Stationary Solution of Duffing Oscillator Driven by Additive and Multiplicative Colored Noise Excitations

Siu-Siu Guo[1]
School of Civil Engineering,
Xi'an University of Architecture and Technology,
Xi'an 710055, China
e-mail: siusiuguo@hotmail.com

Qingxuan Shi
Professor
School of Civil Engineering,
Xi'an University of Architecture and Technology,
Xi'an 710055, China
e-mail: shiqx@xauat.edu.cn

A bistable Duffing oscillator subjected to additive and multiplicative Ornstein–Uhlenbeck (OU) colored excitations is examined. It is modeled through a set of four first-order stochastic differential equations by representing the OU excitations as filtered Gaussian white noise excitations. Enlargement in the state-space vector leads to four-dimensional (4D) Fokker–Planck–Kolmogorov (FPK) equation. The exponential-polynomial closure (EPC) method, proposed previously for the case of white noise excitations, is further improved and developed to solve colored noise case, resulting in much more polynomial terms included in the approximate solution. Numerical results show that approximate solutions from the EPC method compare well with the predictions obtained via Monte Carlo simulation (MCS) method. Investigation is also carried out to examine the influence of intensity level on the probability distribution solutions of system responses.

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1 Introduction

Since colored random excitations can be found widely in the scientific and engineering fields, e.g., ecology [1], earthquake modeling and engineering [2,3], wind engineering [4], and ship dynamics [5], there has been a growing interest in studying dynamical systems disturbed by colored random excitations [1–10].

General approach to dealing with non-Markovian systems' responses in the case of colored excitations is filtering approach. This method is to implement the original dynamical system with an auxiliary system excited by a delta-correlated process whose output is modeled as colored excitation [11–14]. Probability distribution solutions of system responses in multidimensional spaces result in. Traditionally, a straightforward way to obtain probability distributions is by solving FPK equation. In recent years, a number of techniques have been proposed to deal with FPK equation. In the case of colored noise excitations, one of the most successful approaches appears to be stochastic-averaging method, which is for dynamical systems with weak damping under colored excitations of small intensities [15–17]. In this paper, the EPC approximate method, which was proposed previously for the general case of white noise excitations [18,19], is further improved and developed to suit the case of colored noise excitations [20,21]. State variables involved are enlarged, causing much more polynomial terms to be produced in the approximate solutions. As a result, solution procedure becomes more complicated and requires more computational time. The efficiency of the developed EPC method is examined by an example of a bistable Duffing oscillator to OU colored excitations. The influence of intensity level on the probability distribution solutions of system response is considered.

2 Analysis of Stochastic Oscillator to Colored Noise Excitations

2.1 Gaussian Colored Noise. As is well known, real physical processes are generally characterized by correlated functions with finite correlation lengths. One of the most commonly used correlation functions is exponential function. The simplest example for an exponentially correlated noise is OU process $\eta_i(t)$

$$E[\eta_i(t)\eta_i(s)] = \frac{D_i}{\tau_i}\exp\left[-\frac{|t-s|}{\tau_i}\right] = 2D_i \quad \tau_i \to 0 \tag{1}$$

where τ_i denote the correlation time, and D_i express the intensity. Higher D_i correspond to stronger excitations, while smaller τ_i result in broader band widths. And when the correlation time τ_i approach zero, the OU process evolves to Gaussian white noise process. The level of the noise color can be measured directly by the bandwidth parameter τ_i.

Such exponentially correlated Gaussian processes, $\eta_i(t)$, can be obtained by passing Gaussian white noise $W_i(t)$ through first-order filters of the following form:

$$\dot{\eta}_i(t) = -\frac{1}{\tau_i}\eta_i + \frac{1}{\tau_i}W_i(t) \tag{2}$$

where $W_i(t)$ are the Gaussian white noise. Due to the linearity, the filtered white noise OU process is also Gaussian but with power spectral density of finite bandwidth. This simple linear filter can generate colored noises with any required bandwidth, but it usually distributes too much energy in the low frequency range.

