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Nonlinear Estimation Using Risk Sensitive Formulation of Cubature Quadrature Kalman Filter

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***Abstract*— This paper proposes a novel method to minimize the risk sensitive cost function based on cubature quadrature algorithm. The proposed filter is named as risk sensitive cubature quadrature Kalman filter (RSCQKF). The theory and formulation of the RSCQKF have been presented in this paper. The performance of proposed risk sensitive filter is compared with its risk neutral counterpart for a ballistic target tracking problem. The simulation results show that for wrongly modeled process noise parameters, the RSCQKF outperforms the cubature quadrature Kalman filter (CQKF).**

***Index Terms*— Risk sensitive filtering, Nonlinear estimation, Robust filtering.**

I. INTRODUCTION

The risk sensitive estimator minimizes the exponential quadratic cost function scaled by a risk sensitive parameter. Hence the risk sensitive estimators are expected to be more robust in presence of uncertainty in process model or in process noise in comparison to its risk neutral counterpart. For linear Gaussian signal models, a closed-form solution exists and the algorithm of risk-sensitive estimator (RSE) has been formulated as the Kalman filter like recursion and well documented in earlier literature [1]- [4]. Such filters are generalization of standard risk neutral filter in the sense that as risk sensitive parameter approaches to zero, the risk sensitive filters converge with their risk neutral counterparts.

For nonlinear risk sensitive estimation no closed form solution exists. Initially the problem had been solved based on the extended Kalman filter (EKF) approach. The method is known as the risk sensitive extended Kalman filter (RSEKF) [5]. However the limitations associated with EKF are also inherited to the RSEKF and it frequently diverges for severely nonlinear systems. To address the limitations of the RSEKF, under Bayesian framework, several nonlinear risk sensitive filters such as the risk sensitive unscented Kalman filter (RSUKF) [6], the risk sensitive central difference filter (RSCDF) [7], the risk sensitive particle filter (RSPF) [8] , the risk sensitive adaptive grid filter (RSAGF) [9], [10] and the risk sensitive cubature Kalman filter (RSCKF) [11] *etc.* have evolved.

In this paper, we have formulated a risk sensitive filter based on the cubature quadrature Kalman filter [12], [13]. The cubature quadrature Kalman filter (CQKF) is more generalized form of the cubature Kalman filter (CQF) [14] where multivariate moment integrals, encountered under Bayesian filtering framework are calculated using third degree of cubature rule and multiple Gauss-Laguerre quadrature points. The proposed robust filter has been applied to a ballistic target tracking problem which is severely nonlinear in nature. The simulation results show the improvement in tracking performance for wrongly modeled process noise parameters.

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II. FILTERING UNDER BAYESIAN FRAMEWORK

Let us consider the nonlinear plant described by the state and measurement equations as follows:

$$x_{k+1} = \phi(x_k) + \eta_k \tag{1}$$

$$y_k = \gamma(x_k) + v_k \tag{2}$$

where $x_k \in \mathbb{R}^n$ denotes the state of the system, $y_k \in \mathbb{R}^p$ is the measurement at the instant k, where $k = \{0, 1, 2, 3, ..., N\}$, $\phi(x_k)$ and $\gamma(x_k)$ are known nonlinear functions of x_k and k. The process noise $\eta_k \in \mathbb{R}^n$ and measurement noise $v_k \in \mathbb{R}^p$ are assumed to be mutually uncorrelated and normally distributed with covariance Q_k and R_k respectively.

In Bayesian framework, the recursive Bayesian estimator estimates unknown posterior probability density function (pdf) recursively over time using incoming measurements and process model.

The prior probability density can be given by Chapman-Kolmogorov equation:

$$p(x_k|y_{1:k-1}) = \int p(x_k|x_{k-1})p(x_{k-1}|y_{1:k-1})dx_{k-1} \tag{3}$$

The above equation is known as time update equation. The computation of posterior density function is done via Bayes' rule.

$$p(x_k|y_{1:k}) = \frac{p(y_k|x_k)p(x_k|y_{1:k-1})}{p(y_k|y_{1:k-1})} \tag{4}$$

where the normalizing constant

$$p(y_k|y_{1:k-1}) = \int p(y_k|x_k)p(x_k|y_{1:k-1})dx_k \tag{5}$$

For linear Gaussian system the posterior and prior densities remain Gaussian in nature and the estimated value can be obtained optimally by the celebrated Kalman filter. For nonlinear system, the density functions are no longer Gaussian in nature. But many times it is approximated as Gaussian to find out the mean and covariance of prior as well as posterior density function.

