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Transient Characteristics of Simple Systems to Modulated Random Noise

Discussed are the mean-square response exceedance characteristics of a single-tuned system to amplitude modulated noise. The results bear on the accuracy of spectral estimates of nonstationary data, and subsequently, relate directly to the design, analysis, and testing of structural systems in environments as gusts, earthquakes, and ignition transients. For noise correlated as an exponentially damped cosine, the nonstationary response may exceed its stationary value by a factor in excess of two. A time-varying shaping filter explanation is offered for this behavior. For white noise, such exceedances do not occur.

Introduction

Transient response properties of unimodal systems to modulated random noise are fundamental to understanding and subsequently solving statistical problems common not only to structural design, but to data processing as well. Such results are applicable directly to response predictions in nonstationary environments as gusts, earthquakes, and ignition transients. Since this work relates to the filtered output of a "weighted" signal or, correspondingly, the results of a Fourier analysis of a weighted data stream, our results are relevant in a more general sense to methods of time series analysis.

The form of the input considered is amplitude modulated random noise. For a more rapidly varying, nonrepetitive modulation function of limited time duration, the resultant noise frequently is called a random shock pulse. The system is a mechanical single-degree-of-freedom system or, equivalently, a single-tuned bandpass filter. Interest here focuses on the time-varying character of the system mean-square response.

The basic formulation for this class of problems has been established previously by the authors [1][1] and others [3–7], and much of the mathematics carried out for both white noise and noise correlated in the form of a damped harmonic. A leading result was that the output response of the system for the correlated noise may exceed its stationary value. Except for two studies of somewhat specialized interests [2, 6], explanations for this behavior and plots to predict its occurrence have not been advanced. It is the intent of this paper to fill these voids.

Problem Definition

We seek the variance of $y(t)$ given the equation of motion

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = \frac{1}{m} f(t) \qquad (1)$$

where the input excitation is the modulated noise expression

$$f(t) = e(t)n(t) \qquad (2)$$

with $n(t)$ assumed Gaussian with a mean value of zero. For conciseness, we concentrate only on the modulation function

$$e(t) = u(t) \qquad (3)$$

and the two noise correlation functions

$$\begin{aligned} R_n(\tau) &= 2\pi R_0 \delta(\tau) \\ R_n(\tau) &= R_0 e^{-\alpha|\tau|} \cos \rho\tau \end{aligned} \qquad (4)$$

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[1] Numbers in brackets designate References at end of paper.

where $u(t)$ denotes the unit step function and τ is the time lag of the autocorrelation function. The delta function identifies white noise and the exponentially decaying harmonic expression defines the correlated noise.

Solution Formulations

Procedures frequently used to compute the expectation $E[y^2(t)]$ for a modulated noise input are founded upon some form of either unit impulse formulations or/and spectral formulations. We concentrate upon the latter. Since the detailed mathematics are outlined elsewhere [1, 2],[2] we omit many intermediate steps and quote expressions essential to an understanding of our solution. However, for continuity and completeness in this discussion, expressions previously reported may be repeated here. Implicit in what follows are the assumptions of system linearity, and separability of the input excitation with $e(t)$ a real function.

The desired mean-square response is given by

$$\sigma_y^2(t) = E[y^2(t)] = \int_{-\infty}^{\infty} S_y(t, \omega)d\omega \tag{5}$$

where the time-varying spectral density is of the form

$$S_y(t, \omega) = S_n(\omega)|I(t, \omega)|^2 \tag{6}$$

and

$$I(t, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\bar{\omega})F_e(\bar{\omega} - \omega)e^{i\bar{\omega}t}d\bar{\omega} \tag{7}$$

with

$$H(\omega) = \frac{1}{m\omega_n^2}\cdot\frac{1}{1 - \dfrac{\omega^2}{\omega_n^2} + i2\zeta\dfrac{\omega}{\omega_n}}$$
$$F_e(\bar{\omega} - \omega) = \int_{-\infty}^{\infty} e(t)e^{-i(\bar{\omega}-\omega)t}dt \tag{8}$$

We make use of the fact that

$$h(t) \leftrightarrow H(\omega) \tag{9}$$

where

$$h(t) = \frac{1}{am}e^{-bt}\sin at \tag{10}$$

and

$$a = \omega_n[1 - \zeta^2]^{1/2}$$
$$b = \zeta\omega_n \tag{11}$$

[2] The serious reader is urged to examine reference [2] as many useful integral evaluations are listed in the Appendix.