2.2 Formulation of the Four-Dimensional Fokker–Planck–Kolmogorov Equation. Hereinafter, a bistable Duffing oscillator to additive and multiplicative colored OU excitations is considered

$$\ddot{X} + \alpha\dot{X} - X + \varepsilon X^3 = c_1 X\eta_1(t) + \eta_2(t) \tag{3}$$

where ε represents the degree of nonlinearity, and η_i are the colored OU excitations with band limited power spectral density functions, which can be obtained as filtered white noises in Eq. (2). Then, Duffing oscillator combined with filtered systems can be written concisely by first-order differential equations of Ito form

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\alpha x_2 + x_1 - \varepsilon x_1^3 + c_1 x_1 x_4 + x_3 \\
\dot{x}_3 &= -\frac{1}{\tau_3}x_3 + \frac{1}{\tau_3}W_3(t) \\
\dot{x}_4 &= -\frac{1}{\tau_4}x_4 + \frac{1}{\tau_4}W_4(t)
\end{aligned} \tag{4}$$

where Gaussian white noise excitations are with $E[W_3(t)W_3(t+\tau)] = S_3\delta(\tau)$ and $E[W_4(t)W_4(t+\tau)] = S_4\delta(\tau)$. Enlarged system response $\mathbf{X} = [x_1(t), x_2(t), x_3(t), x_4(t)]$ is the 4D state-space vector and leads to 4D FPK equation, which is expressed as

[1]Corresponding author.

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$$x_2 \frac{\partial p}{\partial x_1} - \frac{\partial}{\partial x_2}\left(\alpha x_2 - x_1 + \varepsilon x_1^3 - c_1 x_1 x_4 - x_3\right) - \frac{1}{\tau_3}\frac{\partial}{\partial x_3}(x_3 p) - \frac{1}{\tau_4}\frac{\partial}{\partial x_4}(x_4 p) - \frac{S_3}{2\tau_3^2}\frac{\partial^2 p}{\partial x_3^2} - \frac{S_4}{2\tau_4^2}\frac{\partial^2 p}{\partial x_4^2} = 0 \quad (5)$$

Proceeding with the FPK equation, we found that the exact solution to the 4D FPK equation in Eq. (5) is not available. Approximate methods have to be adopted and herein EPC method is considered.

2.3 Solution Procedure of the Developed Exponential-Polynomial Closure Method. The EPC method is previously proposed for general single degree-of-freedom systems under white noise excitations. In the case of colored noise, it is noticed that state variables involved in Eq. (5) are increased to four. The EPC method has to be improved and developed to suit such case. Consequently, much more polynomial terms are produced in the approximate solution

$$\tilde{p}(\boldsymbol{a}; \boldsymbol{x}) = \exp[Q_n(a_{ijkm}; x_1, x_2, x_3, x_4)] \quad (6)$$

where Q_n are the polynomial functions of four state variables

$$Q_n(a_{ijkm}; x_1, x_2, x_3, x_4) = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}\sum_{m=0}^{n} a_{ijkm} x_1^i x_2^j x_3^k x_4^m \quad i+j+k+m = 0, 1, 2, \ldots, n \quad (7)$$

It should be noticed that the number of polynomial terms goes up to $N_p = 52$ when the polynomial order $n = 4$, which is as much as three times of the ones $n(n+3)/2$ in the case of white noise excitation. In this sense, the procedure for coding the computer programs becomes much complicated and requires much more computational effort.

By substituting approximate solution (Eq. (6)) into the FPK equation in Eq. (5), inevitable residual error results

$$\begin{aligned}\triangle(x_1, x_2, x_3, x_4; \tilde{p}) &= \Bigg\{ x_2 \frac{\partial Q_n}{\partial x_1} - \left(\alpha x_2 - x_1 + \varepsilon x_1^3 - c_1 x_1 x_4 - x_3\right)\frac{\partial Q_n}{\partial x_2} \\ &\quad - \alpha - \frac{1}{\tau_3} - \frac{1}{\tau_4} - \frac{x_3}{\tau_3}\frac{\partial Q_n}{\partial x_3} - \frac{x_4}{\tau_4}\frac{\partial Q_n}{\partial x_4} - \frac{S_3}{2\tau_3^2}\left[\frac{\partial^2 Q_n}{\partial x_3^2} + \left(\frac{\partial Q_n}{\partial x_3}\right)^2\right] \\ &\quad - \frac{S_4}{2\tau_4^2}\left[\frac{\partial^2 Q_n}{\partial x_4^2} + \left(\frac{\partial Q_n}{\partial x_4}\right)^2\right]\Bigg\}\tilde{p}(\boldsymbol{a}; \boldsymbol{x}) \\ &= \delta(a_{ijkm}; x_1, x_2, x_3, x_4)\tilde{p}(\boldsymbol{a}; \boldsymbol{x}) \end{aligned} \quad (8)$$

