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Research Article

Tail Dependence for Regularly Varying Time Series

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We use tail dependence functions to study tail dependence for regularly varying (RV) time series. First, tail dependence functions about RV time series are deduced through the intensity measure. Then, the relation between the tail dependence function and the intensity measure is established: they are biuniquely determined. Finally, we obtain the expressions of the tail dependence parameters based on the expectation of the RV components of the time series. These expressions are coincided with those obtained by the conditional probability. Some simulation examples are demonstrated to verify the results we established in this paper.

1. Introduction

Copula is a useful tool for handling multivariate distributions with given univariate margins. A copula C is a distribution function, defined on the unit cube $[0,1]^d$, with uniform one-dimensional margins U_i. For any $(u_1,\ldots,u_d) \in [0,1]^d$, $C(u_1,\ldots,u_d) = P\{U_1 \le u_1,\ldots,U_d \le u_d\}$; the survival copula is $\widehat{C}(u_1,\ldots,u_d) = P\{U_1 \ge 1-u_1,\ldots,U_d \ge 1-u_d\}$, the joint survival function of copula C is $\overline{C}(u_1,\ldots,u_d) = \widehat{C}(1-u_1,\ldots,1-u_d)$. Given a copula C, let

$$F(t_1,\ldots,t_d) = C(F_1(t_1),\ldots,F_d(t_d)), \quad \text{where } (t_1,\ldots,t_d) \in \mathbb{R}^d, \tag{1.1}$$

then F is a multivariate distribution with univariate margins $F_1,\ldots,F_d$. On the other hand, given a distribution F with margins $F_1,\ldots,F_d$, there exists a copula C such that (1.1) holds. And copula C is unique if $F_1,\ldots,F_d$ are all continuous (Sklar [1], Nelsen [2]).

In generally, copula forms a natural way to describe the dependence between series when making abstraction of their marginal distributions. Overviews of the probabilistic and statistical properties of copula are to be found in [1–6].

Tail dependence plays an important role among dependence measures due to its ability to describe dependence among extreme values (Frahm et al. [7], Resnick [8, 9], and Nikoloulopoulos et al. [10]) which is introduced by Joe [4]. The issue of tail dependence is mainly for heavy tailed phenomena, heavy tailed phenomenon in fractal time series. It is extensively studied and applied in insurance, risk management, traffic management and engineering management, and so forth. [11–27].

Researchers find various multivariate distributions with heavy tails to describe the extremal or tail dependence, see, Pisarenko and Rodkin [13], Hult and Lindskog [28], and Fang et al. [29]. Many interesting tail quantities have been derived via standard methods: coefficients of tail dependence [30–37] and tail dependence copulas (Charpentier and Segers [38]).

In this paper, we are interested in the tail behavior of the time series $X_1, \ldots, X_t$ which have the form:

$$\mathbf{X} = (X_1, \ldots, X_t) = (RZ_1, \ldots, RZ_t), \tag{1.2}$$

where the scale variable R is independent of random vector $(Z_1, \ldots, Z_t)$. And $\mathbf{X}$ is multivariate regularly varying with distribution function F having copula C.

This distribution is a generalized class, including, for example, multivariate Pareto and multivariate elliptical distribution as special ones. Especially, the multivariate t distribution is included in it. As an example, we will justify the results through multivariate t copula.

In order to analyze the tail dependence behavior of (1.2), we first study the tail dependence functions via intensity measure. Then using the relation between tail dependence parameter and the tail dependence functions, we explore the explicit representations of the tail dependence parameters.

The outline of this paper is as follows. After some preliminaries about multivariate regularly varying series and dependence functions in Section 2, detailed results for the tail dependence functions are discussed in Section 3, the expressions of tail dependence parameters for RV time series are demonstrated in Section 4, and multivariate t distribution is demonstrated as an example in Section 5.

Throughout, $(X_1, \ldots, X_t)$ is a random vector with joint distribution function F and copula C. Minima and maxima will be denoted by $\wedge$ and $\vee$, respectively. The Cartesian product $\prod_{i=1}^{t}[a_i, b_i]$ is denoted by $[\mathbf{a}, \mathbf{b}]$ for any $\mathbf{a}, \mathbf{b} \in \overline{R}^t$.

2. Preliminaries

Definition 2.1. The t-dimensional random vector $\mathbf{X}$ is said to be regularly varying with index $\alpha \geq 0$ if there exists a random vector Θ with values in $\mathbb{S}^{t-1}$ a.s., where $\mathbb{S}^{t-1}$ denotes the unit sphere in R^d with respect to the norm $|\cdot|$, such that, for all $u > 0$,

$$\frac{P\{|\mathbf{X}| > ux, \mathbf{X}/|\mathbf{X}| \in \cdot\}}{P\{|\mathbf{X}| > x\}} \xrightarrow{v} u^{-\alpha} P\{\Theta \in \cdot\}, \tag{2.1}$$

as $x \to \infty$. The symbol $\xrightarrow{v}$ stands for vague convergence on $\mathbb{S}^{t-1}$; vague convergence of measures is treated in detail in Kallenberg [39]. The distribution of Θ is referred to as the spectral measure of $\mathbf{X}$. For further information on multivariate regular variation we refer to Resnick [8, 9].