The quantity $h(t)$ is the unit impulse response of the system with $H(\omega)$ as the corresponding frequency response function, $F_e(\bar{\omega} - \omega)$ defines a transformation associated with the modulation function and $S_n(\omega)$ is the (two-sided) spectral density function of the input noise.

Defining

$$K(t, \omega) = \left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H(\bar{\omega})}{H(\omega)}F_e(\bar{\omega} - \omega)e^{i\bar{\omega}t}d\bar{\omega}\right|^2, \tag{12}$$

Equation (7) is reduced to the form

$$|I(t, \omega)|^2 = |H(\omega)|^2K(t, \omega) \tag{13}$$

and

$$S_y(t, \omega) = K(t, \omega)S_y(\omega) \tag{14}$$

where

$$S_y(\omega) = |H(\omega)|^2S_n(\omega) \tag{15}$$

Note the product $|H(\omega)|^2S_n(\omega)$ corresponds to the integrand for stationary response so that

$$\sigma_y = \int_{-\infty}^{\infty} S_y(\omega)d\omega. \tag{16}$$

The term $K(t, \omega)$ is dependent only on the system and the shape of the modulation function. It is dubbed a "shaping" filter as it acts to alter (in time) the spectral content of what otherwise represents a stationary response. This function ultimately governs the time variation of $\sigma_y(t)$ and, subsequently, response overshoot.

Unit Step Modulation

For the unit step modulation function,

$$K(t, \omega) = 1 + A(t) + B(t)\left(\frac{b^2 - a^2 + \omega^2}{a^2}\right) - 2C(t)\cos\omega t - \frac{2\omega}{a}(D(t)\sin\omega t) \tag{17}$$

with the time-varying coefficients

$$A(t) = e^{-2bt}\left(1 + \frac{b}{a}\sin 2at\right)$$
$$B(t) = e^{-2bt}(\sin^2 at)$$
$$C(t) = e^{-bt}\left(\cos at + \frac{b}{a}\sin at\right) \tag{18}$$
$$D(t) = e^{-bt}(\sin at)$$

Nomenclature

- $a = \omega_n(1 - \zeta^2)^{1/2}$ = damped natural frequency of system
- $b = \zeta\omega_n$ = exponential decay coefficient in system response to a unit impulse
- c_0 = stationary coefficient
- $e(t)$ = envelope modulation function
- $f(t)$ = input excitation
- f_n = system natural frequency in Hertz
- $h(t)$ = system response to a unit impulse
- $H(\omega)$ = frequency response function of the system
- $H_0(\omega) = m\omega_n^2H(\omega)$
- $K(t,\omega)$ = shaping filter for unit step modulation
- m = mass of system
- $n(t)$ = Gaussian noise with zero mean
- $Q = 1/2\zeta$ = measure of system damping
- $R_n(\tau)$ = autocorrelation function of $n(t)$
- R_0 = autocorrelation constant
- $s_1 = -s_2^* = a + ib$
- $s_3 = -s_4^* = \rho + i\alpha$
- $S_n(\omega)$ = two-sided power spectral density function of input noise
- $S_0(\omega) = (a/R_0)S_n(\omega)$
- $S_0 = R_0/\pi$ = spectral magnitude for white noise
- $u(t)$ = unit step function
- α = exponential decay coefficient for correlated noise
- ζ = damping factor of system
- ρ = frequency of autocorrelation function for correlated noise
- σ_y = stationary root-mean-square response
- $\sigma_y(t)$ = nonstationary response
- σ_{pk} = maximum value of $\sigma_y(t)$
- $\omega_n = 2\pi f_n$ = system natural frequency in rad per sec

where the quantities a and b are those defined earlier.

