Math. Z. (2012) 272:921–936
DOI 10.1007/s00209-011-0964-x

Mathematische Zeitschrift

Functional inequalities and subordination: stability of Nash and Poincaré inequalities

René L. Schilling · Jian Wang

Received: 21 April 2011 / Accepted: 26 September 2011 / Published online: 26 November 2011

Abstract We show that certain functional inequalities, e.g. Nash-type and Poincaré-type inequalities, for infinitesimal generators of C_0 semigroups are preserved under subordination in the sense of Bochner. Our result improves earlier results by Bendikov and Maheux (Trans Am Math Soc 359:3085–3097, 2007, Theorem 1.3) for fractional powers, and it also holds for non-symmetric settings. As an application, we will derive hypercontractivity, supercontractivity and ultracontractivity of subordinate semigroups.

Keywords Subordination · Bernstein function · Nash-type inequality · Super-Poincaré inequality · Weak Poincaré inequality

Mathematics Subject Classification (2010) 47D60 · 60J35 · 60J75 · 60J25 · 60J27

1 Introduction

In this note we show that certain functional inequalities are preserved under subordination in the sense of Bochner.

Bochner's subordination is a method to get new semigroups from a given one. Let us briefly summarize the main facts about subordination; our main reference is the monograph [12], in particular Chapter 12. Let $(T_t)_{t\geq 0}$ be a strongly continuous (C_0) contraction semigroup on a Banach space $(\mathscr{B}, \|\cdot\|)$. The infinitesimal generator is the operator

R. L. Schilling
Institut für Mathematische Stochastik, TU Dresden, 01062 Dresden, Germany
e-mail: rene.schilling@tu-dresden.de

J. Wang (✉)
School of Mathematics and Computer Science, Fujian Normal University,
Fuzhou 350007, People's Republic of China
e-mail: jianwang@fjnu.edu.cn

$$Au := \lim_{t\to 0} \frac{u - T_t u}{t},$$
$$D(A) := \left\{ u \in \mathscr{B} \,:\, \lim_{t\to 0} \frac{u - T_t u}{t} \text{ exists in the strong sense} \right\}.$$

A *subordinator* is a vaguely continuous convolution semigroup of sub-probability measures $(\mu_t)_{t\geq 0}$ on $[0, \infty)$. Subordinators are uniquely characterized by the Laplace transform:

$$\mathscr{L}\mu_t(\lambda) = \int_{[0,\infty)} e^{-s\lambda}\,\mu_t(ds) = e^{-tf(\lambda)} \quad \text{for all } t \geq 0 \text{ and } \lambda \geq 0.$$

The characteristic exponent $f : (0, \infty) \to (0, \infty)$ is a *Bernstein function*, i.e. a function of the form

$$f(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-t\lambda})\,\nu(dt), \tag{1}$$

where $a, b \geq 0$ are nonnegative constants and ν is a nonnegative measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\,\nu(dt) < \infty$. There are one-to-one relations between the triplet (a, b, ν), the Bernstein function f and the subordinator $(\mu_t)_{t\geq 0}$. Among the most prominent examples of Bernstein functions are the fractional powers $f_\alpha(\lambda) = \lambda^\alpha$, $0 < \alpha \leq 1$. The Bochner integral

$$T_t^f u := \int_{[0,\infty)} T_s u\,\mu_t(ds), \quad t \geq 0,\ u \in \mathscr{B},$$

defines a strongly continuous contraction semigroup on $\mathscr{B}$. We call $(T_t^f)_{t\geq 0}$ *subordinate* to $(T_t)_{t\geq 0}$ (with respect to the subordinator $(\mu_t)_{t\geq 0}$ or the Bernstein function f). Subordination preserves many additional properties of the original semigroup. For example, on a Hilbert space, $(T_t^f)_{t\geq 0}$ inherits symmetry from $(T_t)_{t\geq 0}$ and on an ordered Banach space $(T_t^f)_{t\geq 0}$ is sub-Markovian whenever $(T_t)_{t\geq 0}$ is. Let us write $(A^f, D(A^f))$ for the generator of $(T_t^f)_{t\geq 0}$; it is known that $D(A)$ is an operator core of A^f and that A^f is given by Phillips' formula

$$A^f u = au + b\,Au + \int_{(0,\infty)} (u - T_s u)\,\nu(ds), \quad u \in D(A). \tag{2}$$

Here (a, b, ν) is the defining triplet for f as in (1).

Bochner's subordination gives rise to a functional calculus for generators of C_0 contraction semigroups. In many situations this functional calculus coincides with classical functional calculi, e.g. the spectral calculus in Hilbert space or the Dunford–Taylor spectral calculus in Banach space, cf. [4,12]. It is, therefore, natural to write $f(A)$ instead of A^f.

From now on we will use $\mathscr{B} = L^2(X, m)$ where (X, m) is a measure space with a σ-finite measure m. We write $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_2$ for the scalar product and norm in L^2, respectively; $\|\cdot\|_1$ denotes the norm in $L^1(X, m)$. To compare our result with [2, Theorem 1.3], we start with Nash-type inequalities. For the study of Nash-type inequalities and ultracontractivity of the associated operator semigroups we refer to the paper [1] and the references therein. Our main contribution to this type of functional inequalities are the following two results.

