ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.7(2009) No.1,pp.108-112

Recurrent Set and R Stability of Non-wandering Operator

Huan Qian *
Nonlinear Scientific Research Center, Jiangsu University, Zhenjiang, Jiangsu,212013, China
(Received 5 February 2008, accepted 21 March 2008)

Abstract: In the paper the stability of non-wandering operator on recurrent set is studied. By using the method of functional analysis, a sufficient condition for being non-wandering operator on recurrent set is given. Meanwhile, it is proved that invertible non-wandering operator on recurrent set is structural stable.

Keywords: non-wandering operator; recurrent set; structural stability; pseudo orbit tracing property; no-cycle condition

1 Introduction

It is well known that linear operators in finite-dimensional linear spaces can't be chaotic but the nonlinear operator may be. Only in infinite-dimensional linear spaces can linear operators have chaotic properties. This has attracted wide attention (see [1-6]). Non-wandering operators are new linear chaotic operators. They are relative to hypercyclic operators, but different from the later(see [5]). Some hypercyclic operators are not non-wandering operators (see [5] Remark 3.5 (2)). There also exists a non-wandering operator, which does not belong to hypercyclic operators(see [5], Remark 3.5 (3)). Hence they are different operators. Suppose that bounded linear operator T is invertible. If T is a hypercyclic operator, then $\sigma(T)\bigcap\partial D \neq \emptyset$ (see [8], Remark 4.3 (2)); if T is a non-wandering operator, then $\sigma(T)\bigcap\partial D = \emptyset$ where ∂D is unit circle (see [5], Theorem 4.2). When linear operator is not invertible, there exist operators being not only non-wandering operator but also hypercyclic operator. In recent years, the study of non-wandering operators has got a rapid progress. Jiangbo Zhou, etc discussed the hereditayily hypercyclic decomposition of non-wandering operators in infinite dimensional Frechet space (see [7]); Xun Liu, etc discussed non-wandering semigroup (see [8]); Shaoguang Shi, etc obtained the invariance of non-wandering operator under small perturbation (see[9]) and Lihong Ren, etc studied n-multiple non-wandering operator (see [10]).

The paper is organized as follows. In Section 2, the basic notations and definitions are listed. Then in Section 3, some properties about recurrent set is shown. And a sufficient condition for being non-wandering operator on recurrent set is given. Meanwhile, it is proved that invertible non-wandering operator on recurrent set is structural stable.

2 Basic notation and definitions

Let $(X, \|\cdot\|)$ be an infinite dimensional separable Banach space on real number field or complex number field K. Let $L(X)$ be the set of all bounded linear operators over X. N, Z, Q, R and C will be referred to as the sets of positive integers, rational numbers, and the real and complex scalar fields, respectively.

We introduce the following notations. For $y \in X$, Let $(X, \|\cdot\|)$ be an infinite dimensional separable Banach space on real number field or complex number field K.

We introduce the following notations. For $y \in X$, let

$$W^u_\eta(y) = \{x \in X | \|T^k(y-x)\| > \eta, (k = 0, 1, 2, \cdots) \lim_{k\to+\infty} \|T^{-k}(y-x)\| = 0\}$$

*E-mail address: qh109@163.com

IJNS.2009.02.15/206

$$W^s_\eta(y) = \{x \in X | \|T^k(y-x)\| < \eta, (k = 0,1,2,\cdots) \lim_{k\to+\infty} \|T^k(y-x)\| = 0\}$$
$$W^u(y) = \{x \in X | \lim_{k\to+\infty} \|T^{-k}(y-x)\| = 0\}, W^s(y) = \{x \in X | \lim_{k\to+\infty} \|T^k(y-x)\| = 0\}$$
$$W^u(E) = \{x \in X | \liminf_{k\to+\infty} \|T^{-k}(y-x)\| = 0, x \in E\}$$
$$W^s(E) = \{x \in X | \liminf_{k\to+\infty} \|T^k(y-x)\| = 0, x \in E\}.$$

Definition 1 *Suppose $T \in L(X)$. T is a linear chaotic operator or a linear chaotic map if it satisfies the following two conditions:*
(1) T is topologically transitive, i.e., T has a dense orbit in X.
(2) The set of periodic points $Per(T)$ for T is dense in X.