Time update:
The *prior estimate* is the mean of prior probability density

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function obtained from *time update* equation. So

$$\hat{x}_{k|k-1} = E[x_k|y_{1:k-1}]$$
$$= E[(\phi(x_{k-1}) + \eta_k)|y_{1:k-1}]$$
$$= E[\phi(x_{k-1})|y_{1:k-1}]$$

or,

$$\hat{x}_{k|k-1} = \int \phi(x_{k-1}) p(x_{k-1}|y_{1:k-1}) dx_{k-1}$$
$$= \int \phi(x_{k-1}) \aleph(x_{k-1}; \hat{x}_{k-1|k-1}, P_{k-1|k-1}) dx_{k-1}$$

$$P_{k|k-1} = E[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^T|y_{1:k-1}]$$
$$= \int \phi(x_{k-1})\phi^T(x_{k-1}) \aleph(x_{k-1}; \hat{x}_{k-1|k-1}, P_{k-1|k-1}) dx_{k-1}$$
$$- \hat{x}_{k|k-1}\hat{x}^T_{k|k-1} + Q_k$$

Measurement update:

$$p(y_k|y_{1:k-1}) = \aleph(y_k; \hat{y}_{k|k-1}, P_{yy,k|k-1})$$

where

$$\hat{y}_{k|k-1} = \int \gamma(x_k) \aleph(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) dx_k$$

$$P_{yy,k|k-1} = \int \gamma(x_k)\gamma^T(x_k) \aleph(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) dx_k$$
$$- \hat{y}_{k|k-1}\hat{y}^T_{k|k-1} + R_k$$

Cross covariance

$$P_{xy,k|k-1} = \int x_k\gamma^T(x_k) \aleph(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) dx_k$$
$$- \hat{x}_{k|k-1}\hat{y}^T_{k|k-1}$$

On the receipt of new measurement y_k, the posterior density

$$p(x_k|y_{1:k}) = \aleph(x_k; \hat{x}_{k|k}, P_{k|k})$$

where

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - \hat{y}_{k|k-1})$$

$$P_{k|k} = P_{k|k-1} - K_k P_{yy,k|k-1} K^T_k$$

$$K_k = P_{xy,k|k-1} P^{-1}_{yy,k|k-1}$$

From the equations, it is clear that to obtain state estimation the previous integrals need to be evaluated. Further, the accuracy of the estimation depends on the accuracy of the approximate evaluation of the integrals.

III. RISK SENSITIVE FILTERING

The error cost function with risk sensitive parameter for the estimated sequence $\hat{\Phi}_1, \hat{\Phi}_2, \cdots, \hat{\Phi}_k$ is defined as:

$$C(\hat{\Phi}_1, \hat{\Phi}_2, \cdots, \hat{\Phi}_k) = E\left[exp\left(\mu_1 \sum_{i=1}^{k-1} \rho_1(\Phi(x_i) - \hat{\Phi}_i) + \mu_2\rho_2(\Phi(x_k) - \hat{\Phi}_k)\right)\right] \quad (6)$$

where $\mu_1 \geq 0$ and $\mu_2 > 0$ are two risk sensitive parameters. Functions $\rho_1(.)$ and $\rho_2(.)$ are strictly convex, continuous and bounded from below attaining global minima at 0.

We assume $\hat{\Phi}_i$ for all times $0, 1, \cdots, k-1$ to be known as $\hat{\Phi}^*_i$ and $\hat{\Phi}_k$ to be unknown. Hence the cost function is given as

$$C(\hat{\Phi}_k) = E\left[exp\left(\mu_1 \sum_{i=1}^{k-1} \rho_1(\Phi(x_i) - \hat{\Phi}^*_i) + \mu_2\rho_2(\Phi(x_k) - \hat{\Phi}_k)\right)\right] \quad (7)$$