When $S_n(\omega) \rightarrow S_0$,

$$\sigma_y^2(t) = \frac{\pi S_0}{2\zeta m^2\omega_n^3}\left\{1 - e^{-2bt}\right.$$
$$\left.\times\left(1 + \frac{b}{a}\sin 2at + 2\left(\frac{b}{a}\right)^2\sin^2 at\right)\right\} \quad (19)$$

while for the correlated noise,

$$\sigma_y^2(t) = \frac{R_0}{m^2}\{R_1T_1 - X_1T_2 + R_3T_3 - X_3T_4\} \quad (20)$$

where

$$\begin{aligned} R_1 &= \text{Re}\,(Z_1) \\ R_3 &= \text{Re}\,(Z_3) \\ X_1 &= \text{Im}\,(Z_1) \\ X_3 &= \text{Im}\,(Z_3) \end{aligned} \quad (21)$$

with

$$\begin{aligned} Z_1 &= \frac{\alpha}{a^2}\left(\frac{\rho^2 + \alpha^2 + s_1^2}{s_1(s_1^2 - s_3^2)(s_1^2 - s_4^2)}\right) \\ Z_3 &= \frac{1}{(s_3^2 - s_1^2)(s_3^2 - s_2^2)} \end{aligned} \quad (22)$$

The remaining terms are given by

$$T_1 = \frac{a}{2b}\,[1 - A(t)]$$

$$T_2 = -B(t)$$

$$T_3 = \left[1 + A(t) + \left(\frac{b^2 - a^2 + \rho^2 - \alpha^2}{a^2}\right)B(t)\right.$$
$$\left. - 2\left(C(t) + \frac{\alpha}{a}D(t)\right)e^{-\alpha t}\cos\rho t - 2\frac{\rho}{a}D(t)e^{-\alpha t}\sin\rho t\right]$$

$$T_4 = \left[2\left(\frac{\rho\alpha}{a^2}\right)B(t) - 2\left(C(t) + \frac{\alpha}{a}D(t)\right)e^{-\alpha t}\sin\rho t\right.$$
$$\left. + 2\frac{\rho}{a}D(t)e^{-\alpha t}\cos\rho t\right] \quad (23)$$

where $A(t)$, $B(t)$, $C(t)$, and $D(t)$ are those defined previously.

For the stationary response, equation (20) reduces to

$$\sigma_y^2 = \frac{R_0}{m^2a^4}c_0^2 \quad (24)$$

where

$$c_0 = \left[\left(\frac{a}{2b}\right)\bar{R}_1 + \bar{R}_3\right]^{1/2} \quad (25)$$

with the normalized values

$$\begin{aligned} \bar{R}_1 &= a^4R_1 \\ \bar{R}_3 &= a^4R_3 \end{aligned} \quad (26)$$

When $s_1 = s_3$, the solution offered by equation (20) becomes unbounded. Under such circumstances, the mean-square response reduces to

$$\sigma_y^2(t) = \frac{R_0}{8m^2\omega_n^4}\left\{\left(\frac{a^2}{b^2} + 3\right)\right.$$
$$- e^{-2b}\left(\left(\frac{a^2 + 3b^2 + 2\omega_n^2bt}{b^2}\right) + \frac{2}{a}(b + \omega_n^2t)\sin 2at\right.$$
$$\left.\left. - \frac{2}{a^2}(a^2 - b^2 - 2bt\omega_n^2)\sin^2 at\right)\right\} \quad (27)$$

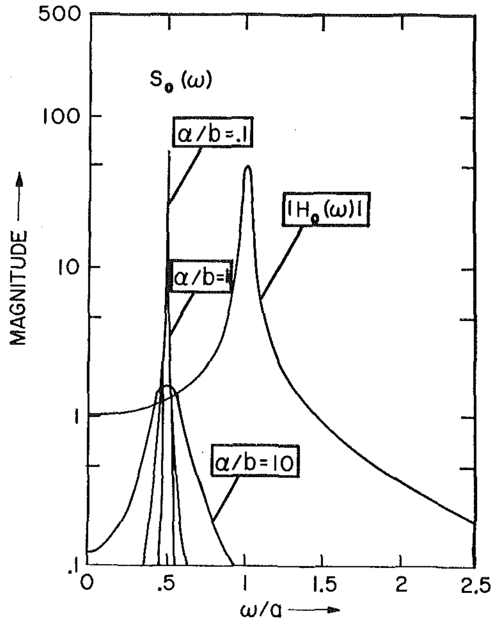

Fig. 1 Normalized spectral plots of $|H_0(\omega)|$ for $Q = 50$ and $S_0(\omega)$ for $\rho/a = 0.5$

and has the stationary value

$$\sigma_y^2 = \frac{R_0}{8m^2\omega_n^4}\left(3 + \frac{a^2}{b^2}\right) \quad (28)$$

Results

For purposes of display, it is convenient to define the functions

$$\begin{aligned} S_0(\omega) &= \frac{a}{R_0}S_n(\omega) \\ H_0(\omega) &= m\omega_n^2H(\omega) \end{aligned} \quad (29)$$

Plots of these normalized quantities are shown in Fig. 1. The effect on $S_0(\omega)$ of a change in the center frequency is a corresponding translation with no alteration in shape. Thus, for $\rho/a = 1.5$, $S_0(\omega)$ would appear but translated to $\omega/a = 1.5$.

