon}(t-\tau,x-\cdot)\|_{L^1}\|k_\epsilon(\tau,\cdot)\|_{L^\infty}\|\widetilde{(D)}^\alpha L_\epsilon(\tau,\cdot)\|_{L^\infty}d\tau\\ &+\int_0^t \|E_{n\epsilon}(t-\tau,x-\cdot)\|_{L^1}\|\widetilde{(D)}^\alpha N_\epsilon(\tau,\cdot)\|_{L^\infty}d\tau.\end{aligned}$$

By the Gronwall inequality we obtain

$$\|\widetilde{(D)}^\alpha L_\epsilon(t,\cdot)\|_{L^\infty} \leq C\epsilon^r \exp(CT(|\ln\epsilon|)^b) \leq C\epsilon^r.$$

Then $\|\widetilde{(D)}^\alpha L_\epsilon(t,\cdot)\|_{L^\infty} \leq C\epsilon^r, \quad \forall r\in\mathbb{R},\ 0<b<1,\ x\in\mathbb{R}^n,\ t\in[0,T],\ \epsilon\in(0,1].$ □

References

[1] J.F. Colombeau, *New generalized functions and Multiplication of distributions,* North-Holland, Amsterdam, 1984.

[2] J.F. Colombeau and A.Y.L. Roux, *Multiplications of distributions in elasticity and hydrodynamics,* J. Math. Phys. 29 (1988) 315–319.

[3] J.F. Colombeau, *Elementary Introduction to New Generalized Functions,* North-Holland Math. Studies Vol. 113, North-Holland, Amsterdam, 1985.

[4] F. Mainardi, Y. Luchko and G. Pagninil, *The fundamental solution of the space-time fractional diffusion equation* , Frac. Calculus Appl. Anal. 4 (2001) 153–192.

[5] D. Rajter-ćirić, *Fractional derivatives of Colombeau Generalized stochastic processes defined on* $\mathbb{R}^+$, Appl. Anal. Discrete Math. 5 (2011) 283–297.

[6] M. Stojanović, *Extension of Colombeau algebra to derivatives of arbitrary order* $D^\alpha, \alpha\in\mathbb{R}_+\cup\{0\}$. *Application to ODEs and PDEs with entire and fractional derivatives,* Nonlinear Anal. 5 (2009) 5458–5475.