In fact, (2.1) is equivalent to the following expression

$$nP\left\{a_n^{-1}\mathbf{X} \in \cdot\right\} \xrightarrow{v} \mu(\cdot), \tag{2.2}$$

where μ is an intensity measure or Radon measure on $R/\{0\}$ and a_n is a sequence an of nonnegative numbers.

From the Definition 2.1, we can see that the regularly varying distribution is connected with intensity measure μ. The following lemma yields the explicit relation between them which can be found in [8].

Lemma 2.2. *Let random vector* $\mathbf{X}$ *be regularly varying with index* $\alpha \geq 0$ *and distribution function* F*, then it is equivalent to the following.*

(1) *There exists an intensity measure* μ *on* $R^t/\{0\}$*, such that for every Borel set* $B \subset R^t/\{0\}$ *bounded away from the origin that satisfies* $\mu(\partial B) = 0$*,*

$$\lim_{u\to\infty} \frac{P\{\mathbf{X} \in uB\}}{P\{|\mathbf{X}| > u\}} = \mu(B), \tag{2.3}$$

with the homogeneous condition $\mu(uB) = u^{-\alpha}\mu(B)$.

(2) *There exists an intensity measure* μ *on* $R^t/\{0\}$*, such that*

$$\lim_{u\to\infty} \frac{1 - F(ux_1, ux_2, \ldots, ux_t)}{1 - F(u, u, \ldots, u)} = \frac{P\left\{\mathbf{X}/u \in [0, \mathbf{x}]^c\right\}}{P\left\{\mathbf{X}/u \in [0, 1]^c\right\}} = \mu\left([0, \mathbf{x}]^c\right), \tag{2.4}$$

for all continuous points $\mathbf{x}$ *of* μ*. According to* Lemma 2.2*, one notices that for any nonnegative multivariate regularly varying random vector* $\mathbf{X}$*, its nondegenerate univariate margins* X_i *have regularly varying right tails and with the same index of* $\mathbf{X}$ *also, that is,*

$$\overline{F}_i(x) = P\{\mathbf{X}_i > x\} = x^{-\alpha}L_i(x), \quad x > 0, \tag{2.5}$$

where $L_i(x)$ *is a slowly varying function.*

Lemma 2.3 (Breiman [40]). *Let* ξ *and* η *be two independent nonnegative random variables,* η *be regularly varying with index* α*. If there exists a* $\gamma > \alpha$*, such that* $E\xi^\gamma < \infty$*, then*

$$P\{\xi\eta > x\} \sim E(\xi^\alpha)P\{\eta > x\}. \tag{2.6}$$

The multivariate version of the Lemma belongs to Basrak et al. [41]. It is said that, if $\mathbf{X}$ *is regularly varying in the sense of* (2.2), $\mathbf{A}$ *is a random* $t \times t$ *matrix, independent of* $\mathbf{X}$, *with* $0 < E\|\mathbf{A}\|^{\gamma} < \infty$ *for some* $\gamma > \alpha$, *then*

$$nP\left\{a_n^{-1}\mathbf{A}X \in \cdot\right\} \xrightarrow{v} \tilde{\mu}(\cdot) := E\left(\mu \circ \mathbf{A}^{-1}(\cdot)\right), \tag{2.7}$$

where $\xrightarrow{v}$ *denotes vague convergence on* $\mathbb{R}^t/\{0\}$.

Definition 2.4 (Kluppelberg et al. [42]). Let F be the distribution function of random vector $\mathbf{X}$ with continuous margins $F_i, 1 \le i \le t$ and copula C. For any $\mathbf{w} = (w_1, w_2, \dots, w_t) \in R_+^t$, the lower dependence function is defined as

$$l(\mathbf{w}; C) = \lim_{x \to 0^+} \frac{C(xw_1, xw_2, \dots, xw_t)}{x}, \tag{2.8}$$

and the upper dependence function is defined as

$$u(\mathbf{w}; C) = \lim_{x \to 0^+} \frac{\overline{C}(1 - xw_1, 1 - xw_2, \dots, 1 - xw_t)}{x}. \tag{2.9}$$

The upper exponent function is defined as

$$u^*(\mathbf{w}; C) = \sum_{\emptyset \neq S \subset I} (-1)^{|S|-1} u_S(w_S; C_S), \tag{2.10}$$

where $u_S(w_S; C_S) = \lim_{x \to 0^+} \overline{C}(1 - xw_j, \forall j \in S)/x$.