Theorem 1 (symmetric case) *Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup of symmetric operators on $L^2(X, m)$ and assume that for each $t \geq 0$, $T_t|_{L^2(X,m)\cap L^1(X,m)}$ has an extension which is a contraction on the space $L^1(X, m)$, i.e. we have $\|T_t u\|_1 \leq \|u\|_1$ for all*

$u \in L^1(X, m) \cap L^2(X, m)$. *Suppose that the generator* $(A, D(A))$ *satisfies the following Nash-type inequality:*

$$\|u\|_2^2\, B(\|u\|_2^2) \le \langle A\,u, u\rangle, \quad u \in D(A),\ \|u\|_1 = 1, \tag{3}$$

where $B : (0, \infty) \to (0, \infty)$ *is any increasing function. Then, for any Bernstein function* f, *the generator* $f(A)$ *of the subordinate semigroup satisfies*

$$\frac{\|u\|_2^2}{2}\, f\left(B\left(\frac{\|u\|_2^2}{2}\right)\right) \le \langle f(A)\,u, u\rangle, \quad u \in D(f(A)),\ \|u\|_1 = 1. \tag{4}$$

Remark 2 For fractional powers A^α, $0 < \alpha < 1$, the result of Theorem 1 is due to Bendikov and Maheux [2, Theorem 1.3]; this corresponds to the Bernstein functions $f(\lambda) = \lambda^\alpha$. Our result is valid for *all* Bernstein functions, hence, for *all subordinate generators* $f(A)$. Note that [2, Theorem 1.3] claims that

$$c_1\|u\|_2^2\,(B(c_2\|u\|_2^2))^\alpha \le \langle A^\alpha\,u, u\rangle, \quad u \in D(A^\alpha),\ \|u\|_1 = 1,$$

holds for all $0 < \alpha < 1$ with $c_1 = c_2 = 1$, but a close inspection of the proof in [2] reveals that one has to assume, in general, $c_1, c_2 \in (0, 1)$. Note that Theorem 1 yields $c_1 = c_2 = 1/2$.

If $(T_t)_{t\ge0}$ is not symmetric, we still have the following result.

Theorem 3 (non-symmetric case) *Let* $(T_t)_{t\ge0}$ *be a strongly continuous contraction semigroup on* $L^2(X, m)$ *and assume that for each* $t \ge 0$, $T_t|_{L^2(X,m)\cap L^1(X,m)}$ *has an extension which is a contraction on* $L^1(X, m)$. *Suppose that the generator* $(A, D(A))$ *satisfies the following Nash-type inequality:*

$$\|u\|_2^2\, B(\|u\|_2^2) \le \mathrm{Re}\langle A\,u, u\rangle, \quad u \in D(A),\ \|u\|_1 = 1, \tag{5}$$

where $B : (0, \infty) \to (0, \infty)$ *is any increasing function. Then, for any Bernstein function* f, *the generator* $f(A)$ *of the subordinate semigroup satisfies*

$$\frac{\|u\|_2^2}{4}\, f\left(2B\left(\frac{\|u\|_2^2}{2}\right)\right) \le \mathrm{Re}\langle f(A)\,u, u\rangle, \quad u \in D(f(A)),\ \|u\|_1 = 1. \tag{6}$$

Remark 4 (i) The assumption that T_t is a contraction both in $L^2(X, m)$ and $L^1(X, m)$ is often satisfied in concrete situations. Assume that $(T_t)_{t\ge0}$ is a strongly continuous contraction semigroup on $L^2(X, m)$ such that the operators T_t are symmetric and sub-Markovian—i.e. $0 \le T_t v \le 1$ a.e. for all $0 \le v \le 1$ m-a.e. Then the following argument shows that $T_t|_{L^2(X,m)\cap L^1(X,m)}$ is a contraction on $L^1(X, m)$:

$$\langle T_t u, v\rangle = \langle u, T_t v\rangle \le \langle |u|, \|v\|_\infty\rangle = \|v\|_\infty \|u\|_1 \quad u \in L^2 \cap L^1,\ v \in L^1 \cap L^\infty.$$

In general, a sub-Markovian L^2-contraction operator T_t is also an L^1-contraction if, and only if, the L^2-adjoint T_t^* is a sub-Markovian operator, cf. [11, Lemma 2].

(ii) From (1) it follows that Bernstein functions are subadditive, thus

$$\frac{1}{2}\, f(2x) \le f(x).$$

This shows that, for symmetric semigroups, (4) implies (6).

The remaining part of this paper is organized as follows. Section 2 contains some preparations needed for the proof of Theorems 1 and 3, in particular a one-to-one relation between Nash-type inequalities and estimates for the decay of the semigroups. These estimates are needed for the proof of Theorems 1 and 3 in Sect. 3. Section 4 contains several applications of our main result, e.g. the super-Poincaré and weak Poincaré inequality for subordinate semigroups and the hyper-, super- and ultracontractivity of subordinate semigroups.

2 Preliminaries

In this section we collect a few auxiliary results for the proof of Theorems 1 and 3. We begin with a differential and integral inequality, which is a consequence of [6, Appendix A, Lemma A.1, p. 193]. Note that the right hand side of the inequality (7) below is negative. This is different from the usual Gronwall–Bellman–Bihari inequality, see e.g. [5, Section 3], [3, Chapter 4 §§ 4,5] and [13, I.1.VI, I.6.IX], but it is essential for our purposes. For the sake of completeness, we include the short proof whose idea follows [6, Appendix A, Remark A.3, p. 194].

Recall that for an increasing function $G : [0, \infty) \to \mathbb{R}$ the *generalized (right continuous) inverse* $G^{-1} : \mathbb{R} \to [0, +\infty]$ is defined as

$$G^{-1}(y) := \inf\{t > 0 \,:\, G(t) > y\}, \qquad \inf \emptyset := \infty.$$

If G is strictly increasing, then G^{-1} coincides with the usual inverse.

Lemma 5 *Let $h : [0, \infty) \to [0, \infty)$ be a differentiable function. Suppose that there exists an increasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) > 0$ for $t > 0$, $\int_{0+} 1/\varphi(t)\,dt = \infty$ and*

$$h'(t) \le -\varphi(h(t)) \quad \textit{for all} \;\; t \ge 0. \tag{7}$$

Then, we have

$$h(t) \le G^{-1}(G(h(0)) - t) \quad \textit{for all} \;\; t \ge 0,$$

where G^{-1} is the (generalized right continuous) inverse of

$$G(t) = \begin{cases} \displaystyle\int_1^t \frac{du}{\varphi(u)}, & \textit{if } t \ge 1, \\ \displaystyle -\int_t^1 \frac{du}{\varphi(u)}, & \textit{if } t \le 1. \end{cases}$$

