ACADIMIC World Academic Union

ISSN 1479-3889 (print), 1479-3897 (online)
International Journal of Nonlinear Science
Vol.3(2007) No.1,pp.68-73

Boundary Control of an Unstable Heat Equation

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(Received 26 November 2006, accepted 7 December 2006)

Abstract: In this paper, a feedback boundary controller for an unstable heat equation is designed. The boundary control law is designed in the form of an integral operator with a known, continuous kernel function which can be interpreted as a back stepping control law. This interpretation provides a Lyapunov function for proving stability of the system. The control is applied by insulating one end of the rod and applying either Dirichlet or Neumann boundary actuation on the other.

Key words: Backstepping; boundary control; distributed parameter systems; unstable heat equation; exponential stabilization

1 Introduction

Boundary control is one kind of distributed parameter controls,which has been emphasized in the control theory and has been extensively studied and developed.Rencently more and more attention has been paid to boundary control of K-S equation[1, 2],Burgers equation[3],KdV-Burgers equation,C-H equation[5, 6],heat equation[7, 8, 9, 10].In [1],the authors studied the boundary control of the K-S equation withan external excitation, mainly using Banach contraction fixed point and semi-group theory to prove the uniqueness and existence of the solution.In [3],according to Galerkin method and suitable performance index $J(y, u)$,the authors presented an optimal controler and proved the existence of solution.In [14],using the numerical methods,the authors showen the multiple solution for the equation $-\Delta u = \lambda f(u)$ with Dirichlet boundary condition in a bounded domain Ω.In[4],A new simple controller is proposed for controlling the Chen's chaotic system.

The heat equation is a typical parabolic equation,which has rich physics background. In recent years,many researchers have tried their best to study the parabolic equation , particularly heat equation .For example,Burns and Rubio (in[7]);A.Shidfar and Karamali[8] studied the Numerical solution of inverse heat equation conduction problem; .Doubova, E.Fernandez-Cara, and Gonzalez-Burgos[9]drew some condition about the controllability of the heat equation with nonlinear boundary Fourier conditions; Dejan M. Boskovic' Miroslav Krstic' and Weijiu Liu[10] analyze the boundary feedback control of heat equation.

In this paper, a problem of temperature stabilization is addressed for a model of a thin rod with not only the loss of heat to a surrounding medium but also the destabilizing heat generation inside the rod. We develop a backstepping [11] control law involving infinitely many steps for a PDE. An inherent danger in applying infinitely many steps of backstepping is that the feedback gains may go to infinity. This is prevented here by choosing the transformed system in a special way ,which not only makes the feedback kernel a continuous function but also a known, closed-form function.

We analyze the most general case when the effects of the heat loss and the heat generation are significant and have to be modeled. In this case the system can have only one constant temperature distribution along

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IJNS.2007.01.15/063

the rod which can be either stable or unstable, depending on whether the heat loss dominates the effect of heat generation or not. In this paper we focus on the "unstable" heat equation (heat generation dominates the heat loss) for which we will be able to design a control law that stabilizes the system.

The control objective is achieved by applying either Dirichlet or Neumann boundary control on one end and insulating the other. In addition, an exact range of the positive parameter λ multiplying the linear term is found. We will construct a nonsingular coordinate transformation to convert the original system into a new set of coordinates where we can design a control law that achieves stabilization using homogeneous boundary control.

Consider the heat equation with boundary condition:

$$\begin{cases} u_t = u_{xx} + \lambda u + u_x & (x,t) \in (0,1) \times (0,\infty) \\ u(0,t) = 0, u_x(0,t) = 0 & t \in (0,\infty) \\ u(x,0) = u^0(x) & x \in (0,1) \end{cases} \tag{1.1}$$

where $\lambda \geq 0$ is a constant parameter, and $u^0(x)$ denotes the initial data. Under the homogeneous Dirichlet boundary condition at $x = 1$ $(u(1,t) = 0, t \in (0,\infty))$, equation (1.1) is unstable when $\lambda > \frac{\pi^2}{4}$ since for $\lambda = 0, \frac{\pi^2}{4}$ is the first eigenvalue of (1.1) with $u(1,t) = 0$ This becomes obvious if we introduce a new variable$v(x,t) = u(x,t)e^{-\lambda t}$. Therefore, a natural question to ask is: Can one find a Dirichlet boundary feedback control law $u(1,t)$ that exponentially stabilizes the system (3.1) if $\lambda > \frac{\pi^2}{4}$.