From the definition, we can verify the elementary properties listed in Proposition 2.5 of the tail dependence function. We denote $\tau_J = \lim_{x \to 1^-} P\{F_j(X_j) > x, \forall j \notin \mathbf{J} \mid F_i(X_i) > x, \forall i \in \mathbf{J}\}$ and $\xi_J = \lim_{x \to 0^+} P\{F_j(X_j) < x, \forall j \notin \mathbf{J} \mid F_i(X_i) < x, \forall i \in \mathbf{J}\}$ are the upper tail and lower dependence parameters of $\mathbf{X}$, respectively, where J is a nonempty subset of $I = \{1, \dots, t\}$. C_J is the margin of C with component indexes in J.

Proposition 2.5. (1) *For any* $1 \le i, j \le t$,

$$\tau_{ij} = u(1, 1; C_{ij}); \qquad \xi_{ij} = l(1, 1; C_{ij}), \tag{2.11}$$

where C_{ij} *is the margin copula of* X_i, X_j.

(2) *For any nonempty* $J \subset I$,

$$\tau_J = \frac{u(1, 1, \dots, 1; C)}{u(1, 1, \dots, 1; C_J)}; \qquad \xi_J = \frac{l(1, 1, \dots, 1; C)}{l(1, 1, \dots, 1; C_J)}, \tag{2.12}$$

(3)

$$u(\mathbf{w};C)=\lim_{x\to 0^+}\frac{\widehat{C}(xw_1,xw_2,\ldots,xw_t)}{x}=l\left(\mathbf{w};\widehat{C}\right). \tag{2.13}$$

Proof. (1) According to the definition of τ_{ij}, we get

$$\begin{aligned}\tau_{ij}&=\lim_{x\to 1^-}P\left\{F_j\left(X_j\right)>x\mid F_i(X_i)>x\right\}=\lim_{x\to 0^+}\frac{P\left\{F_j\left(X_j\right)>1-x,F_i(X_i)>1-x\right\}}{P\{F_i(X_i)>1-x\}}\\&=\lim_{x\to 0^+}\frac{\overline{C}_{ij}(1-x,1-x)}{x}=u\left(1,1;C_{ij}\right);\end{aligned} \tag{2.14}$$

similarly,

$$\begin{aligned}\xi_{ij}&=\lim_{x\to 0^+}P\left\{F_j\left(X_j\right)<x\mid F_i(X_i)<x\right\}=\lim_{x\to 0^+}\frac{P\left\{F_j\left(X_j\right)<x,F_i(X_i)<x\right\}}{P\{F_i(X_i)<x\}}\\&=\lim_{x\to 0^+}\frac{C_{ij}(x,x)}{x}=l\left(1,1;C_{ij}\right).\end{aligned} \tag{2.15}$$

(2) Note that

$$\tau_J=\lim_{x\to 1^-}\frac{P\left\{F_j\left(X_j\right)>x,\forall j\in I\right\}}{P\{F_i(X_i)>x,\forall i\in \mathbf{J}\}}=\lim_{x\to 0^+}\frac{\overline{C}(1-x,\ldots,1-x)/x}{\overline{C_J}(1-x,\ldots,1-x)/x} \tag{2.16}$$

combined with (2.9), the first part is determined. The second part can be verified similarly.

(3) We can obtained the proof only paying attention to $\overline{C}(u_1,\ldots,u_t)=\widehat{C}(1-u_1,\ldots,1-u_t)$.

From the proposition, the upper tail dependence function of copula C is the lower one of its survival copula $\widehat{C}$. And in most fractal time series, from the point of view of either theory or applications, people only need to understand the right tail of the data, so we focus on the upper tail function $u(\mathbf{w};C)$ and coefficient τ_J in the following.

We first study the upper tail dependence function of multivariate regularly varying time series in (1.2) using the intensity measure. □

3. The Upper Tail Dependence Function for RV Time Series

Theorem 3.1. *Let $\mathbf{X}_1,\ldots,X_t$ be RV time series with regularly varying index α, distribution function F, copula C, and the stochastic representation as* (1.2)*. If the margins are tail equivalent as $x\to\infty$, then the upper tail dependence function can be written as*

$$u(\mathbf{w};C)=\frac{\mu\left(\prod_{i=1}^{t}\left[w_i^{-1/\alpha},\infty\right]\right)}{\mu\left([1,\infty]\times\overline{R}^{t-1}\right)}, \tag{3.1}$$

and the upper exponent function can be written as

$$u^*(\mathbf{w};C)=\frac{\mu\left(\left(\prod_{i=1}^{t}\left[0,w_i^{-1/\alpha}\right]\right)^c\right)}{\mu\left(\left([0,1]\times\overline{R}^{t-1}\right)^c\right)}. \tag{3.2}$$

Proof. For any $\mathbf{w}=(w_1,\ldots,w_t)\in R_+^t$,

$$u(\mathbf{w};C)=\lim_{x\to 0^+}\frac{P\left\{\overline{F}_i(X_i)\le xw_i,\forall i\in I\right\}}{P\left\{\overline{F}_1(X_1)\le x\right\}}=\lim_{x\to 0^+}\frac{P\left\{X_i>\overline{\mathbf{F}}_i^{-1}(xw_i),\forall i\in I\right\}}{P\left\{X_1>\overline{\mathbf{F}}_1^{-1}(x)\right\}}. \tag{3.3}$$