Proof Since $h'(t) \le -\varphi(h(t)) \le 0$, the set $I = \{t : h(t) > 0\}$ is a (bounded or unbounded) interval; for $t \in I$, the function $h(t)$ is strictly decreasing. With the convention $\int_a^b = -\int_b^a$, we see for all $t \in I$ that

$$\begin{aligned} G(h(t)) &= \int_1^{h(t)} \frac{1}{\varphi(u)}\,du \\ &= G(h(0)) + \int_{h(0)}^{h(t)} \frac{1}{\varphi(u)}\,du \end{aligned}$$

$$= G(h(0)) + \int_0^t \frac{h'(u)\,du}{\varphi(h(u))}$$
$$\leq G(h(0)) - t.$$

If $t \notin I$, $G(h(t)) = G(0) = -\infty$, and the above inequality is trivial. The claim follows from the definition of the generalized inverse G^{-1}. □

Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup of (not necessarily symmetric) operators on $L^2 = L^2(X, m)$. Denote by $(A, D(A)$ the infinitesimal generator. Since $\frac{d}{dt} T_t u = -T_t A u$ for all $u \in D(A)$, we have

$$\frac{d}{dt}\, \|T_t u\|_2^2 = -2\,\mathrm{Re}\langle A\, T_t u, T_t u\rangle, \quad u \in D(A).$$

Proposition 6 *Let $(T_t)_{t\geq 0}$ be a C_0 contraction semigroup on $L^2(X, m)$ and assume that each $T_t|_{L^2(X,m)\cap L^1(X,m)}$, $t \geq 0$, has an extension which is a contraction on $L^1(X, m)$, i.e. $\|T_t u\|_1 \leq \|u\|_1$ for all $u \in L^1(X, m) \cap L^2(X, m)$. Then the following Nash-type inequality*

$$\|u\|_2^2\, B(\|u\|_2^2) \leq \mathrm{Re}\langle A\, u, u\rangle, \quad u \in D(A),\ \|u\|_1 = 1 \tag{8}$$

with some increasing function $B : (0, \infty) \to (0, \infty)$ holds if, and only if,

$$\|T_t u\|_2^2 \leq G^{-1}(G(\|u\|_2^2) - t) \quad \text{for all } t \geq 0 \text{ and } u \in D(A),\ \|u\|_1 = 1 \tag{9}$$

where

$$G(t) = \begin{cases} \displaystyle\int_1^t \frac{ds}{2s B(s)}, & \text{if } t \geq 1, \\ \displaystyle -\int_t^1 \frac{ds}{2s B(s)}, & \text{if } t \leq 1. \end{cases}$$

Proof Assume that (8) holds. Then,

$$\|u\|_2^2\, B\left(\frac{\|u\|_2^2}{\|u\|_1^2}\right) \leq \mathrm{Re}\langle A\, u, u\rangle, \quad u \in D(A).$$

For all $u \in D(A)$ with $\|u\|_1 = 1$ we have

$$\frac{d}{dt}\, \|T_t u\|_2^2 = -2\,\mathrm{Re}\langle A\, T_t u, T_t u\rangle \leq -2\, \|T_t u\|_2^2\, B\left(\frac{\|T_t u\|_2^2}{\|T_t u\|_1^2}\right),$$

Since the function B is increasing and $\|T_t u\|_1 \leq \|u\|_1 = 1$, we have

$$\frac{d}{dt}\, \|T_t u\|_2^2 \leq -2\, \|T_t u\|_2^2\, B(\|T_t u\|_2^2).$$

This, together with Lemma 5, proves (9).

For the converse we assume that (9) holds. Then, for all $u \in D(A)$ with $\|u\|_1 = 1$,

$$\begin{aligned}\mathrm{Re}\langle Au, u\rangle &= -\frac{1}{2}\frac{d}{dt}\,\|T_t u\|_2^2\,\Big|_{t=0}\\ &= \frac{1}{2}\lim_{t\to 0}\frac{\|u\|_2^2 - \|T_t u\|_2^2}{t}\\ &\geq \frac{1}{2}\lim_{t\to 0}\frac{\|u\|_2^2 - G^{-1}(G(\|u\|_2^2) - t)}{t}\\ &= -\frac{1}{2}\frac{d}{dt}\,G^{-1}(G(\|u\|_2^2) - t)\,\Big|_{t=0}\\ &= [G^{-1}(G(\|u\|_2^2) - t)\cdot B(G^{-1}(G(\|u\|_2^2) - t))]\,\Big|_{t=0}\\ &= \|u\|_2^2\, B(\|u\|_2^2),\end{aligned}$$

which is just the Nash-type inequality (8). □

Finally we need some elementary estimate for Bernstein functions.

Lemma 7 *Let f be a Bernstein function given by (1) where $a = b = 0$ and with representing measure ν. Set*

$$\nu_1(x) := \int_0^x \nu(s,\infty)\,ds.$$

Then for $x > 0$,

$$\frac{e-1}{e}\,x\,\nu_1\left(\frac{1}{x}\right) \leq f(x) \leq x\,\nu_1\left(\frac{1}{x}\right).$$

Proof By Fubini's theorem we find

$$\begin{aligned}x\,\nu_1\left(\frac{1}{x}\right) &= x\int_0^{1/x}\nu(s,\infty)\,ds = \int_0^1 \nu\left(\frac{t}{x},\infty\right)dt\\ &= \int_0^1\int_{t/x}^\infty \nu(dy)\,dt\\ &= \int_0^\infty (xy\wedge 1)\,\nu(dy),\end{aligned}$$

see also Ôkura [9, (1.5)]. Using the following elementary inequalities

$$\frac{e-1}{e}(1\wedge r) \leq 1 - e^{-r} \leq 1\wedge r \quad \text{for } r \geq 0,$$

we conclude

$$\frac{e-1}{e}\,x\,\nu_1\left(\frac{1}{x}\right) = \int_0^\infty \frac{e-1}{e}\,(xy\wedge 1)\,\nu(dy) \leq \int_0^\infty (1 - e^{-xy})\,\nu(dy) = f(x).$$

The upper bound follows similarly. □

3 Proof of the main theorems

Proof of Theorem 1. Since $D(A)$ is an operator core for $(f(A), D(f(A)))$, it is enough to prove (4) for $u \in D(A)$. Using Phillips' formula (2) we find for all $u \in D(A)$

$$\langle f(A)\, u, u\rangle = a\, \|u\|_2^2 + b\, \langle A\, u, u\rangle + \int_{(0,\infty)} \langle u - T_s u, u\rangle\, \nu(ds).$$

This formula and the representation (1) for f show that we may, without loss of generality, assume that $a = b = 0$.