The minimum risk sensitive estimate (MRSE) at time k is given by

$$\hat{\Phi}^*_k = arg\ min\ C(\hat{\Phi}_k) \quad (8)$$

If we assume the variable to be estimated as the state variables themselves ($\Phi(x) = x$) and the covex functions $\rho_j(\Phi(x) - \hat{\Phi}) = [x - \hat{x}]^T[x - \hat{x}]$ where $j = 1, 2$, then the cost function given in (7) can be written as

$$C(\hat{x}_k) = E\left[exp\left(\mu_1 \sum_{i=1}^{k-1} [x_i - \hat{x}_i{}^*]^T[x_i - \hat{x}_i{}^*] + \mu_2[x_k - \hat{x}_k]^T[x_k - \hat{x}_k]\right)\right] \quad (9)$$

and the minimum risk sensitive estimate (MRSE) at time k is given by

$$\hat{x}^*_k = arg\ min\ C(\hat{x}_k) \quad (10)$$

The solution to the MRSE may be given by the following recursive relation:

$$\sigma_{k|k-1} = \int_{-\infty}^{+\infty} p(x_k|x_{k-1}) exp(\mu_1[x_k - \hat{x}_{k|k-1}]^T[x_k - \hat{x}_{k|k-1}])\sigma_{k-1|k-1} dx_{k-1} \quad (11)$$

and

$$\sigma_{k|k} = p(y_k|x_k)\sigma_{k|k-1}$$
$$= p(y_k|x_k) \times \int_{-\infty}^{+\infty} p(x_k|x_{k-1}) exp(\mu_1[x_k - \hat{x}_{k|k-1}]^T[x_k - \hat{x}_{k|k-1}])\sigma_{k-1|k-1} dx_{k-1} \quad (12)$$

where $\sigma_{k|k-1}$ and $\sigma_{k|k}$ represent information state. The risk sensitive posterior estimate is given by

$$\hat{x}_{k|k} = \underbrace{arg\ min}_{\alpha \in \mathbb{R}^n} \int_{-\infty}^{+\infty} exp(\mu_2[x_k - \alpha]^T[x_k - \alpha])\sigma_{k|k} dx_k \quad (13)$$

It should be noted that when $\mu_1 = 0$, $\mu_2 > 0$, the cost function has only instantaneous error (It does not include the stored error over past time.) and it merges with the MMSE Kalman filter.

If we consider

$$\sigma^+_{k-1|k-1} = exp(\mu_1[x_k - \hat{x}_{k|k-1}]^T[x_k - \hat{x}_{k|k-1}])\sigma_{k-1|k-1} \quad (14)$$

then,

$$\sigma_{k|k-1} = \int_{-\infty}^{+\infty} p(x_k|x_{k-1})\sigma^+_{k-1|k-1} dx_{k-1} \quad (15)$$

and

$$\sigma_{k|k} = p(y_k|x_k)\sigma_{k|k-1} \quad (16)$$

Equation (14) may be considered as risk sensitive update step. Equation (15) and (16) may be considered as time update and measurement update respectively. It should be noted that if Gaussian approximation is assumed to be maintained, the optimal estimate obtained from (13) is simply the mean value of the distribution.

IV. FORMULATION OF RISK SENSITIVE CUBATURE QUADRATURE KALMAN FILTER

In [12], [13], we have developed an algorithm to evaluate the intractable integrals encountered in filtering under Bayesian framework, using third degree of spherical radial cubature rule and multiple Gauss-Laguerre quadrature points. The estimator is named as the cubature quadrature Kalman filter (CQKF).

A. Cubature Quadrature Rule

1) Cubature rule:

Theorem 1: *For an arbitrary function $f(x)$, $X \in \mathbb{R}^n$ the integral $I(f) = \frac{1}{\sqrt{|\Sigma|(2\pi)^n}} \int_{\mathbb{R}^n} f(x) e^{-\frac{1}{2}(X-\mu)^T \Sigma^{-1}(X-\mu)} dX$ can be expressed in spherical coordinate system as*

$$I(f) = \frac{1}{\sqrt{(2\pi)^n}} \int_{r=0}^{\infty} \int_{U_n} [f(CrZ+\mu) ds(Z)] r^{n-1} e^{r^2/2} dr \tag{17}$$

where $X = CrZ + \mu$, C is the Cholesky decomposition of Σ, $\|Z\| = 1$, μ and Σ are the mean and covariance respectively and U_n is the surface of unit hyper-sphere.