Since every margin F_i is regularly varying with the same index α, we obtain that

$$\overline{F}_i(y)=\frac{L_i(y)}{y^\alpha},\quad y>0, \tag{3.4}$$

where $L_i(y)$ is slowing varying function. So for any $w_i>0$, as $x\to 0^+$,

$$\overline{F}_i\left(w_i^{1/\alpha}y\right)=\frac{L_i\left(w_i^{1/\alpha}y\right)}{w_iy^\alpha}=\frac{1}{w_i}\cdot\frac{L_i\left(w_i^{1/\alpha}y\right)}{L_i(y)}\cdot\frac{L_i(y)}{y^\alpha}=\frac{1}{w_i}\cdot h_i(w_i,y)\cdot\overline{F}_i(y), \tag{3.5}$$

where $h_i(w_i,y)=L_i(w_i^{1/\alpha}y)/L_i(y)\to 1$ as $y\to\infty$. So the equation becomes

$$\overline{F}_i\left(w_i^{1/\alpha}y\right)=\frac{1}{w_i}\cdot h_i(w_i,y)\cdot\overline{F}_i(y), \tag{3.6}$$

in other words,

$$w_i^{1/\alpha}y=\overline{\mathbf{F}}_i^{-1}\left(\frac{1}{w_i}\overline{F}_i(y)h_i(w_i,y)\right). \tag{3.7}$$

Now we let $\overline{F}_i(y)=xw_i$, then

$$w_i^{1/\alpha}\overline{\mathbf{F}}_i^{-1}(xw_i)=\overline{\mathbf{F}}_i^{-1}\left(xh_i\left(w_i,\overline{\mathbf{F}}_i^{-1}(xw_i)\right)\right), \tag{3.8}$$

so, $\overline{\mathbf{F}}_i^{-1}(xw_i)=w_i^{-1/\alpha}\overline{\mathbf{F}}_i^{-1}(xh_i(w_i,\overline{\mathbf{F}}_i^{-1}(xw_i)))$.

As $x\to 0^+$, $h_i(w_i,\overline{\mathbf{F}}_i^{-1}(xw_i))\to 1$, so we get that

$$\overline{\mathbf{F}}_i^{-1}(xw_i)\approx w_i^{-1/\alpha}\overline{\mathbf{F}}_i^{-1}(x). \tag{3.9}$$

And since the margins are equivalent, that is, $\overline{F}_i(y)/\overline{F}_1(y) \to 1$ as $y \to \infty$. We have $\overline{\mathbf{F}}_i^{-1}(x)/\overline{\mathbf{F}}_1^{-1}(x) \to 1$ as $x \to 0^+$ (Resnick [8]). So for sufficient small x, $\overline{\mathbf{F}}_i^{-1}(x) \approx \overline{\mathbf{F}}_1^{-1}(x)$, and $z = \overline{\mathbf{F}}_1^{-1}(x)$, combining (3.3) and (2.3), we obtain that

$$\begin{aligned} u(\mathbf{w};C) &= \lim_{x\to 0^+} \frac{P\left\{X_i > w_i^{-1/\alpha}\overline{\mathbf{F}}_i^{-1}(x), \forall i \in I\right\}}{P\left\{X_1 > \overline{\mathbf{F}}_1^{-1}(x)\right\}} = \lim_{z\to\infty} \frac{P\left\{X_i > w_i^{-1/\alpha}z, \forall i \in I\right\}}{P\{X_1 > z\}} \\ &= \frac{\mu\left(\prod_{i=1}^{t}\left[w_i^{-1/\alpha}, \infty\right]\right)}{\mu\left([1,\infty] \times \overline{R}^{t-1}\right)}. \end{aligned} \tag{3.10}$$

In order to calculate $u^*(\mathbf{w};C)$, we recall *the inclusion-exclusion formula*, it says that

$$P\{\cap_{i\in I} A_i\} = \sum_{\emptyset \neq S \subset I} (-1)^{|S|-1} P\{\cup_{j\in S} A_j\} \tag{3.11}$$

is valid for any finite set I and arbitrary events A_i, where $i \in I$.