Assume that (3) holds. Proposition 6 shows for $t \geq 0$ and $u \in D(A)$ with $\|u\|_1 = 1$,

$$\frac{\langle T_t u, u\rangle}{\|u\|_2^2} = \frac{\|T_{t/2} u\|_2^2}{\|u\|_2^2} \leq \frac{G^{-1}(G(\|u\|_2^2) - t/2)}{\|u\|_2^2}.$$

Then,

$$\begin{aligned}
\langle f(A)\, u, u\rangle &= \int_{(0,\infty)} \langle u - T_s u, u\rangle\, \nu(ds) \\
&= \|u\|_2^2 \int_{(0,\infty)} \left(1 - \frac{\langle T_s u, u\rangle}{\|u\|_2^2}\right) \nu(ds) \\
&\geq \int_{(0,\infty)} \left(\|u\|_2^2 - G^{-1}\left(G(\|u\|_2^2) - \frac{s}{2}\right)\right)\, \nu(ds) \\
&= g(\|u\|_2^2),
\end{aligned}$$

where

$$g(r) = \int_{(0,\infty)} \left(r - G^{-1}\left(G(r) - \frac{s}{2}\right)\right)\, \nu(ds).$$

Furthermore, for all $r > 0$,

$$\begin{aligned}
g(r) &= \int_{(0,\infty)} \left(r - G^{-1}\left(G(r) - \frac{s}{2}\right)\right)\, \nu(ds) \\
&= \int_{(0,\infty)} \left(\int_{G(r)-s/2}^{G(r)} dG^{-1}(u)\right) \nu(ds) \\
&= \int_{-\infty}^{G(r)} \nu(2(G(r) - u, \infty))\, dG^{-1}(u) \\
&= \int_0^r \nu(2(G(r) - G(u)), \infty)\, du.
\end{aligned}$$

For the last equality we used that B is increasing, $G(x) > -\infty$ for all $x > 0$ and $G(0) = -\infty$; this follows from

$$G(0) = -\int_0^1 \frac{du}{uB(u)} \le \frac{-1}{B(1)} \int_0^1 \frac{du}{u} = -\infty.$$

Using again the monotonicity of B, we find from the mean value theorem

$$\frac{1}{2uB(u)} \ge \frac{G(r) - G(u)}{r - u} \ge \frac{1}{2rB(r)} \quad \text{for all} \quad 0 < u < r. \tag{10}$$

Therefore,

$$\begin{aligned} g(r) &\ge \int_0^r \nu\left(\frac{1}{uB(u)}(r-u), \infty\right) du \\ &\ge \int_{r/2}^r \nu\left(\frac{1}{uB(u)}(r-u), \infty\right) du \qquad (11) \\ &\ge \int_0^{r/2} \nu\left(\frac{2v}{rB(r/2)}, \infty\right) dv \\ &= \frac{1}{2} rB(r/2) \int_0^{1/B(r/2)} \nu(s, \infty)\, ds. \end{aligned}$$

A similar calculation, now using the lower bound in (10), yields

$$g(r) \le rB(r) \int_0^{1/B(r)} \nu(s, \infty)\, ds.$$

Now we can use Lemma 7 to deduce that

$$\frac{e}{e-1} rf(B(r)) \ge g(r) \ge \frac{r}{2} f\left(B\left(\frac{r}{2}\right)\right) \quad \text{for all} \quad r > 0,$$

and the proof is complete. □

Remark 8 (i) In the proof of Theorem 1, at the line (11), we can replace $r/2$ by εr for any $\varepsilon \in (0, 1)$. Then we get

$$g(r) \ge \sup_{\varepsilon \in (0,1)} \left[(1-\varepsilon)\, r\, f\left(\frac{\varepsilon B(\varepsilon r)}{1-\varepsilon}\right)\right],$$

which shows that we can improve (4) by

$$\sup_{\varepsilon \in (0,1)} \left[(1-\varepsilon)\, \|u\|_2^2\, f\left(\frac{\varepsilon B(\varepsilon \|u\|_2^2)}{1-\varepsilon}\right)\right] \le \langle f(A)\, u, u\rangle, \quad u \in D(f(A)),\ \|u\|_1 = 1.$$

(ii) A close inspection of our proof shows that Theorem 1 remains valid if we replace the norming condition $\|u\|_1 = 1$ in (3) and (5) by the more general condition $\Phi(u) = 1$. Here $\Phi : L^2(X, m) \to [0, \infty]$ is a measurable functional satisfying $\Phi(cu) = c^2\Phi(u)$ and $\Phi(T_t u) \le \Phi(u)$ for all $t \ge 0$ and $\Phi(u) = 0$ if, and only if, $u = 0$.