Proof: Let us transform the integral $I(f)$ to a spherical coordinate system [15]. Let $X = CY + \mu$, $Y \in \mathbb{R}^n$, where $\Sigma = CC^T$ *i.e.* C is the Cholesky decomposition of Σ. Then $(X-\mu)^T \Sigma^{-1} (X-\mu) = Y^T C^T C^{-T} C^{-1} C Y = Y^T Y$ and $dX = |C| dY = \sqrt{|\Sigma|} dY$. So the desired integral,

$$I(f) = \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} f(CY+\mu) e^{-\frac{1}{2} Y^T Y} dY \tag{18}$$

Now let $Y = rZ$, with $\|Z\| = \sqrt{Z^T Z} = 1$, $Y^T Y = Z^T rrZ = r^2$. The elementary volume of hyper-sphere at n dimensional space is $dY = r^{n-1} dr ds(Z)$ where $ds(.)$ is the area element on U_n. U_n is the surface of hyper-sphere defined by $U_n = \{Z \in \mathbb{R}^n | ZZ^T = 1\}$; $r \in [0, \infty)$. Hence

$$I(f) = \frac{1}{\sqrt{(2\pi)^n}} \int_{r=0}^{\infty} \int_{U_n} f(CrZ+\mu) e^{-r^2/2} r^{n-1} dr ds(Z)$$
$$= \frac{1}{\sqrt{(2\pi)^n}} \int_{r=0}^{\infty} \int_{Un} [f(CrZ+\mu) ds(Z)] r^{n-1} e^{-r^2/2} dr \quad \square$$

Now to compute the integration $I(f)$ as described above, first we need to compute

$$\int_{U_n} f(CrZ+\mu) ds(Z) \tag{19}$$

The integral (19) can be approximately calculated by third degree fully symmetric spherical radial cubature rule. If we consider zero mean unity variance, (19) can be approximated as [14].

$$\int_{U_n} f(rZ) ds(Z) \approx \frac{2\sqrt{\pi^n}}{2n\Gamma(n/2)} \sum_{i=1}^{2n} f[ru]_i \tag{20}$$

where $[u]_i$ $(i = 1, 2, ..., 2n)$ are the cubature points located at the intersections of the unit hyper-sphere and its axes. For example, in single dimension, the two cubature points will be on $+1$ and -1. For two dimensions, the four cubature points will be on $(+1,0)$, $(-1,0)$, $(0,+1)$ and $(0,-1)$. For Gaussian distribution with non zero mean and non unity covariance, the cubature points will be located at $(C[u]_i + \mu)$.

2) Gauss-Laguerre Quadrature rule: Any integral in the form of,

$$\int_{\lambda=0}^{\infty} f(\lambda) \lambda^{\alpha} e^{-\lambda} d\lambda \tag{21}$$

can be approximately evaluated using quadrature points and weights associated with them. The error associated with the approximate evaluation of integral depends on the number of quadrature points. The quadrature points can be determined from the roots of n' order of Chebyshev-Laguerre polynomial equation [16], [17].

$$L_{n'}^{\alpha}(\lambda) = (-1)^{n'} \lambda^{-\alpha} e^{\lambda} \frac{d^{n'}}{d\lambda^{n'}} \lambda^{\alpha+n'} e^{-\lambda} = 0 \tag{22}$$

Let the quadrature points be $\lambda_{i'}$. The weights can be determined as

$$A_{i'} = \frac{n'! \Gamma(\alpha+n'+1)}{\lambda_{i'} [\dot{L}_{n'}^{\alpha}(\lambda_{i'})]^2}$$

So the integral (21) can be written approximately using quadrature rule as

$$\int_{\lambda=0}^{\infty} f(\lambda) \lambda^{\alpha} e^{-\lambda} d\lambda \approx \sum_{i'=1}^{n'} A_{i'} f(\lambda_{i'})$$

3) Cubature Quadrature rule: Combining the equation (17) and (20) we get

$$I(f) = \frac{1}{\sqrt{(2\pi)^n}} \times \frac{2\sqrt{\pi^n}}{2n\Gamma(n/2)} \int_{r=0}^{\infty} (\sum_{i=1}^{2n} f[ru]_i) r^{n-1} e^{-r^2/2} dr \tag{23}$$