Using this formula, (2.10) becomes

$$\begin{aligned} u^*(\mathbf{w};C) &= \lim_{x\to 0^+} \frac{P\{F_j(X_j) > 1 - xw_j, \exists j \in I\}}{x} = \lim_{x\to 0^+} \frac{P\left\{\overline{F}_j(X_j) \le xw_j, \exists j \in I\right\}}{P\left\{\overline{F}_1(X_1) \le x\right\}} \\ &= \lim_{x\to 0^+} \frac{P\left\{X_j > \overline{\mathbf{F}}_j^{-1}(xw_j), \exists j \in I\right\}}{P\left\{X_1 > \overline{\mathbf{F}}_1^{-1}(x)\right\}}. \end{aligned} \tag{3.12}$$

By using the same method of (3.3), the following equation holds:

$$u^*(\mathbf{w};C) = \lim_{z\to\infty} \frac{P\left\{X_i > w_i^{-1/\alpha}z, \exists i \in I\right\}}{P\{X_1 > z\}} = \frac{\mu\left(\left(\prod_{i=1}^{t}\left[0, w_i^{-1/\alpha}\right]\right)^c\right)}{\mu\left([1,\infty] \times \overline{R}^{t-1}\right)}. \tag{3.13}$$

□

Corollary 3.2. *Under the same conditions as Theorem 3.1, the following result holds*

$$\mu\left([1,\infty] \times \overline{R}^{t-1}\right) = \frac{1}{u^*(1,\ldots,1;C)}. \tag{3.14}$$

Proof. By (2.4), one can see that $\mu([0,1]^c) = 1$. So we can get the result immediately by letting all $w_i = 1,\ 1 \le i \le t$ in (3.2).

According to Theorem 3.1 and Corollary 3.2, we can represent the intensity measure through the tail dependence function as the following Corollary. □

Corollary 3.3. *Under the same conditions as Theorem 3.1, one has*

$$\mu([\mathbf{w},\infty]) = \frac{u(w_1^{-\alpha},\ldots,w_t^{-\alpha};C)}{u^*(1,\ldots,1;C)},$$
$$\mu([0,\mathbf{w}]^c) = \frac{u^*(w_1^{-\alpha},\ldots,w_t^{-\alpha};C)}{u^*(1,\ldots,1;C)}. \tag{3.15}$$

4. The Upper Tail Dependence Parameters for Regularly Varying Time Series

According to Proposition 2.5 and Theorem 3.1, we can express the tail dependence parameters by their tail dependence functions. In this section, we will deduce the upper tail dependence parameters of time series with multivariate varying distribution in (1.2) by this method. Hereafter, we let μ be the intensity measure of $\mathbf{R} = (R,R,\ldots,R)$ with copula C^R. Where R is regularly varying at ∞ with index α, with survival function $\overline{F}_R(r) = L(r)/r^{\alpha}$, and $L(\cdot)$ is a slowly varying function. So for any nonnegative vector $\mathbf{w} = (w_1,\ldots,w_t)$, we have

$$\mu\left([\mathbf{0},\mathbf{w}]^c;C^R\right) = \lim_{r\to\infty}\frac{P\{R > r\wedge_{i=1}^t w_i\}}{P\{R>r\}} = \lim_{r\to\infty}\frac{\overline{F}_R\left(r\bigwedge_{i=1}^t w_i\right)}{\overline{F}_R(r)}, \tag{4.1}$$

by inserting $\overline{F}_R(r\wedge_{i=1}^t w_i) = L(r\bigwedge_{i=1}^t w_i)/(r\bigwedge_{i=1}^t w_i)^{\alpha}$ and $\overline{F}_R(r) = L(r)/r^{\alpha}$ into the representation, then,

$$\mu\left([\mathbf{0},\mathbf{w}]^c;C^R\right) = \frac{1}{\left(\bigwedge_{i=1}^t w_i\right)^{\alpha}} = \left(\bigvee_{i=1}^t w_i\right)^{-\alpha}. \tag{4.2}$$

Similarly, we have,

$$\mu\left([\mathbf{w},\infty];C^R\right) = \bigwedge_{i=1}^t w_i^{-\alpha}. \tag{4.3}$$

Consequently, we get the main result as follows.

Theorem 4.1. *Let $X_1,\ldots,X_t$ be regularly varying time series with the same regularly varying index α and the stochastic representation given in* (1.2)*, the margins are tail equivalent as $x\to\infty$. If there exists a $\gamma > \alpha$ holds for $0 < E(Z_{i+}^{\gamma}) < \infty$, then the upper tail dependence parameter of $X_1,\ldots,X_t$ is*

$$\tau_J = \frac{E\left(\bigwedge_{i=1}^t (Z_{i+}^{\alpha}/E(Z_{i+}^{\alpha}))\right)}{E\left(\bigwedge_{i\in J}(Z_{i+}^{\alpha}/E(Z_{i+}^{\alpha}))\right)}. \tag{4.4}$$

Proof. We first calculate the tail dependence function of $\mathbf{X} = (RZ_1, \ldots, RZ_t)$. In the following, let C^X and C^Y be the copula of $\mathbf{X}$ and $\mathbf{Y}$, respectively. Denote

$$(Y_1, \ldots, Y_t)^T = \mathbf{A}(R, \ldots, R)^T, \tag{4.5}$$

where

$$\mathbf{A} = \operatorname{diag}\left(\frac{Z_{1+}}{\left(E\left(Z_{1+}^{\alpha}\right)\right)^{1/\alpha}}, \ldots, \frac{Z_{t+}}{\left(E\left(Z_{1+}^{\alpha}\right)\right)^{1/\alpha}}\right). \tag{4.6}$$