In order to vanish the residual error in Eq. (8), Galerkin method is adopted

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(a_{ijkm}; x_1, x_2, x_3, x_4) w_h dx_1 dx_2 dx_3 dx_4 = 0, \quad (h = 1, 2, \ldots, N_p) \quad (9)$$

where w_h are the weighting functions. Numerical analysis evidences that effective choice for weighting functions is

$$w_h = x_1^i x_2^j x_3^k x_4^m f(x_i; m_i, \sigma_{ij}), \quad (i+j+k+m = 1, 2, \ldots, n) \quad (10)$$

where $f(x_i; m_i, \sigma_{ij})$ are the multivariate Gaussian probability distribution functions (PDFs) with parameters statistical means m_i and statistical variances σ_{ij} obtained from moment equations

$$\begin{aligned} &E\left[x_2 \frac{\partial M_k}{\partial x_1}\right] - E\left[\left(\alpha x_2 - x_1 + \varepsilon x_1^3 - c_1 x_1 x_4 - x_3\right)\frac{\partial M_k}{\partial x_2}\right] \\ &\quad - \frac{1}{\tau_3}E\left[x_3 \frac{\partial M_k}{\partial x_3}\right] - \frac{1}{\tau_4}E\left[x_4 \frac{\partial M_k}{\partial x_4}\right] + \frac{S_3}{2\tau_3^2}E\left[\frac{\partial^2 M_k}{\partial x_3^2}\right] \\ &\quad + \frac{S_4}{2\tau_4^2}E\left[\frac{\partial^2 M_k}{\partial x_4^2}\right] = 0, \quad M_k = x_1^i x_2^j x_3^m x_4^n, \quad k = i+j+m+n \end{aligned} \quad (11)$$

where M_k are the kth-order moment equations, and $E[\cdot]$ is the statistical mean of variable $[\cdot]$. Multivariable Gaussian moments of different orders needed in the multifold integration procedure of Eq. (9) can be computed by the statistical variables (m_i, σ_{ij}) obtained from Eq. (11). As a result, Eq. (9) is transformed to N_p nonlinear algebraic equations, which can be solved by numerical methods.

Besides, since there is no exact PDF solution, MCS with a sample size of 2×10^7 is performed to verify the efficiency of the EPC method.

3 Numerical Analysis and Discussion

It is discussed earlier that approximate PDF of the system response in the case of colored noise excitation is not accurate for all the values of the parameters used. Fronzoni et al. reported that the approximation is only good near the white noise limit and retains accuracy for large correlation time at small noise intensity [22].

In this example, system parameters are set as $\alpha = 0.2$, $\varepsilon = 1.0$, and $c_1 = 0.1$. Then, the corresponding relaxation time for the Duffing system is $\tau_{\text{rel}} = 2/\alpha = 10$. Important parameter correlated time τ_i, which are measured by comparison to the relaxation time τ_{rel}, with small values $\tau_3 = \tau_4 = 0.5$ are considered. But strong intensity levels $D_3 = D_4 = 2S_3 = 2S_4 = (1,\ 2)$ are involved. The EPC method and MCS method are employed separately to analyze such system. The results are shown in Figs. 1–4. It is explicitly found that the results from the EPC method agree well with the ones from the MCS method, especially those at the tail regions. It is further observed that the shape of the PDFs for displacement is obviously changed. It becomes more spread as the intensity of OU excitation increases.