Now to integrate the rest of the term, we use Gauss-Laguerre quadrature formula described above. To cast the integration in that form of (21), let us assume $t = r^2/2$. With this transformation the equation (23) becomes

$$I(f) = \frac{1}{2^{n/2} n\Gamma(n/2)} \times 2^{(n/2-1)} \int_{t=0}^{\infty} (\sum_{i=1}^{2n} f[\sqrt{2t}u]_i) t^{(n/2-1)} e^{-t} dt \tag{24}$$

Now the integration $\int_{t=0}^{\infty} f(t) t^{(n/2-1)} e^{-t} dt$ is approximated using multiple quadrature points with $\alpha = n/2 - 1$. As per earlier discussion, the accuracy of the estimator depends on the order of quadrature rule. For i' number of quadrature points denoted as $\lambda_{i'}$ the integral (24) becomes

$$I(f) = \frac{1}{2n\Gamma(n/2)} \times [\sum_{i=1}^{2n} \sum_{i'=1}^{n'} A_{i'} f(\sqrt{2\lambda_{i'}}) [u]_i]$$

For n dimension of state space problem solved with third order spherical cubature rule and n' order Gauss-Laguerre quadrature points, total $2nn'$ cubature quadrature points and associated weights need to be calculated.

B. RSCQKF algorithm

Step (i) Filter initialization

- Initialize the filter with $\hat{x}_{0|0}$ and $P_{0|0}$.
- Calculate the cubature quadrature (CQ) points, ξ_j, and their corresponding weights, $w_j (j = 1, 2, ..., 2nn')$.

Step (ii) Predictor step

- Perform the Cholesky decomposition of posterior error covariance
$$P_{k|k} = S_{k|k} S_{k|k}^T$$
- Evaluate cubature quadrature points
$$\chi_{j,k|k} = S_{k|k}\xi_j + \hat{x}_{k|k}$$
- Update cubature quadrature points
$$\chi_{j,k+1|k} = \phi(\chi_{j,k|k})$$
- Compute time updated mean and covariance
$$\overline{\sigma}_{k+1|k} = \sum_{j=1}^{2nn'} w_j \chi_{j,k+1|k}$$
$$P_{k+1|k} = \sum_{j=1}^{2nn'} w_j [\chi_{j,k+1|k} - \overline{\sigma}_{k+1|k}][\chi_{j,k+1|k} - \overline{\sigma}_{k+1|k}]^T + Q_k$$
- The risk sensitive mean remains the same and the risk sensitive covariance is updated with
$$P_{k+1|k}^+ = (P_{k+1|k}^{-1} - 2\mu_1 I)^{-1}$$

Step (iii) Corrector step or measurement update

- Perform the Cholesky decomposition of prior error covariance
$$P_{k+1|k}^+ = S_{k+1|k} S_{k+1|k}^T$$
- Evaluate cubature quadrature points
$$\chi_{j,k+1|k} = S_{k+1|k}\xi_j + \overline{\sigma}_{k+1|k}$$
where $j = 1, 2, ..., 2nn'$.
- Find the predicted measurements at each cubature quadrature points
$$Y_{j,k+1|k} = \gamma(\chi_{j,k+1|k})$$
- Estimate the predicted measurement
$$\hat{y}_{k+1} = \sum_{j=1}^{2nn'} w_j Y_{j,k+1|k}$$
- Calculate the covariances
$$P_{y_{k+1}y_{k+1}} = \sum_{j=1}^{2nn'} w_j [Y_{j,k+1|k} - \hat{y}_{k+1}][Y_{j,k+1|k} - \hat{y}_{k+1}]^T + R_k$$
$$P_{x_{k+1}y_{k+1}} = \sum_{j=1}^{2nn'} w_j [\chi_{j,k+1|k} - \overline{\sigma}_{k+1|k}][Y_{j,k+1|k} - \hat{y}_{k+1}]^T$$
- Calculate Kalman gain
$$K_{k+1} = P_{x_{k+1}y_{k+1}} P_{y_{k+1}y_{k+1}}^{-1}$$
- Compute posterior state values
$$\overline{\sigma}_{k+1|k+1} = \overline{\sigma}_{k+1|k} + K_{k+1}(y_{k+1} - \hat{y}_{k+1})$$
- Posterior error covariance matrix is given by
$$P_{k+1|k+1} = P_{k+1|k}^+ - K_{k+1} P_{y_{k+1}y_{k+1}} K_{k+1}^T$$
- The posterior risk sensitive estimate is given by
$$\hat{x}_{k+1|k+1} = \overline{\sigma}_{k+1|k+1}$$

Note: The condition $(P_{k+1|k}^{-1} - 2\mu_1 I)^{-1} > 0$ need to be satisfied at each step for risk sensitive error covariance to be positive definite. The condition limits the upper value of μ_1.