Note that $Y_i = (Z_{i+}/(E(Z_{i+}^{\alpha}))^{1/\alpha})R$ is strictly increasing transformation of $X_i > 0$, for all $i \in I$, and the tail dependence function and the parameter are all copula properties. Hence $\mathbf{Y}$ and $\mathbf{X}$ have the same tail dependence functions. By Lemma 2.3, one can see that the marginal variables Y_i of vector $\mathbf{Y}$ are tail equivalent and regularly varying with the same index as $\mathbf{X}$ as $x \to \infty$. Denote the intensity measures of $\mathbf{Y}$ and $\mathbf{R}$ by $\tilde{\mu}(\cdot)$ and $\mu(\cdot)$, respectively. According to (2.7),

$$\tilde{\mu}(\cdot) = E\left(\mu\left(\mathbf{A}^{-1}\cdot\right)\right). \tag{4.7}$$

Now by (4.6), we see that,

$$\mathbf{A}^{-1} = \operatorname{diag}\left(\frac{\left(E\left(Z_{1+}^{\alpha}\right)\right)^{1/\alpha}}{Z_{1+}}, \ldots, \frac{\left(E\left(Z_{t+}^{\alpha}\right)\right)^{1/\alpha}}{Z_{t+}}\right), \tag{4.8}$$

combining this with (4.3), for any nonnegative $\mathbf{w}$, we obtain the intensity measure given by

$$\begin{aligned}\tilde{\mu}([\mathbf{w}, \infty]) = E\left(\mu\left(\mathbf{A}^{-1}[\mathbf{w}, \infty]\right)\right) &= E\left(\mu\left(\prod_{i=1}^{t}\left[\frac{\left(E\left(Z_{i+}^{\alpha}\right)\right)^{1/\alpha}}{Z_{i+}} w_i, \infty\right]; C^R\right)\right) \\ &= E\left(\bigwedge_{i=1}^{t}\frac{Z_{i+}^{\alpha}}{E\left(Z_{i+}^{\alpha}\right)} w_i^{-\alpha}\right).\end{aligned} \tag{4.9}$$

Hence, we have

$$\tilde{\mu}\left(\prod_{i=1}^{t}\left[\mathbf{w}_i^{-\alpha}, \infty\right]\right) = E\left(\bigwedge_{i=1}^{t}\frac{Z_{i+}^{\alpha}}{E\left(Z_{i+}^{\alpha}\right)} w_i\right). \tag{4.10}$$

Substituting this measure into (3.1), we get the upper tail dependence function of vector $\mathbf{Y}$ as follows:

$$u\left(\mathbf{w}; C^Y\right) = E\left(\bigwedge_{i=1}^{t}\frac{Z_{i+}^{\alpha}}{E\left(Z_{i+}^{\alpha}\right)} w_i\right). \tag{4.11}$$

Since $\mathbf{Y}$ and $\mathbf{X}$ have the same tail dependence functions, we have

$$u\left(\mathbf{w}; C^X\right) = E\left(\bigwedge_{i=1}^{t} \frac{Z_{i+}^{\alpha}}{E(Z_{i+}^{\alpha})} w_i\right). \tag{4.12}$$

By (2) in Proposition 2.5, we obtain the upper tail dependence parameters of vector $\mathbf{X}$. □

5. Examples

Let $\mathbf{Z}$ in (1.2) be $\mathbf{Z} = A(U_1, \ldots, U_n)$, where A is a $t \times n$ matrix with $AA^T = \Sigma$, and Σ is a $t \times t$ semidefinite matrix, $U = (U_1, \ldots, U_n)$ is uniformly distributed on the unit sphere (with respect to Euclidean distance) in R^n. We know that $\mathbf{X}$ conforms to an elliptical contoured distribution (Fang et al. [43]). The tail dependence of the elliptical contoured distribution has been discussed in Schmidt [33]. Here we select the t distribution to display our results in Theorem 4.1 as a special case.

If $\mathbf{X} \sim \mathbb{t}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$, then $\mathbf{X}$ has the stochastic representation ([43]):

$$\mathbf{X} = \boldsymbol{\mu} + \frac{\sqrt{\nu}}{\sqrt{S}} \mathbf{Z}, \tag{5.1}$$

where $S \sim \chi_\nu^2$ and $\mathbf{Z} \sim \mathbb{N}_n(\mathbf{0}, \boldsymbol{\Sigma})$ are independent, $\boldsymbol{\mu} \in R^n$.

Let $R = \sqrt{\nu/S}$. Then $R^2 \sim IG(\nu/2, \nu/2)$ and R is regularly varying with index ν at ∞. So the vector $(X_1, \ldots, X_n)$ is regularly varying according to Schmidt [33].