Proof of Theorem 3 The proof of Theorem 3 is similar to the proof of Theorem 1. Therefore we only outline the differences in the arguments. As before we can assume that the function $f(\lambda) = \int_{(0,\infty)} (1 - e^{-t\lambda})\, \nu(dt)$. Moreover, it is enough to verify (6) for all $u \in D(A)$. Since $(T_t)_{t \geq 0}$ is a contraction on $L^1(X, m) \cap L^2(X, m)$, we see from (5) and Proposition 6 that for all $t \geq 0$ and $u \in D(A)$ with $\|u\|_1 = 1$,

$$\|T_t u\|_2^2 \leq G^{-1}(G(\|u\|_2^2) - t).$$

By the Cauchy–Schwarz inequality,

$$\frac{\mathrm{Re}\langle T_t u, u\rangle}{\|u\|_2^2} \leq \frac{|\langle T_t u, u\rangle|}{\|u\|_2^2} \leq \frac{\|T_t u\|_2 \|u\|_2}{\|u\|_2^2} \leq \frac{\sqrt{G^{-1}(G(\|u\|_2^2) - t)}}{\|u\|_2}.$$

Using (2) yields that for any $u \in D(A)$ with $\|u\|_1 = 1$,

$$\begin{aligned}
\mathrm{Re}\langle f(A)\, u, u\rangle &= \int_{(0,\infty)} \mathrm{Re}\langle u - T_s u, u\rangle\, \nu(ds) \\
&= \|u\|_2^2 \int_{(0,\infty)} \left(1 - \frac{\mathrm{Re}\langle T_s u, u\rangle}{\|u\|_2^2}\right) \nu(ds) \\
&\geq \|u\|_2^2 \int_{(0,\infty)} \left(1 - \frac{\sqrt{G^{-1}(G(\|u\|_2^2) - s)}}{\|u\|_2}\right) \nu(ds) \\
&= \|u\|_2^2 \int_{(0,\infty)} \left(\frac{1 - \frac{G^{-1}(G(\|u\|_2^2) - s)}{\|u\|_2^2}}{1 + \frac{\sqrt{G^{-1}(G(\|u\|_2^2) - s)}}{\|u\|_2}}\right) \nu(ds) \\
&\geq \frac{\|u\|_2^2}{2} \int_{(0,\infty)} \left(1 - \frac{G^{-1}(G(\|u\|_2^2) - s)}{\|u\|_2^2}\right) \nu(ds) \\
&= g(\|u\|_2^2),
\end{aligned}$$

where

$$g(r) = \frac{r}{2} \int_{(0,\infty)} \left(1 - \frac{G^{-1}(G(r) - s)}{r}\right) \nu(ds).$$

A similar calculation as in the proof of Theorem 1 shows

$$g(r) = \frac{1}{2} \int_0^r \nu(G(r) - G(u), \infty)\, du \geq \frac{r}{4}\, f\left(2\, B\left(\frac{r}{2}\right)\right),$$

which is exactly (6). □

4 Applications

We will now give some applications of our results. Throughout this section we retain the notation introduced in the previous sections. In particular, $(T_t)_{t\geq 0}$ will be a strongly continuous contraction semigroup on $L^2(X, m)$ with generator $(A, D(A))$. We assume that $\|T_t u\|_1 \leq \|u\|_1$ for all $u \in L^2(X, m) \cap L^1(X, m)$ and, for simplicity, that the operators T_t, $t \geq 0$, are symmetric. By $\Phi : L^2(X, m) \to [0, \infty]$ we denote a functional on $L^2(X, m)$ such that for all $c, t > 0$ and $u \in L^2(X, m)$

$$\Phi(u) = 0 \Rightarrow u = 0, \Phi(cu) = c^2\Phi(u) \quad \text{and} \quad \Phi(T_t u) \leq \Phi(u);$$

by f we always denote a Bernstein function given by (1).

4.1 Subordinate super-Poincaré inequalities

In this section, we study the analogue of Theorem 1 for super-Poincaré inequalities. For details on super-Poincaré inequalities and their applications we refer to [14–16] or [17, Chapter 3].

Proposition 9 *Assume that $(A, D(A))$ satisfies the following super-Poincaré inequality:*

$$\|u\|_2^2 \leq r\,\langle A\,u, u\rangle + \beta(r)\,\Phi(u), \quad r > 0,\ u \in D(A), \tag{12}$$

where $\beta : (0, \infty) \to (0, \infty)$ is a decreasing function such that $\lim_{r\to 0}\beta(r) = \infty$ and $\lim_{r\to\infty}\beta(r) = 0$; moreover, we set $\beta(0) := \infty$. Then the generator $f(A)$ of the subordinate semigroup also satisfies a super-Poincaré inequality

$$\|u\|_2^2 \leq r\,\langle f(A)\,u, u\rangle + \beta_f(r)\,\Phi(u), \quad r > 0,\ u \in D(f(A)), \tag{13}$$

where

$$\beta_f(r) = 4\beta\left(\frac{1}{2f^{-1}(2/r)}\right).$$

Proof We can rewrite (12) for any $u \in D(A)$ with $\Phi(u) = 1$ in the following form:

$$\|u\|_2^2\, B(\|u\|_2^2) \leq \langle A\,u, u\rangle,$$

where

$$B(x) = \sup_{s>0} \frac{1 - \beta(s)/x}{s}.$$

Clearly, $B(x)$ is an increasing function on $(0, \infty)$. Since $\beta^{-1} : (0, \infty) \to (0, \infty)$, we see from

$$\frac{1}{2\beta^{-1}(x/2)} = \frac{1 - \beta(\beta^{-1}(x/2))/x}{\beta^{-1}(x/2)} \leq B(x) = \sup_{s\geq\beta^{-1}(x)} \frac{1 - \beta(s)/x}{s} \leq \frac{1}{\beta^{-1}(x)} \tag{14}$$

that $B : (0, \infty) \to (0, \infty)$.

Using Theorem 1 and the Remark 8 (ii) yields for any $u \in D(f(A))$ with $\Phi(u) = 1$,

$$\Theta(\|u\|_2^2) \leq \langle f(A)\,u, u\rangle,$$

where

$$\Theta(x) = \frac{x}{2}\, f\left(B\left(\frac{x}{2}\right)\right) = \frac{x}{2} \sup_{s>0} f\left(\frac{1 - 2\beta(s)/x}{s}\right).$$

For $r > 0$, define

$$\widetilde{\beta}(r) = \sup_{s>0}\{\Theta^{-1}(s) - rs\}.$$

Then,

$$\|u\|_2^2 \le r\,\langle f(A)\,u, u\rangle + \widetilde{\beta}(r)\,\Phi(u), \quad r > 0,\ u \in D(f(A)). \tag{15}$$