V. SIMULATION RESULTS

The above described algorithm is used to track a ballistic object in its re-entry phase. Inspired from the earlier work of Ristic *et al.* [18], [19], we have considered two dimensional ballistic target motion described by the following nonlinear discrete time dynamic state equation
$$s_{k+1} = \psi(s_k) + G \begin{bmatrix} 0 \\ -g \end{bmatrix} + \eta_k$$
where the state vector given by
$$s_k = \begin{bmatrix} x_k & \dot{x_k} & y_k & \dot{y_k} \end{bmatrix}^T$$
provides the positions and velocities of target in x and y coordinates at k^{th} time instant and
$$\psi(s_k) = \varphi s_k + G f_k(s_k)$$
where
$$\varphi = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} T^2/2 & 0 \\ T & 0 \\ 0 & T^2/2 \\ 0 & T \end{bmatrix}$$
where T is the time interval between two consecutive radar measurements. The drag force $f_k(s_k)$ is directed opposite to the target speed u and its magnitude is given by $0.5\frac{g}{\beta}\rho u^2$.
$$f_k(s_k) = -0.5\frac{g}{\beta}\rho(\dot{x_k}^2 + \dot{y_k}^2) \times \begin{bmatrix} cos(tan^{-1}(\frac{\dot{y_k}}{\dot{x_k}}) \\ sin(tan^{-1}(\frac{\dot{y_k}}{\dot{x_k}}) \end{bmatrix}$$
$$= -0.5\frac{g}{\beta}\rho\sqrt{\dot{x_k}^2 + \dot{y_k}^2} \begin{bmatrix} \dot{x_k} \\ \dot{y_k} \end{bmatrix}$$
where g is the acceleration due to gravity and β is the ballistic coefficient. It varies with mass, shape and cross sectional area of the target perpendicular to the direction of motion. It is constant for supersonic speed due to formation of shock waves. The shock waves vanish when speed of a target decreases to the speed of sound. Here we assume β to be a constant as the speed of the target is more than the speed of the sound throughout its motion. The air density ρ is given by
$$\rho = c_1 e^{-c_2 y}$$
where
$c_1 = 1.227$ and $c_2 = 1.093 \times 10^{-4}$; for $y < 9144$ m
$c_1 = 1.754$ and $c_2 = 1.49 \times 10^{-4}$; for $y \geq 9144$ m.

The process noise η_k is taken as zero mean white Gaussian with covariance Q given by

$$Q = q\begin{bmatrix}\theta & 0\\ 0 & \theta\end{bmatrix}$$

with

$$\theta = \begin{bmatrix}T^3/3 & T^2/2\\ T^2/2 & T\end{bmatrix}$$

where q is a parameter. This accounts for the possible deviation of the process model from the real situation. The truth is simulated for the target trajectory in the MATLAB as shown in Fig. (1)-(3) with $g = 9.8\ m.s^{-2}$, $\beta = 40000\ Kg.m^{-1}.s^{-2}$, $q = 1\ m^2.s^{-3}$, and $T = 2s$ with number of path samples $N = 60$. Truth is initialized with $x_0 = 232000\ m$, $y_0 = 88000\ m$, $v_0 = 2290\ ms^{-1}$, $\gamma_0 = 190^o$ where γ is the angle between horizontal axis and the direction of motion.