Numerical results for cases with $\tau_3 = \tau_4 = 1.0$ are also computed, and the results are shown in Figs. 5 and 6. It is seen that the results from the EPC method still agree well with the ones from

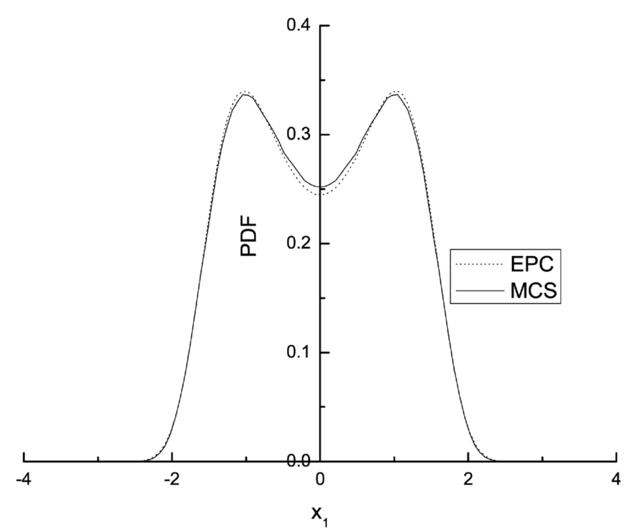

Fig. 1 PDF of displacement for Duffing system under OU processes with $\tau_3 = \tau_4 = 0.5$ and $D_3 = D_4 = 1$

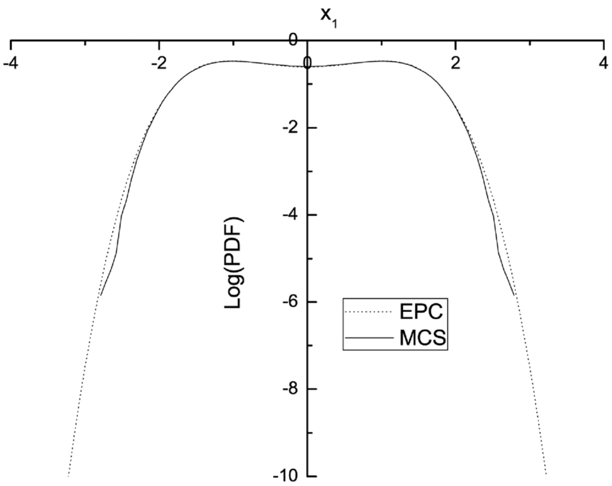

Fig. 2 Log(PDF) of displacement for Duffing system under OU processes with $\tau_3 = \tau_4 = 0.5$ and $D_3 = D_4 = 1$

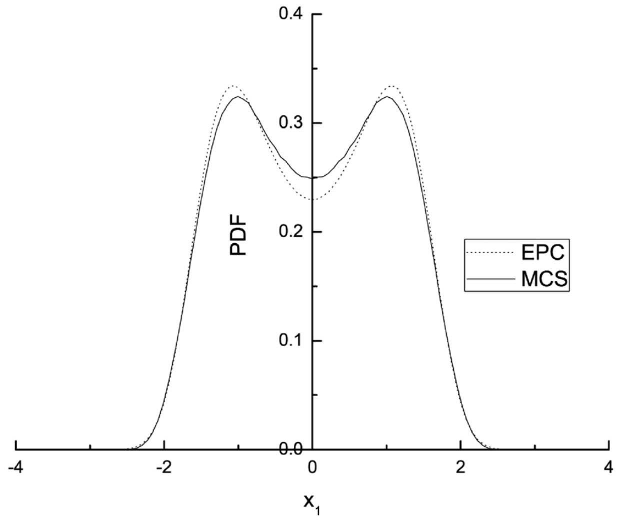

Fig. 5 PDF of displacement for Duffing system under OU processes with $\tau_3 = \tau_4 = 1$ and $D_3 = D_4 = 2$