Using a Lyapunov design, we obtain a Dirichlet boundary feedback law that achieves exponential stability of the following system:

$$\begin{cases} u_t = u_{xx} + \lambda u + u_x & (x,t) \in (0,1) \times (0,\infty) \\ u(0,t) = 0, u_x(0,t) = 0 & t \in (0,\infty) \\ u(1,t) = (-a\tan(a) - \frac{1}{2}) \int_0^1 u(\xi,t)d\xi & t \in (0,\infty) \\ u(x,0) = u^0(x), u_x(x) = 0 & x \in (0,1) \end{cases} \tag{1.2}$$

In this paper,we will study the boundary control and gain the L^2 and H^1exponential stability estimate of system (1.2).

2 Main results and proof

Theorem 2.1 *Assume that* $\lambda \in \left[0, \frac{3}{4}\pi^2\right)$ *, and* $a \in \left(\max\left\{0, sgn\left(\frac{\lambda}{2} - \frac{\pi^2}{8}\right)\sqrt{\left|\frac{\lambda}{2} - \frac{\pi^2}{8}\right|}\right\}, \frac{\pi}{2}\right)$.
i) For arbitrary initial data $u^0(x) \in C(0,1), u_x^0(0) = 0$*, (1.2) has a unique classical solution that satisfies the following* L^2 *exponential stability estimate:*

$$\|u(t)\| \leq M \left\|u^0\right\| e^{-(\frac{\pi^2}{4}+2a^2-\lambda)t} \tag{2.1}$$

where M*is a positive constant independent of* u^0.
ii) For arbitrary initial data $u^0(x) \in H^1(0,1), u_x^0(0) = 0$*, equation (1.2) has a unique strong solution that satisfies the following* H^1 *exponential stability estimate:*

$$\|u(t)\|_{H^1} \leq M \left\|u^0\right\|_{H^1} e^{-(\frac{\pi^2}{4}+2a^2-\lambda)t/2} \tag{2.2}$$

where M *is a positive constant independent of* u^0.

Proof: We first consider the following equation:

$$\begin{cases} \omega_t = \omega_{xx} - c(x)\omega + \omega_x & (x,t) \in (0,1) \times (0,\infty) \\ \omega_x(0,t) = 0, \omega(1,t) = 0 & t \in (0,\infty) \\ \omega(x,0) = \omega^0(x), \omega_x^0(x) = 0 & x \in (0,1) \end{cases} \tag{2.3}$$

with

$$c(x) = -\lambda + 2\frac{a^2}{\cos^2(ax)} \tag{2.4}$$

System (2.3) will be shown exponentially stable by finding an invertible coordinate transformation to transform system (1.2) into system (2.3) if $\min_{0\le x\le 1} c(x) = c(0) = -\lambda + 2a^2 > -\frac{\pi^2}{4}$. We need the following Lemma.

Lemma 2.1 *The coordinate transformation*

$$\omega(x,t) = u(x,t) + (a\tan(ax) + \frac{1}{2})\int_0^x u(\xi,t)d\xi \tag{2.5}$$

with $x \in (0,1)$ *and* $0 < a < \frac{\pi}{2}$, *has an inverse*

$$u(x,t) = \omega(x,t) - \frac{1}{2}\left[a\sin(ax)\int_0^x \frac{\omega(\xi,t)}{\cos(a\xi)}d\xi + \frac{1}{2}e^{-\frac{1}{2}x}\int_0^x \frac{\omega(\xi,t)}{e^{-\frac{1}{2}\xi}}d\xi\right] \tag{2.6}$$

and converts the system (1.2) into (2.3).

Proof: To prove that (2.6) is the inverse of (2.5) we set

$$\omega(x,t) = u(x,t) - \beta(x,t)$$
$$\beta(x,t) = -\left[a\tan(ax) + \frac{1}{2}\right]\int_0^x u(\xi,t)d\xi \tag{2.7}$$

For $x = 0$ we get $\beta(0,t)$since$\omega(0,t) = u(0,t)$. For $0 < x \le 1$,we start by finding $\beta_x(x,t)$ from (2.7), and using the variation of constants formula with $\beta(0,t)$, we have

$$\beta(x,t) = -\frac{1}{2}\left[a\sin(ax)\int_0^x \frac{\omega(\xi,t)}{\cos(ax)}d\xi + \frac{1}{2}e^{-\frac{1}{2}x}\int_0^x \frac{\omega(\xi,t)}{e^{-\frac{1}{2}\xi}}d\xi\right] \tag{2.8}$$