Definition 2 *(see [5]).Let $(X, \|\cdot\|)$ be an infinite dimensional separable Banach space.Suppose $T \in L(X)$. Then T is called to be a non-wandering operator relative to E if it satisfies:*
(1) There exists a closed subspace $E \subset X$, which has hyperbolic structure: $E = E^u \bigoplus E^s$,$TE^u = E^u$,$TE^s = E^s$, where E^u, E^s are closed subspaces. In addition, there exist constants $\tau(0 < \tau < 1)$ and $C > 1$, such that $\|T^k\xi\| \geq C\tau^{-k}\|\xi\|$, for any $\xi \in E^u, k \in N$; $\|T^k\eta\| \leq C\tau^k\|\eta\|$, for any $\xi \in E^s, k \in N$;
(2) $Per(T)$ is dense in E.

Remark 1 *(1) T may be invertible or not. When T is invertible, the spectral property of non-wandering operators is different from that of hypercyclic operators(see [5] Theorem 4.2). But when T is not invertible, the case is much complicated.*
(2) If T is a non-wandering operator, then $Per(T) \cap E = \emptyset$.(see [8] Remark 2.6).

Definition 3 *(see[5]). Let $E \subset X$ be a closed linear subspace of T. If there exist countable closed invariant subsets $E_1, E_2, \cdots, E_n, \cdots$ (any two of them are never intersected) such that $E = \cup_{i=1}^{\infty} E_i$. And for arbitrary nonempty open subsets $U, V \subset E_i$, there exists $n \in N$, such that $T^n U \bigcap V \neq \emptyset$. Then it is called the spectra decomposition of T for E, and $E_1, E_2, \cdots, E_n, \cdots$ are called basic sets.*

Definition 4 *Let T be a non-wandering operator relative to closed subspace $E \subset X$. Suppose that $\{x_i\}_{i=a}^b$ is a sequence in E and $\alpha > 0$. If for each $i = a, \cdots, b-1$ ($a = -\infty$ or $b = +\infty$ is also allowed), we have $\|Tx_i - x_{i+1}\| < \alpha$, then $\{x_i\}_{i=a}^b$ is supposed to be a $\alpha-$pseudo orbit of T. For a given $\beta > 0$, if there is $y \in E$ such that $\|T^i y - X_i\| < \beta$ for each $i = a, \cdots, b$, then we say that the $\alpha-$pseudo orbit is $\beta-$ traced by the orbit sending from y. T is called the pseudo orbit tracing property, if for each $\beta > 0$, there is $\alpha = \alpha(\beta) > 0$ such that each $\alpha-$pseudo orbit of can be $\beta-$traced by some point in E.*

Definition 5 *Let $E_i(T)(i = 1, 2, \cdots)$ be the basic sets of T for $E(T)$, we define the relationship "$\succ$" as follows:$E_i \succ E_j \Longleftrightarrow (W^u(E_i \setminus E_j)) \cap (W^s(E_j \setminus E_j)$. Moreover, $E_i(T)$ is called no-cycle if there aren't distinct indices such that $E_{i_1} \succ E_{i_2} \succ \cdots \succ E_{i_r} \succ\succ \cdots \succ E_{i_1}$.*

Remark 2 *If $i \neq j$, then $W^u(E_i) \bigcap E_j = \emptyset, E_i \bigcap W^s(E_j) = \emptyset$. If we define the relation "$\rhd$" : $E_i \rhd E_j \Leftrightarrow W^u(E_i \bigcap W^s(E_j)) \neq \emptyset$, then the no-cycle condition in Def. 2.5 turns to:*
(a) There aren't distinct indices $i_1, i_2, \cdots, i_r, \cdots$ such that $E_{i_1} \rhd E_{i_2} \rhd \cdots \rhd E_{i_r} \rhd \cdots E_{i_1}$
(b) $W^u(E_i) \bigcap W^s(E_i) = E_i (i = 1, 2, \cdots)$.