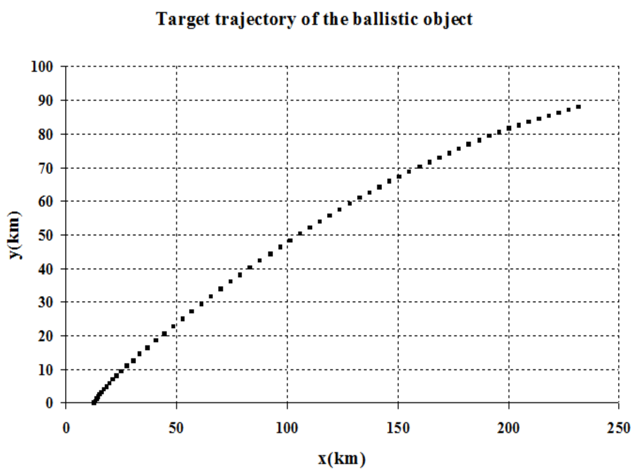

Fig. 1. Target trajectory of the ballistic object

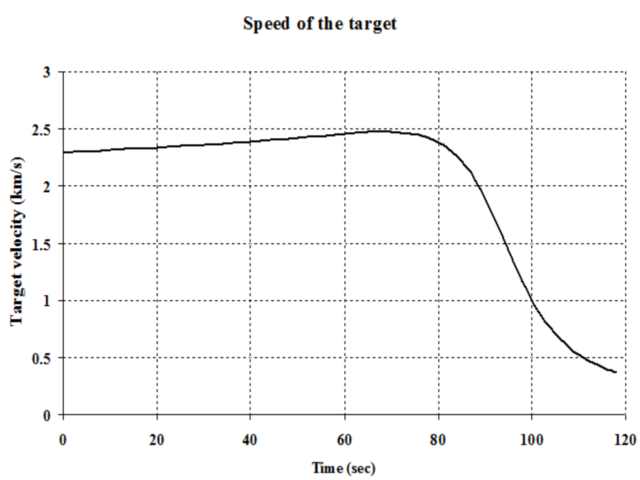

Fig. 2. Speed of the target versus time

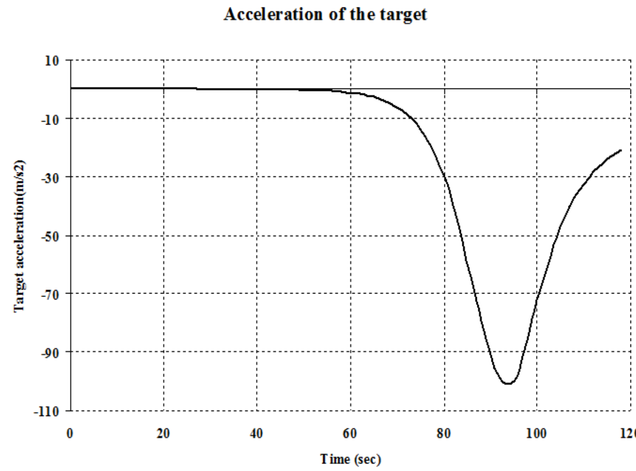

Fig. 3. Acceleration of the target versus time

The measurement equation is given as

$$z_k = Hs_k + v_k$$

where $H = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$, $z_k = \begin{bmatrix}d_k & h_k\end{bmatrix}^T$ is the radar measurement in cartesian coordinate. Considering radar to be located at the origin and measurements collected are the range r, and elevation ε, $d = r\ cos\varepsilon$ and $h = r\ sin\varepsilon$ are measurement of the positions along x and y coordinates respectively. v_k is the measurement noise which is white Gaussian with zero mean and covariance, R_k, given by

$$R_{k11} = \sigma_d^2 = \sigma_r^2 cos^2(\varepsilon) + r^2\sigma_\varepsilon^2 sin^2(\varepsilon)$$
$$R_{k22} = \sigma_h^2 = \sigma_r^2 sin^2(\varepsilon) + r^2\sigma_\varepsilon^2 cos^2(\varepsilon)$$
$$R_{k12} = R_{k21} = \sigma_{dh} = (\sigma_r^2 - r^2\sigma_\varepsilon^2)sin(\varepsilon)cos(\varepsilon)$$

where σ_r and σ_ε are the standard deviations of radar measurement for range and elevetion. The above expression for R_k is derived by measurement conversion from polar to cartesian coordinate and is good approximation for linear measurement.