The stationary, mean-square response for any $S_0(\omega)$ and $H_0(\omega)$ is given by the integral

$$\sigma_y^2 = \frac{R_0}{am^2\omega_n^4}\int_{-\infty}^{\infty}|H_0(\omega)|^2S_0(\omega)d\omega \quad (30)$$

which can be reduced to equation (24). The coefficient, c_0, is proportional to the foregoing integral and is plotted for $Q = 50$ in Fig. 2. The maximum response of σ_y occurs at "resonance," that is, where $\omega/a = 1$ and $\rho/a = 1$.

Typical response time histories are shown in Fig. 3. Since arguments of $2a$, $(a + \rho)$, and $(a - \rho)$ are embedded in the expressions for $\sigma_y(t)$, the oscillatory nature of the response is not surprising. For $n(t)$ where $\alpha/b \geq 5$, the response oscillation is twice the damped natural frequency of the system which is typical of $\sigma_y(t)$ when $n(t)$ is white noise. The "arrowed" values represent stationary response magnitudes for the indicated α/b ratios.

For a spectral input with broadband characteristics as for $\alpha/b = 10$, exceedance of the stationary value is not observed. Overshoot of stationary values is noted for the smaller values of α/b. In such cases, the spectral character of the input is highly peaked and prominent oscillations in the response are particularly noticeable. Such oscillations are governed by the center frequency

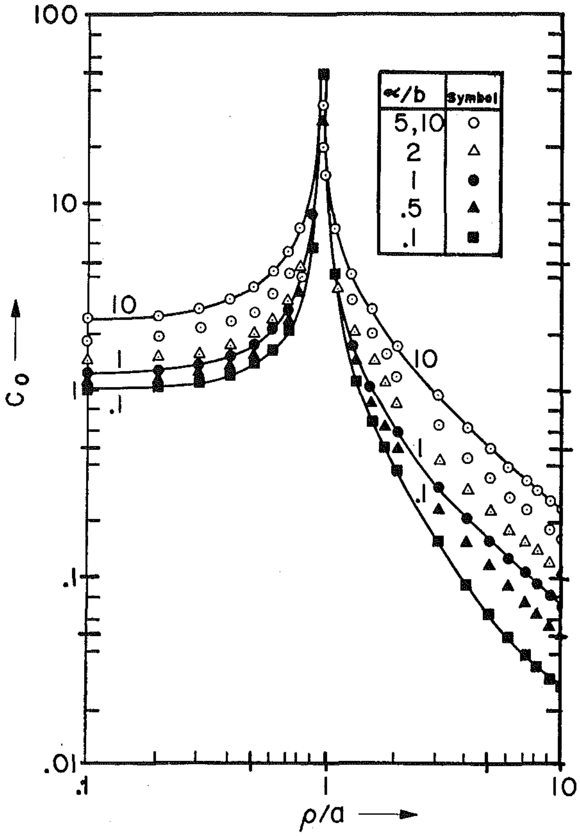

Fig. 2 Normalized stationary value for correlated noise inputs, $Q = 50$

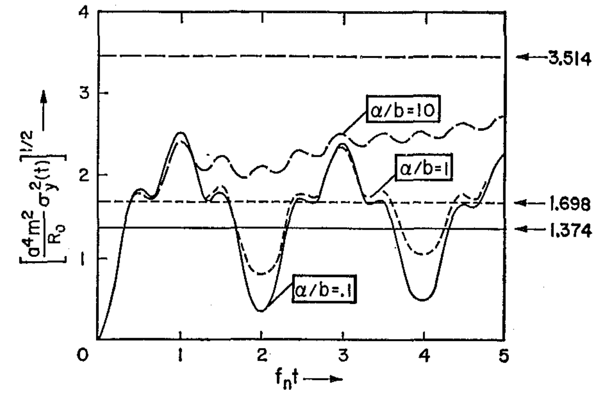

Fig. 3 Normalized system response to correlated noise modulated by unit step function, $Q = 50$, $\rho/a = 0.5$