This proves the first part of the lemma. To prove the second part ,let us first put(2.5) into

$$\omega(x,t) = u(x,t) - k(x)\int_0^x u(\xi,t)d\xi$$

where

$$k(x) = -a\tan(ax) - \frac{1}{2} \tag{2.9}$$

We now look for conditions that $k(x)$ and $c(x)$ should satisfy in order to make sure that if $u(x,t)$ satisfies system (1.2), $\omega(x)$ satisfies system (2.3). Taking one time partial derivative of (2.9) with respect to t, two times derivatives with respect to x, and substituting the obtained expressions into (2.3), we have

$$\left[\lambda + 2k'(x) + c(x)\right]u(x,t) + \left[k''(x) - \lambda k(x) - c(x)k(x) + k'(x)\right]\int_0^x u(\xi,t)d\xi = 0 \tag{2.10}$$

Therefore, if $k(x)$ and $c(x)$satisfy

$$\begin{cases} \lambda + 2k'(x) + c(x) = 0 \\ k''(x) - \lambda k(x) + c(x)k(x) + k'(x) = 0 \end{cases} \tag{2.11}$$

the theorem will be proven. By substitution of (2.4) and (2.9) we verify that (2.11) is indeed satisfied. Finally, the boundary condition for $\omega_x(0,t) = 0$ is obtained by differentiating (2.5) and substituting $u_x(0,t) = 0$, while the Dirichlet feedback boundary control $u(1,t) = -[a\tan(a) + \frac{1}{2}]\int_0^1 u(\xi,t)d\xi$ is obtained by substituting $x = 1$ into (2.5) together with the fact that $\omega(1,t) = 0$. This completes the proof.

Before continuing, it is necessary to clarify that system (1.2) is well posed.Since transformation (2.5) is invertible and the problem defined by (2.3) is well posed (see [11]).In addition, by (2.6), there exists a positive constant $\mu > 0$such that$\|u(t)\|_{L^2} \le \mu\|\omega(t)\|_{L^2}$, $\|u(t)\|_{H^1} \le \mu\|u(t)\|_{H^1}$, and by (2.5) there exists a positive constant $\upsilon > 0$ such that $\|\omega(t)\|_{L^2} \le \upsilon \|u(t)\|_{L^2}$, $\|\omega(t)\|_{H^1} \le \upsilon \|\omega(t)\|_{H^1}$. So, it is sufficient to prove (2.1) and (2.2) for the solution ω of (2.3).

Define

$$E(\omega,t) = \frac{1}{2}\int_0^1 \omega(x,t)^2 dx \tag{2.12}$$

with $\omega_x(0) = \omega(1) = 0$,the inequality $(\frac{\pi^2}{4})\|\omega\|^2 \le \|\omega_x\|^2$ holds ($\frac{\pi^2}{4}$ is the smallest eigenvalue of the operator $-\frac{\partial^2}{\partial x^2}$ with the same boundary conditions), since $\min_{0\le x\le 1} c(x) > -\frac{\pi^2}{4}$, we get

$$\dot{E}(\omega,t) \le -2(\frac{\pi^2}{4} + 2a^2 - \lambda)E(\omega,t) \tag{2.13}$$

which implies

$$E(\omega,t) \le E(\omega,0)e^{-2(\frac{\pi^2}{4}+2a^2-\lambda)t}, t \ge 0 \tag{2.14}$$

Set

$$V(t) = \int_0^1 \omega_x(x,t)^2 dx \tag{2.15}$$

Using the definition (2.12), we deduce that (the following $C's$ denote various positive constants that may vary from line to line)

$$\dot{E}(t) + V(t) \le CE(t) \tag{2.16}$$

Multiplying (2.16) by $e^{(\frac{\pi^2}{4}+2a^2-\lambda)t}$ and integrating it from 0 to t gives

$$e^{(\frac{\pi^2}{4}+2a^2-\lambda)t}E(t) + \int_0^t e^{(\frac{\pi^2}{4}+2a^2-\lambda)s}V(s)ds \le CE(0) \tag{2.17}$$

Multiplying the first equation of (2.3) by ω_{xx} and integrating from 0 to 1 by parts we obtain

$$\dot{V}(t) \le -2\int_0^1 \omega_{xx}^2 dx + \int_0^1 \omega_{xx}^2 dx + C\int_0^1 \omega^2 dx \le C\int_0^1 \omega^2 dx \le CE(t) \tag{2.18}$$

which implies that

$$\frac{d}{dt}(V(t)e^{(\frac{\pi^2}{4}+2a^2-\lambda)t}) \le C[E(t)+V(t)]e^{(\frac{\pi^2}{4}+2a^2-\lambda)t} \tag{2.19}$$

Integrating (2.19) from 0 to t, together with (2.14) and (2.17) we get

$$V(t)e^{(\frac{\pi^2}{4}+2a^2-\lambda)t} \le C[V(0)+E(0)] \tag{2.20}$$

which implies that (2.2) holds.