Next, we will estimate $\widetilde{\beta}(r)$. By (14),

$$\Theta(x) \ge \frac{x}{2}\, f\left(\frac{1}{2\,\beta^{-1}(x/4)}\right) := \Theta_0(x),$$

which in turn implies that

$$\Theta^{-1}(x) \le \Theta_0^{-1}(x).$$

By the definition of $\Theta_0(x)$, $\Theta_0 : (0,\infty) \to (0,\infty)$ is a strictly increasing function such that $\lim_{x\to 0}\Theta_0(x) = 0$ and $\lim_{x\to\infty}\Theta_0(x) = \infty$, and so

$$\Theta_0^{-1}(x) = 2x\left[f\left(\frac{1}{2\,\beta^{-1}(\Theta_0^{-1}(x)/4)}\right)\right]^{-1}. \tag{16}$$

On the other hand,

$$\widetilde{\beta}(r) \le \sup_{s>0}\{\Theta_0^{-1}(s) - rs\} = \sup_{s>0,\ \Theta_0^{-1}(s)\ge rs} \Theta_0^{-1}(s).$$

From (16) we see that $\Theta_0^{-1}(s) \ge rs$ is equivalent to

$$\frac{1}{2f^{-1}(2/r)} \le \beta^{-1}\left(\frac{\Theta_0^{-1}(s)}{4}\right).$$

Since β is decreasing, we can rewrite this as

$$\Theta_0^{-1}(s) \le 4\beta\left(\frac{1}{2\,f^{-1}(2/r)}\right),$$

and so

$$\widetilde{\beta}(r) \le \sup_{s>0,\ \Theta_0^{-1}(s)\le 4\beta\left(\frac{1}{2\,f^{-1}(2/r)}\right)} \Theta_0^{-1}(s) \le 4\beta\left(\frac{1}{2\,f^{-1}(2/r)}\right). \tag{17}$$

The proof is complete if we combine (15) and (17). □

4.2 Subordinate weak Poincaré inequalities

We can also consider the subordination for weak Poincaré inequalities; for details we refer to [10] or [17, Chapter 4].

Proposition 10 *Assume that $(A, D(A))$ satisfies the following weak Poincaré inequality:*

$$\|u\|_2^2 \le \alpha(r)\,\langle A\,u, u\rangle + r\,\Phi(u), \quad r > 0,\ u \in D(A), \tag{18}$$

where $\alpha : (0,\infty) \to (0,\infty)$ is a decreasing function. Then the generator $f(A)$ of the subordinate semigroup also satisfies a weak Poincaré inequality

$$\|u\|_2^2 \le \alpha_f(r)\,\langle f(A)\,u, u\rangle + r\,\Phi(u), \quad r > 0,\ u \in D(f(A)), \tag{19}$$

where

$$\alpha_f(r) = 2\Big/\left[f\left(\frac{1}{2\,\alpha(r/4)}\right)\right].$$

Proof Suppose that (18) holds. As in the proof of Proposition 9 we find that

$$\|u\|_2^2 \le \widetilde{\alpha}(r)\langle f(A)\,u, u\rangle + r\,\Phi(u), \quad r > 0,\ u \in D(f(A)), \tag{20}$$

where

$$\widetilde{\alpha}(r) = \sup_{s>0}\left\{\frac{\Theta^{-1}(s) - r}{s}\right\} \quad \text{and} \quad \Theta(x) = \frac{x}{2}\sup_{s>0} f\left(\frac{1 - 2s/x}{\alpha(s)}\right).$$

If we set $s = x/4$,

$$\Theta(x) \ge \frac{x}{2}\, f\left(\frac{1}{2\,\alpha(x/4)}\right) =: \Theta_0(x),$$

and this gives us

$$\widetilde{\alpha}(r) = \sup_{s>0}\left\{\frac{\Theta_0^{-1}(s) - r}{s}\right\} \le \sup_{s>0,\ \Theta_0^{-1}(s)\ge r} \frac{\Theta_0^{-1}(s)}{s}. \tag{21}$$

According to the definition of $\Theta_0(x)$, we have

$$\frac{\Theta_0^{-1}(x)}{x} = 2\left[f\left(\frac{1}{2\,\alpha(\Theta_0^{-1}(x)/4)}\right)\right]^{-1}.$$

Since α is decreasing,

$$\begin{aligned}\sup_{s>0,\ \Theta_0^{-1}(s)\ge r} \frac{\Theta_0^{-1}(s)}{s} &= \sup_{s>0,\ \Theta_0^{-1}(s)\ge r} 2\left[f\left(\frac{1}{2\,\alpha(\Theta_0^{-1}(s)/4)}\right)\right]^{-1}\\ &\le 2\left[f\left(\frac{1}{2\,\alpha(r/4)}\right)\right]^{-1}.\end{aligned} \tag{22}$$

The required inequality (19) follows from (20), (21) and (22). □

4.3 The converses of Theorem 1 and Propositions 9 and 10

If A is a nonnegative self-adjoint operator, then it is possible to show a converse to the assertions of Theorem 1 and Propositions 9 and 10.

Proposition 11 *Let A be a nonnegative self-adjoint operator on $L^2(X, m)$, and f be some non-degenerate Bernstein function. Let $\Phi : L^2(X, m) \to [0, \infty]$ be a measurable functional satisfying $\Phi(cu) = c^2\Phi(u)$ and $\Phi(T_t u) \le \Phi(u)$ for all $t \ge 0$ and $\Phi(u) = 0$ if, and only if, $u = 0$, where $(T_t)_{t\ge0}$ is the semigroup generated by A. If the following Nash-type inequality*

$$\|u\|_2^2\, f(B(\|u\|_2^2)) \le \langle f(A)\,u, u\rangle, \quad u \in D(f(A)),\ \Phi(u) = 1$$

holds for some increasing function $B : (0, \infty) \to (0, \infty)$, then

$$\|u\|_2^2\, B(\|u\|_2^2) \le \langle A\,u, u\rangle, \quad u \in D(A),\ \Phi(u) = 1.$$