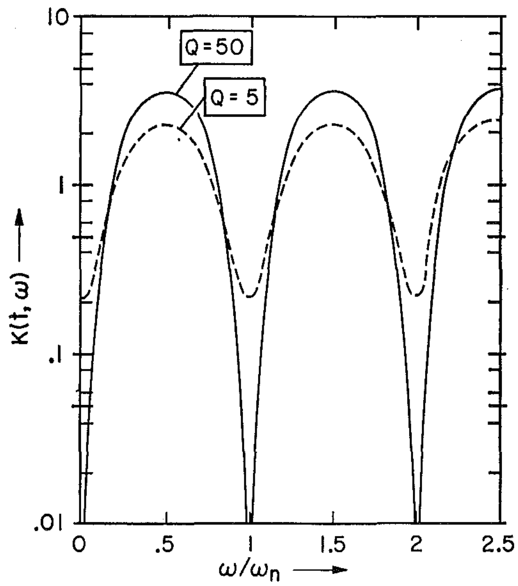

Fig. 4 Shaping filter $K(t, \omega)$ with $f_n t = 1$

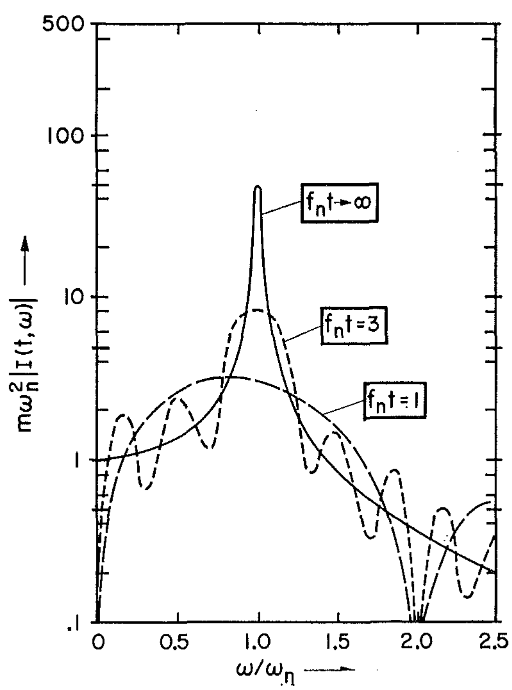

Fig. 5 $m\omega_n^2|I(t, \omega)| = |H_0(\omega)|K(t, \omega)$ with $Q = 50$

associated with either $S_0(\omega)$ or $H_0(\omega)$, or sums and differences thereof.

A requisite of overshoot, therefore, is a highly peaked spectral input. A lightly damped system is catalytic but not essential as exceedances do occur for systems with moderate damping ($Q = 5$, for example). As $S_0(\omega)$ and $H_0(\omega)$ collectively cannot dictate overshoot, we are left with $K(t, \omega)$.

The spectral character of $K(t, \omega)$ is determined by the modulation and system functions, and is clearly time-dependent. It is highly selective (many peaks and valleys) very early in the response time history and gradually resolves to a constant as the system achieves stationarity. A plot of this function at one instant in time is shown in Fig. 4. For interest, the product of $K(t, \omega)$ and the system function is shown in Fig. 5.

At any instant in time, the value of $\sigma_y(t)$ is computed by integration over ω of the integrand $|H_0(\omega)|^2 S_0(\omega)K(t, \omega)$, a triple product. For a system tuned to resonance, the peaks of $S_0(\omega)$ and $H_0(\omega)$ are coincident so that the nonstationary integrand is affected mainly by the character of $K(t, \omega)$ near resonance. Such is notched for all $f_n t$ and appears as nearly a constant to the product $|H_0(\omega)|^2 S_0(\omega)$, albeit a different magnitude at each value of $f_n t$. When stationarity is achieved, $K(t, \omega)$ reduces to a single constant value. The notching effect (magnitude) is most pronounced for the early values of $f_n t$ and gradually becomes less severe with increasing time. At resonance, therefore, the response is governed by $K(t, \omega)$ evaluated over an extremely narrow frequency band; $\sigma_y(t)$ thus gradually builds up (in time) to its stationary value and overshoot is not experienced.