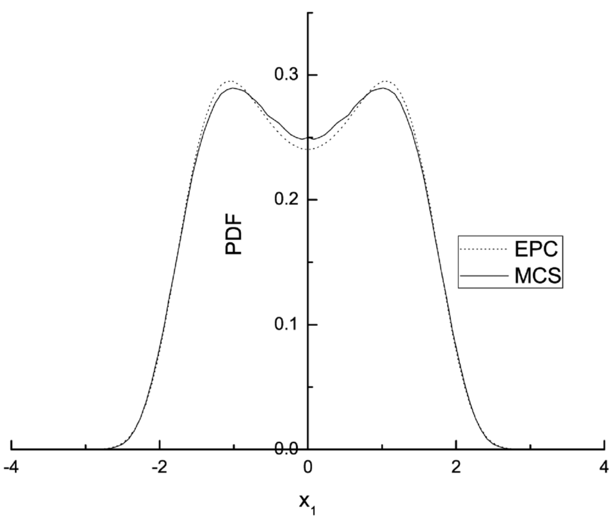

Fig. 3 PDF of displacement for Duffing system under OU processes with $\tau_3 = \tau_4 = 0.5$ and $D_3 = D_4 = 2$

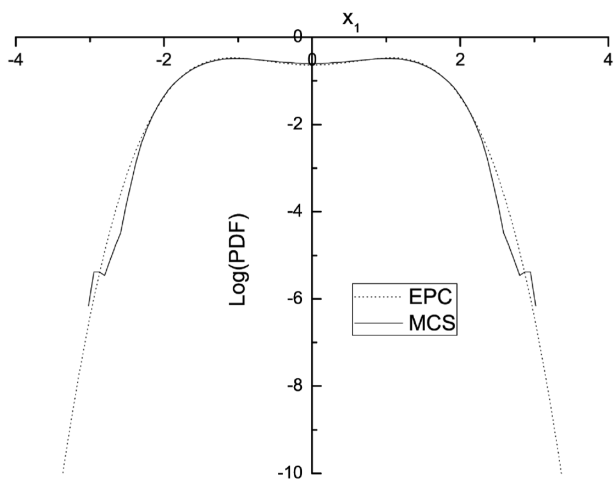

Fig. 6 Log(PDF) of displacement for Duffing system under OU processes with $\tau_3 = \tau_4 = 1$ and $D_3 = D_4 = 2$

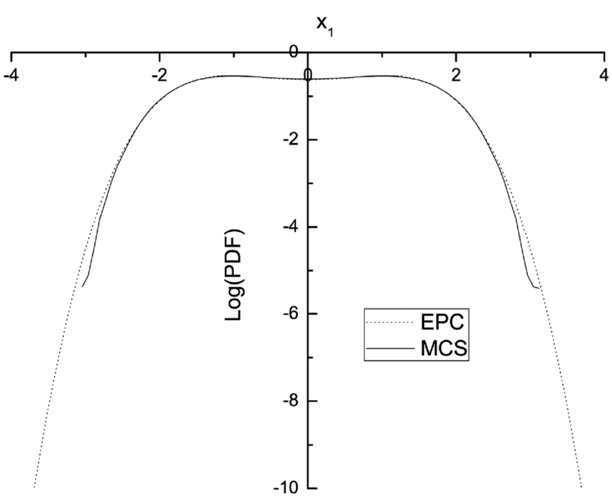

Fig. 4 Log(PDF) of displacement for Duffing system under OU processes with $\tau_3 = \tau_4 = 0.5$ and $D_3 = D_4 = 2$

the MCS method. As the correlated time becomes larger, the PDF of displacement becomes narrower. For the case of more colored noise with larger correlated time, it is found that unstable approximate solutions are obtained with the developed EPC method.

4 Conclusions

The behavior of a bistable Duffing oscillator driven by additive and multiplicative OU excitations is examined. The developed EPC method involves much more polynomial terms in the approximate solutions to solve such case associated with 4D FPK equations. Numerical results show that approximate solutions from the EPC method compare well with the numerical ones in the case of OU excitations with small correlated time at strong noise intensity. As the correlated time becomes large, the PDF distribution becomes narrow. For the case of more colored noise with larger correlated time, nonlinear algebraic equations in Eq. (9) for the vanishing of numerical errors associated with the FPK equation by the EPC method cannot be solved closely. Adjusted convergent condition may improve such situation or the EPC method may be not suitable for such case. The influence of OU excitation intensity level is also investigated. It alters the shape of PDF of displacement and makes the PDF more distributed. The developed EPC method can also be extended to the cases of nonlinear systems subjected to excitations in polynomial forms of filtered normal or non-normal delta-correlated processes.

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