Proof Every non-degenerate (i.e. non-constant) Bernstein function f is strictly increasing and concave. Thus f^{-1} is strictly increasing and convex. Let $(E_\lambda)_{\lambda\geq 0}$ be the spectral resolution of the self-adjoint operator A. Using Jensen's inequality we get for all $u \in D(A)$ with $\|u\|_1 = 1$

$$\begin{aligned} B(\|u\|_2^2) &= f^{-1} \circ f(B(\|u\|_2^2)) \\ &\leq f^{-1}\left(\frac{\langle f(A)u, u\rangle}{\|u\|_2^2}\right) \\ &= f^{-1}\left(\int_{[0,\infty)} f(\lambda)\,\frac{dE_\lambda(u,u)}{\|u\|_2^2}\right) \\ &\leq \int_{[0,\infty)} f^{-1}\circ f(\lambda)\,\frac{dE_\lambda(u,u)}{\|u\|_2^2} \\ &= \frac{\langle Au, u\rangle}{\|u\|_2^2}, \end{aligned}$$

cf. also [2, Proposition 2.3]. □

Using Proposition 11 we can get the converses of Propositions 9 and 10. For example, if the following super-Poincaré inequality

$$\|u\|_2^2 \leq r\,\langle f(A)\,u, u\rangle + \beta_f(r)\,\Phi(u), \quad r>0,\ u\in D(f(A))$$

holds for some decreasing function $\beta_f : (0,\infty) \to (0,\infty)$, then

$$\|u\|_2^2 \leq r\,\langle A\,u, u\rangle + \beta(r)\,\Phi(u), \quad r>0,\ u\in D(A),$$

where

$$\beta(r) = 2\,\beta_f\left(\frac{1}{2\,f(1/r)}\right).$$

4.4 On-diagonal estimates for subordinate semigroups: Nash type inequalities

In this section X is the n-dimensional Euclidean space $\mathbb{R}^n$ equipped with Lebesgue measure $m(dx) = dx$.

Proposition 12 *Assume that* $(A, D(A))$ *satisfies the following Nash-type inequality*

$$\|u\|_2^2\, B(\|u\|_2^2) \leq \langle Au, u\rangle, \quad u\in D(A),\ \|u\|_1 = 1,$$

where $B : (0,\infty)\to(0,\infty)$ *is some increasing function. Then, if for any* $t>0$,

$$\eta(t) := \int_t^\infty \frac{du}{u\, f(B(u))} < \infty,$$

the subordinate semigroup $(T_t^f)_{t\geq 0}$ *has a bounded kernel* $p_t^f(x,y)$ *with respect to Lebesgue measure, and the following on-diagonal estimate holds:*

$$\operatorname*{ess\,sup}_{x,y\in\mathbb{R}^d} p_t^f(x,y) = \|T_t^f\|_{1\to\infty} \leq 2\,\eta^{-1}\left(\frac{t}{2}\right).$$

Proof By Theorem 1 we know that the generator $f(A)$ of the subordinate semigroup $(T_t^f)_{t\ge 0}$ satisfies

$$\frac{\|u\|_2^2}{2}\, f\circ B\left(\frac{\|u\|_2^2}{2}\right)\le \langle f(A)\,u,u\rangle,\quad u\in D(f(A)),\ \|u\|_1=1. \tag{23}$$

Therefore the required assertion follows from [7, Proposition II.2] or [17, Theorem 3.3.17 (1), p. 158]. □

4.5 Contractivity of subordinate semigroups: Super- and Weak Poincaré inequalities

Let (X,m) be a measure space with a σ-finite measure m. Let $(T_t)_{t\ge 0}$ be a semigroup on $L^2(X,m)$ which is bounded on $L^p(X,m)$ for all $p\in[1,\infty]$. This is, e.g., always the case for symmetric sub-Markovian contraction semigroups on $L^2(X,m)$.

Recall that a semigroup $(T_t)_{t\ge 0}$ is said to be *hypercontractive* if $\|T_t\|_{2\to 4}<\infty$ for some $t>0$, *supercontractive* if $\|T_t\|_{2\to 4}<\infty$ for all $t>0$, and *ultracontractive* if $\|T_t\|_{1\to\infty}<\infty$ for all $t>0$. The example below improves [2, Theorem 3.1].

Proposition 13 *Let f be a Bernstein function and $(T_t)_{t\ge 0}$ be an ultracontractive symmetric sub-Markovian semigroup on $L^2(X,m)$ such that for all $t>0$,*

$$\|T_t\|_{1\to\infty}\le \exp(\lambda\, t^{-1/(\delta-1)})$$

for some $\lambda>0$ and $\delta>1$. Then, we have the following statements for the subordinate semigroup $(T_t^f)_{t\ge 0}$:

(i) *If $\int_1^\infty \frac{dr}{f(r^\delta)}<\infty$, then $(T_t^f)_{t\ge 0}$ is ultracontractive.*
(ii) *If $\lim_{r\to\infty}\frac{f^{-1}(\lambda)}{\lambda^\delta}=0$, then $(T_t^f)_{t\ge 0}$ is supercontractive.*
(iii) *If $\lim_{r\to\infty}\frac{f^{-1}(\lambda)}{\lambda^\delta}\in(0,\infty)$, then $(T_t^f)_{t\ge 0}$ is hypercontractive.*
(iv) *If $\lim_{r\to\infty}\frac{f^{-1}(\lambda)}{\lambda^\delta}=\infty$, then $(T_t^f)_{t\ge 0}$ is not hypercontractive.*

Proof Denote by A and $f(A)$ the generators of the semigroups $(T_t)_{t\ge 0}$ and $(T_t^f)_{t\ge 0}$, respectively. By [7, Proposition II. 4] and [17, Proposition 3.3.16, p. 157], we know that the following super-Poincaré inequality holds:

$$\|u\|_2^2\le r\,\langle A\,u,u\rangle+\beta(r)\,\|u\|_1^2,\quad r>0,\ u\in D(A),$$

where

$$\beta(r)=c_1[\exp(c_2\,r^{-1/\delta})-1]$$

for some $c_1,c_2>0$. By Proposition 9,

$$\|u\|_2^2\le r\,\langle f(A)\,u,u\rangle+\beta_f(r)\,\|u\|_1^2,\quad r>0,\ u\in D(f(A)),$$

where

$$\beta_f(r)=4c_1\{\exp[c_3(f^{-1}(2/r))^{1/\delta}]-1\}$$

for some constant $c_3>0$. Therefore, the required assertions follow from [17, Theorem 3.3.14, p. 156 and Theorem 3.3.13, p. 155] and the comment after Proposition 11.