For $S_0(\omega)$ away from $H_0(\omega)$, it is mainly the interplay of the notches and peaks of $K(t, \omega)$ with a peaked $S_0(\omega)$ which produces response spectra that fluctuate widely in time. Upon integration, such spectra yield fluctuating response values which may exceed the stationary value. The conditions under which $\sigma_y(t)$ exceeds its stationary value for a system with $Q = 50$ are summarized in Fig. 6. For $\alpha/b \geq 5$, no overshoot is noted over the range $0.1 \leq \rho/a \leq 10$ and, as expected, no overshoot occurs at

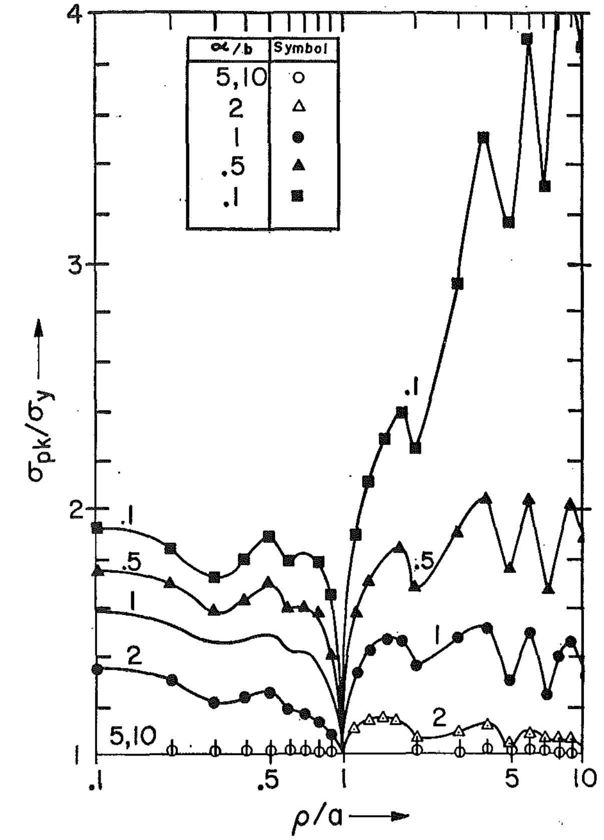

Fig. 6 Response overshoot for correlated noise inputs modulated by $e(t) = u(t)$, $Q = 50$

resonance. Exceedance values greater than two are found over $1 \leq \rho/a \leq 10$ and for $0.1 \leq \alpha/b \leq 0.50$.

Concluding Remarks

The mathematics is reviewed for computing the nonstationary response of a single-tuned system to amplitude modulated noise of damped harmonic correlation. Previously established is that the system response does not exceed its stationary value for white noise. For correlated noise inputs where $\alpha/b > 5$, a lightly damped system perceives the noise spectrum as nearly white and overshoot of the stationary value does not occur.

For highly peaked spectral inputs, overshoots of the stationary value commonly occur. This phenomenon is dependent upon the relative interaction of the system parameters, the noise parameters, and the modulation function. It is governed mainly by the properties of the function $K(t, \omega)$, dubbed a time-varying shaping filter. Stationary response properties are summarized completely in Fig. 2, with Fig. 3 needed for clarity.

The formulation presented is general and may be applied, in theory, to any real modulation function. The rectangular step and damped exponential modulations have been studied in detail elsewhere [2, 5]. For rectangular step modulation, the residual response is sensitive to the step duration as might be expected. The maximum response may occur after termination of the pulse and may be higher than the peak response for a unit step modulation. It may not only exceed the stationary value, but the peak value during the primary response as well. For exponentially damped modulation, response overshoot may be controlled by varying the decay coefficient, the system damping, or both.

For some envelope functions, it may be useful to modify $K(t, \omega)$ as defined by equation (12). This was done for the damped exponential modulation [2] in order to make a direct comparison with the results for a unit step modulation. Such resulted in a fictitious system function which, interestingly, was subject to ready interpretation.

Although the work shown is analytically precise, the functions are not necessarily convenient to evaluate for all practical applications. Much attention must be given to mathematical detail, particularly for modulation functions of finite duration. Accordingly, approximate formulations have been the subject of investigation by various authors [3, 5, 7], each with varying degrees of success.

Acknowledgments

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