Now we extend the results to Neumann boundary control.

When $\lambda > 0$, and system (1.1)is unstable under the Neumann boundary condition $u_x(1,t) = 0$we propose a Neumann boundary feedback control law such that the following close-loop system is exponentially stable ,when $\alpha > 2$.

$$\begin{cases} u_t = u_{xx} + \lambda u + u_x & (x,t) \in (0,1)\times(0,\infty) \\ u(x,0) = 0, u_x(0,t) = 0 & t \in (0,\infty) \\ u_x(1,t) = -(\alpha + a\tan(a))u(1,t) - (\alpha a\tan(a) + \frac{a^2}{\cos^2 a})\int_0^1 u(\xi,t)d\xi & t \in (0,\infty) \\ u(x,0) = u^0(x), u_x^0(x) = 0 & x \in (0,1) \end{cases} \tag{2.21}$$

Theorem 2.2 *Assume that* $\lambda \in [0, 1+\frac{\pi^2}{2})$,*and* $a \in (\max\{0, sgn(\frac{\lambda}{2}-\frac{1}{2})\sqrt{|\frac{\lambda}{2}-\frac{1}{2}|}\}, \frac{\pi}{2}), \lambda > 2$,
i) For arbitrary initial data $u^0(x) \in C(0,1), u_x^0(0) = 0$,*(2.21) has a unique classical solution that satisfies the following* L^2 *exponential stability estimate:*

$$\|u(t)\| \le M\left\|u^0\right\| e^{-(1+2a^2-\lambda)t} \tag{2.22}$$

where M is a positive constant independent of u^0.
ii) For arbitrary initial data $u^0(x) \in H^1(0,1), u_x^0(0) = 0$, equation (2.21) has a unique strong solution that satisfies the following H^1 exponential stability estimate:

$$\|u(t)\|_{H^1} \leq M \left\|u^0\right\|_{H^1} e^{-(1+2a^2-\lambda)t/2} \tag{2.23}$$

where M is a positive constant independent of u^0.

The proof of Theorem 2.2 is similar to that of Theorem 2.1 We outline only the differences. We consider

$$\begin{cases} \omega_t = \omega_{xx} - c(x)\omega + \omega_x & (x,t) \in (0,1) \times (0,\infty) \\ \omega_x(0,t) = 0, \omega_x(1,t) = -\alpha\omega(1,t) & t \in (0,\infty) \\ \omega(x,0) = \omega^0(x), \omega_x^0(x) = 0 & x \in (0,1) \end{cases} \tag{2.24}$$

We can get (2.24)is exponential stability if $c(x)$ satisfies $\min\limits_{0\leq x\leq 1} c(x) = c(0) > -1$.

Using inequality

$$\int_0^1 \omega(x,t)^2 dx \leq 2\omega(1,t)^2 + 2\int_0^1 (1-x)dx \int_0^1 \omega_x(x,t)^2 dx \leq 2\omega(1,t)^2 + \int_0^1 \omega_x(x,t)^2 dx \tag{2.25}$$

And

$$E(\omega,t) \leq E(\omega,0)e^{-2(1+2a^2-\lambda)t}, t \geq 0 \tag{2.26}$$

We get

$$\dot{E}(\omega,t) \leq (2-\alpha)\omega(1,t)^2 - 2(1+2a^2-\lambda)E(\omega,t) \tag{2.27}$$

And replace $\frac{\pi^2}{4} + 2a^2 - \lambda$ by $1 + 2a^2 - \lambda$and replace $V(t)$ by $V(t) = \alpha\omega(1,t)^2 + \int_0^1 w_x(x,t)^2 dx$, other is same to Theorem 2.1. The theorem is proved.

Acknowledgements

This research was supported by Outstanding Personnel Program in Six Fields of Jiangsu Province(NO.6-A-029) and the Teaching and Research Award Program for Outstanding Young Teachers in higher Education Institutions of MOE,P.R.C.(NO.2002-383)

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