□

We close this section with a result that shows how decay properties are inherited under subordination.

Proposition 14 *Let $(T_t)_{t\geq 0}$ be a symmetric sub-Markovian semigroup on $L^2(X, m)$. Assume that there exist two constants $\delta, c_0 > 0$ such that*

$$\|T_t u\|_2^2 \leq \frac{c_0\,\Phi(u)}{t^\delta} \quad \textit{for all} \;\; t > 0,\;\; u \in L^2(X, m),$$

where $\Phi : L^2(X, m) \to [0, \infty]$ is a functional with $\Phi(cu) = c^2\Phi(u)$ and $\Phi(T_t u) \leq \Phi(u)$ for all $c \in \mathbb{R}$ and $t \geq 0$ and $\Phi(u) = 0$ if, and only if, $u = 0$. If

$$\eta(t) := \int_t^\infty \frac{ds}{s f(s)} < \infty \quad \textit{for all} \;\; t > 0,$$

then there are constants $c_1, c_2 > 0$ such that

$$\|T_t^f u\|_2^2 \leq c_1\, [\eta^{-1}(c_2 t)]^\delta\, \Phi(u).$$

Proof Denote by A and $f(A)$ the generators of $(T_t)_{t\geq 0}$ and $(T_t^f)_{t\geq 0}$, respectively. From [17, Corollary 4.1.8 (1), p. 189; and Corollary 4.1.5 (2), p. 186] we know that the following weak Poincaré inequality holds:

$$\|u\|_2^2 \leq \alpha(r)\, \langle A\, u, u\rangle + r\, \Phi(u), \quad r > 0,\; u \in D(A),$$

where

$$\alpha(r) = c_3\, r^{-1/\delta}$$

for some $c_3 > 0$. Proposition 10 shows that

$$\|u\|_2^2 \leq \alpha_f(r)\, \langle f(A)\, u, u\rangle + r\, \Phi(u), \quad r > 0,\; u \in D(f(A)),$$

where

$$\alpha_f(r) = 2[f(c_4 r^{1/\delta})]^{-1}$$

for some constant $c_4 > 0$. Therefore, the assertion follows from [17, Theorem 4.1.7, p. 188]. □

Note added in Proof After we have finished this paper, Patrick Maheux informed us that he and Ivan Gentil have, independently, obtained similar results in their (at that point still forthcoming) preprint [8]; although our findings partially overlap, the methods used here and in [8] are essentially different.

Acknowledgments The authors thank Professors Alexander Bendikov, Mu-Fa Chen and Feng-Yu Wang for helpful comments on earlier versions of the paper. Financial support through DFG (grant Schi 419/5-2) and DAAD (PPP Kroatien) (for R.L. Schilling), the Alexander-von-Humboldt Foundation and the Programme of Excellent Young Talents in Universities of Fujian (No. JA10058 and JA11051) (for Jian Wang) is gratefully acknowledged. Most of this work was done when Jian Wang was a Humboldt fellow at TU Dresden. He is grateful for the hospitality and the good working conditions.

References

1. Bendikov, A., Coulhon, T., Saloff-Coste, L.: Ultracontractivity and embedding into L^∞. Math. Ann. **337**, 817–853 (2007)
2. Bendikov, A., Maheux, P.: Nash-type inequalities for fractional powers of non-negative self-adjoint operators. Trans. Am. Math. Soc. **359**, 3085–3097 (2007)
3. Beckenbach, E.F., Bellman, R.: Inequalities, Ergebnisse der Mathematik und ihrer Grenzgebiete (Neue Folge), vol. 40. Springer, Berlin (1961)
4. Berg, C., Boyadzhiev, Kh., de Laubenfels, R.: Generation of generators of holomorphic semigroups. J. Aust. Math. Soc. (Ser. A) **55**, 246–269 (1993)
5. Bihari, I.: A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations. Acta Math. Hungarica **7**, 81–94 (1956)
6. Chen, M.-F.: Eigenvalues, Inequalities and Ergodic Theory. Springer, London (2005)
7. Coulhon, T.: Ultracontractivity and Nash-type inequalities. J. Func. Anal. **141**, 510–539 (1996)
8. Gentil, I., Maheux, P.: Nash-type inequalities, super-Poincaré inequalities for subordinated semigroups. Preprint. arXiv: 1105. 3095v1
9. Ôkura, H.: Recurrence and transience criteria for subordinated symmetric Markov processes. Forum Math. **14**, 121–146 (2002)
10. Röckner, M., Wang, F.-Y.: Weak Poincaré inequalities and L^2-convergence rates of Markov semigroups. J. Func. Anal. **185**, 564–603 (2001)
11. Schilling, R.L.: A note on invariant sets. Prob. Math. Stat. **24**, 47–66 (2004)
12. Schilling, R.L., Song, R.M., Vondraček, Z.: Bernstein Functions—Theory and Applications. In: Studies in Mathematics, vol. 37. de Gruyter, Berlin (2010)
13. Walter, W.: Differential and Integral Inequalities, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 55. Springer, Berlin (1970)
14. Wang, F.-Y.: Functional inequalities for empty essential spectrum. J. Funct. Anal. **170**, 219–425 (2000)
15. Wang, F.-Y.: Functional inequalities, semigroup properties and spectrum estimates. Infinite Dimens. Anal. Quant. Prob. Rel. Topics **3**, 263–295 (2000)
16. Wang, F.-Y.: Functional inequalities and spectrum estimates: the infinite measure case. J. Funct. Anal. **194**, 288–310 (2002)
17. Wang, F.-Y.: Functional Inequalities, Markov Semigroups and Spectral Theory. Science Press, Beijing (2005)