The above described problem of tracking the ballistic object is solved using the RSCQKF with fourth order Gauss-Laguerre approximation and its performance is compared with the CQKF in terms of root mean square error (RMSE). From the first two measurements, filters are initialized as $\hat{s}_{0|0} = \begin{bmatrix}d_1 & \frac{d_0-d_1}{T} & h_1 & \frac{h_0-h_1}{T}\end{bmatrix}^T$. Initial error covariance matrix is derived as

$$P_{0|0} = \begin{bmatrix}\sigma_d^2 & -\frac{\sigma_d^2}{T} & \sigma_{dh} & -\frac{\sigma_{dh}}{T}\\ -\frac{\sigma_d^2}{T} & 2\frac{\sigma_d^2}{T^2} & -\frac{\sigma_{dh}}{T} & 2\frac{\sigma_{dh}}{T^2}\\ \sigma_{dh} & -\frac{\sigma_{dh}}{T} & \sigma_h^2 & -\frac{\sigma_h^2}{T}\\ -\frac{\sigma_{dh}}{T} & 2\frac{\sigma_{dh}}{T^2} & -\frac{\sigma_h^2}{T} & 2\frac{\sigma_h^2}{T^2}\end{bmatrix}$$

During simulation we use the radar parameters $\sigma_r = 100\ m$, $\sigma_\varepsilon = 0.017\ rad$. The risk sensitive parameter is taken as $\mu_1 = 7\times 10^{-9}$.

The process noise covariance of filter is mismatched with that of truth by changing the value of the parameter $q = 0.01$, (the q for truth remains unity). The RMSE (out of 100 MC runs) of position estimation obtained from the RSCQKF and CQKF have been plotted in Fig.(4)-(5). From the figures it may be concluded that the RSCQKF tracks better compared to the ordinary CQKF. Similar result has been obtained for velocities and not shown here.

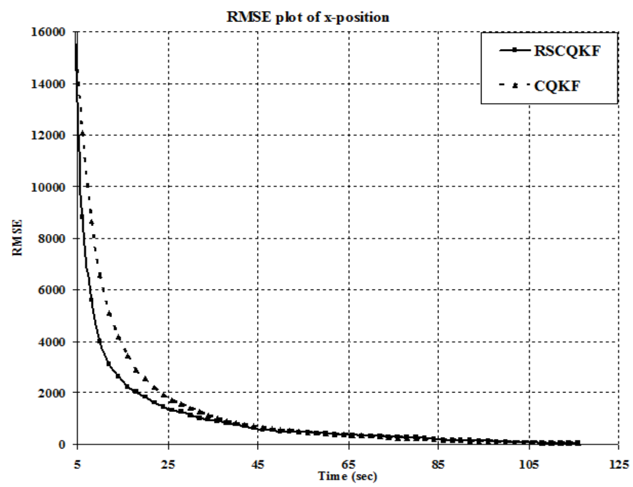

Fig. 4. RMSE plot of RSCQKF (solid line) versus CQKF (dashed line) for x-position of the target

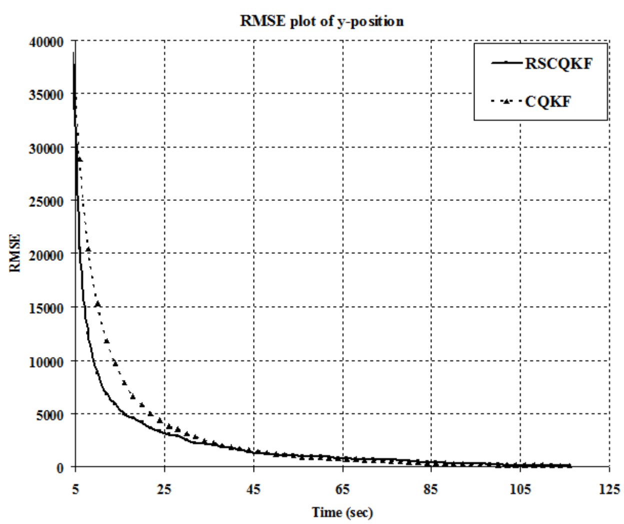

Fig. 5. RMSE plot of RSCQKF (solid line) versus CQKF (dashed line) for y-position of the target

VI. DISCUSSION AND CONCLUSIONS

A risk sensitive cubature quadrature Kalman filter (RSC-QKF) algorithm has been developed in this paper. The developed algorithm is used to solve a ballistic target tracking problem in its re-entry phase. The performance of the RSCQKF algorithm is compared with the CQKF in terms of RMSE. The simulation results reveal that for wrongly modeled process noise parameter, the risk sensitive filter performs better than the CQKF.

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