For the upper tail dependence that only relies on the tail behavior of the random vector, we can focus, without loss of generality, on the random vector $\mathbf{X}$ with zero mean vector. Furthermore, since the strictly increasing transformation of $(X_1, \ldots, X_n)$ does not change the copula, $\Delta^{-1/2}\mathbf{X}$ has the same copula as $\mathbf{X}$, where $\boldsymbol{\Sigma} = (\sigma_{ij})$ and $\Delta = \operatorname{diag}(\sigma_{11}, \sigma_{22}, \ldots, \sigma_{nn})$. Thus $\Delta^{-1/2}\mathbf{X} \sim \mathbb{t}_n(0, \Delta^{-1/2}\boldsymbol{\Sigma}\Delta^{-1/2}, \nu)$. It is evident that $\Delta^{-1/2}\boldsymbol{\Sigma}\Delta^{-1/2}$ becomes the correlation matrix of the random vector. Consequently, we may assume that the covariance matrix $\boldsymbol{\Sigma}$ is the correlation matrix. In this situation, all $Z_i's$ have the same margins as $N(0,1)$. So $E(Z_{i+}^{\nu})$ are all equal for any $1 \le i \le n$. Under these assumptions, using (4.4), we get the upper tail dependence parameter of $\mathbf{t}_n(0, \boldsymbol{\Sigma}, \nu)$ as

$$\tau_J = \frac{E\left(\bigwedge_{i=1}^{n} Z_{i+}^{\nu}\right)}{E\left(\bigwedge_{i\in J} Z_{i+}^{\nu}\right)}. \tag{5.2}$$

This is coincided to the one obtained in Shi and Lin [34].

6. Simulations

In Section 4, we obtain the expressions of the tail dependence indexes about RV time series in (1.2). In Section 5, we display our result in the multivariate t distribution as example. In this Section, we will illustrate these results by some Monte Carlo simulated numerical examples.

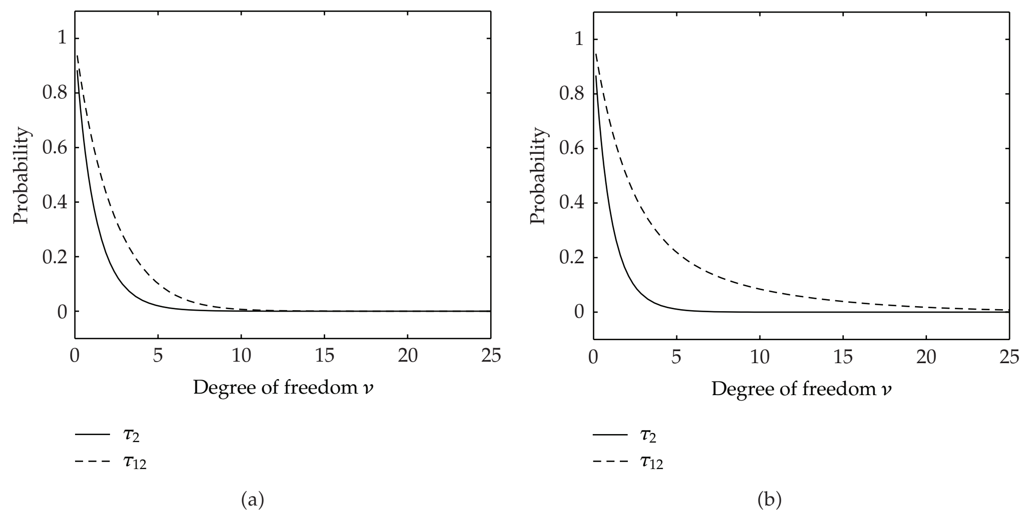

Figure 1: The estimation of τ_2, τ_{12} under AR(1) (the left one) and EX (the right one) correlation structure.

Given that $y^{(1)}, y^{(2)}, \ldots, y^{(m)}$ be generated from the multivariate normal distribution $\mathbf{N}_n(\mathbf{0}, \boldsymbol{\rho})$, then the upper tail dependence indices of $\mathbf{t}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ can be estimated by

$$\widehat{\tau}_J = \frac{\sum_{k=1}^{m} \left(\bigwedge_{j=1}^{n} \left|y_j^{(k)}\right|\right)^{\nu} I\{y^{(k)} > 0 \text{ or } y^{(k)} < 0\}}{\sum_{k=1}^{m} \left(\bigwedge_{i\in J} \left|y_i^{(k)}\right|\right)^{\nu} I\{y^{(k)} > 0 \text{ or } y^{(k)} < 0\}}. \tag{6.1}$$

We estimate the upper tail dependence parameter of 3-dimensional t distribution under autoregressive of order 1 (AR(1)), exchangeable(EX), Toeplitz(TOEP), and unstructured(UN) correlation structure, respectively. For each correlation matrix, we first generate 80,000 pseudorandom vectors, then use (5.2) to estimate tail dependence parameter for different ν. Specifically, we do the following simulations.

$$\sum_1 = \begin{pmatrix} 1 & -0.3 & 0.09 \\ -0.3 & 1 & -0.3 \\ 0.09 & -0.3 & 1 \end{pmatrix}, \qquad \sum_2 = \begin{pmatrix} 1 & -0.3 & -0.3 \\ -0.3 & 1 & -0.3 \\ -0.3 & -0.3 & 1 \end{pmatrix}. \tag{6.2}$$

Let $\mathbf{J} = \{2\}$ and $\{1,2\}$, respectively. The corresponding upper tail dependence parameters are denoted by τ_2 and τ_{12}. $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ are under AR(1) and EX correlation structure, respectively, the simulated values of τ_2, τ_{12} about different ν are computed and plotted in Figure 1. $\boldsymbol{\Sigma}_3$ and $\boldsymbol{\Sigma}_4$ are under TOEP and UN correlation structure, the corresponding results are demonstrated in Figure 2.

From the two figures, in spite of the correlation structure, τ_J decreased and approached 0 quickly as ν increased to ∞, which is the tail dependence index for multivariate normal copula.

Many researchers try to discuss the monotonicity of the tail dependence parameter about the regular varying index. Embrechts et al. [11] proved that the tail dependence of the bivariate t distribution is decreasing about the regular varying index ν, and demonstrated

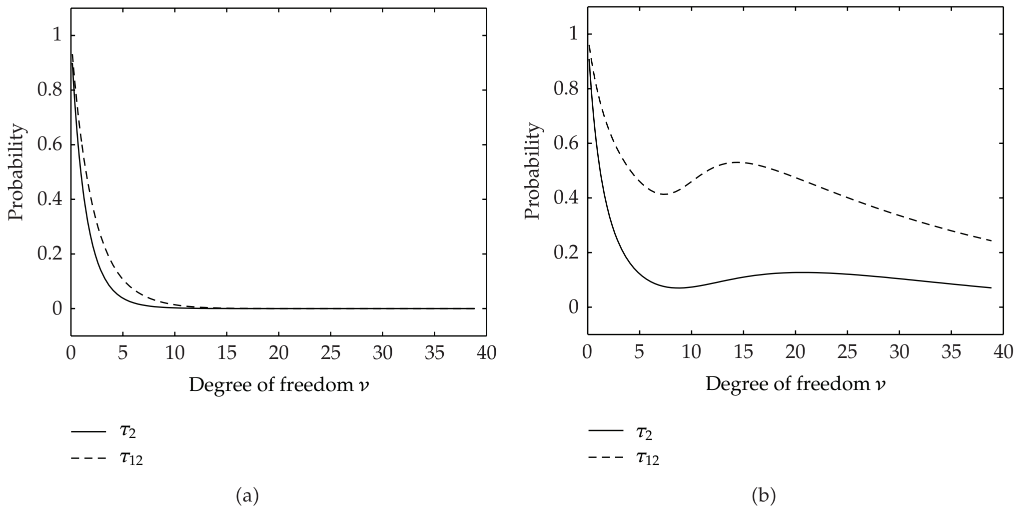

Figure 2: The estimation of τ_2, τ_{12} under TOEP (the left one) and UN (the right one) correlation structure.

that the tail dependence parameter τ_1 is decreasing in ν by numerical results. But From the right graph in Figure 2., these conclusions are not always correct when $t \geq 3$.

$$\sum_3 = \begin{pmatrix} 1 & -0.3 & 0.5 \\ -0.3 & 1 & -0.3 \\ 0.5 & -0.3 & 1 \end{pmatrix}, \qquad \sum_4 = \begin{pmatrix} 1 & 0.3 & 0.5 \\ 0.3 & 1 & 0.7 \\ 0.5 & 0.7 & 1 \end{pmatrix}. \tag{6.3}$$

7. Conclusion

In the paper, we mainly study tail dependence of RV time series in (1.2). We use tail dependence function and intensity measure to express tail dependence parameters. Using tail dependence function, we do not need to consider the explicit representation of the copula. We first discuss the tail dependence function of the RV time series due to the propositions of the regularly varying function, connecting the biuniquely determined property between the tail dependence function and the intensity measure. Then we calculate the explicit formula of the upper tail dependence parameter about the RV time series under some conditions. In fact, we can obtain the extreme upper tail dependence index (Shi and Lin [34]) very similarly to Theorem 4.1, for concise, we omit it here.

Copula of continuous variables is invariant under strictly increasing transformation (Nelsen [2]). In order to obtain the tail dependence function of random vector $\mathbf{X}$, we shift to solve that of $\mathbf{Y}$ in (4.5), which is just a strictly increasing transformation of $\mathbf{X}$.

At last, we select the t distribution as a special case to display our result, they are coincided to the one given in [34]. The monotonicity of the tail dependence parameters about the regular varying index is still an open problem. Under what constraints the tail dependence parameters will be deceasing in the variation index? We are still interested in the problem. We will discuss it in the following work in details. In engineering application, when we confront fractal time series and seasonal data, we can model the tail dependence property via the tail dependence function if the data is consistent with the constraint conditions in our work.

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