Definition 6 *Let $(X, \|\cdot\|)$ be an infinite dimensional separable Banach space. Suppose $T \in L(X)$. Point x is called recurrent point, if for $\forall \varepsilon > 0$, there exists periodic $\alpha-$pseudo orbit by point x. The set of all recurrent points of T is called recurrent set of T, denoted by $R(T)$.*

Definition 7 *Let $(X, \|\cdot\|)$ be an infinite dimensional separable Banach space. Suppose $T_1, T_2 \in L(X)$. If there exists a homeomorphism $\varphi : R(T_1) \to R(T_2)$, such that $\varphi \circ T_1 | R(T_1 = T_2 \circ \varphi)$, then T_1 is called $R-$conjugate to T_2.*

Definition 8 *Let $(X, \|\cdot\|)$ be an infinite dimensional separable Banach space. Suppose $T \in L(X)$. T is called $R-$stable, if for $\forall \varepsilon > 0$, there exists a neighborhood $B_\varepsilon(T)$ of T, such that for any $S \in B_\varepsilon(T)$, T and S is $R-$conjugate.*

Definition 9 *Let Λ is the set of all non-empty closed subset on X. We define Hausdorff distance as follows:* $D(A,B) = \sup_{x\in X} |d(x,A) - d(x,B)|$, *here* $d(x,A) = \inf_{y\in A} \|x-y\|$.

3 Non-wandering operator on recurrent set and stability

3.1 properties about recurrent set

Proposition 3 *The sufficient and necessary condition of $x \in R(T)$ is that for $\forall\delta > 0$, there exists periodic $\delta-$pseudo orbit of T in the $\delta-$ball field of x.*

Proof. The necessary is obvious. Then we will proof the sufficiency.

For $\forall\varepsilon > 0$, set $0 < \delta < \frac{\varepsilon}{3}$ such that $p,q \in X, \|p-q\| < \delta \Rightarrow \|Tp - Tq\| < \frac{\varepsilon}{3}$. Let $\{x_i\}_{-\infty}^{\infty}$ is the periodic $\delta-$pseudo orbit which satisfies $\|x_0 - x\| < \delta$ and its period is n. Set $x_i^{'} = x, i = kn, or x_i^{'} = x_i$. Next we will proof $\{x_i^{'}\}_{-\infty}^{\infty}$ is the periodic $\varepsilon-$pseudo orbit by point x, and its period is n.

There are two cases as follows:

(a)$n = 1$. Then $x_k = x_0, \forall k \in Z$. Thus

$$\|x_0 - x\| < \delta < \tfrac{\varepsilon}{3}, \|Tx_0 - x_0\| < \delta < \tfrac{\varepsilon}{3}, \|Tx - Tx_0\| < \delta < \tfrac{\varepsilon}{3},$$

So

$$\|Tx_0^{'} - x_0^{'}\| = \|Tx - x\| \le \|Tx_0 - Tx\| + \|Tx_0 - x_0\| + \|x_0 - x\| < \varepsilon.$$

(b)$n > 1$.Then

$$\|x_0 - x\| < \delta < \tfrac{\varepsilon}{3}, \|Tx - Tx_0\| < \tfrac{\varepsilon}{3},$$
$$\|Tx_0^{'} - x_1^{'}\| \le \|Tx - Tx_0\| + \|Tx_0 - x_1\| < \tfrac{\varepsilon}{3} + \delta < \varepsilon,$$
$$\|Tx_{n-1}^{'} - x_n^{'}\| = \|Tx_{n-1} - x\| \le \|Tx_{n-1} - x_0\| + \|x_0 - x\| < \delta + \delta < \varepsilon.$$

For two cases, we all proof: $\{x_i^{'}\}_{-\infty}^{\infty}$ is the periodic $\varepsilon-$pseudo orbit by point x. ■

Proposition 4 *$R(T)$ is closed set.*

Proof. Let $x \in \overline{R(T)}$. Then there exists periodic $\varepsilon-$pseudo orbit of T in the $\varepsilon-$ball field of x. From Proposition 3.1, we know $x \in R(T)$. ■

Proposition 5 *Let T is invertible, then $R(T)$ is invariant set of T.*

Proof. For $\forall\varepsilon > 0$, set $\delta > 0$, such that $p,q \in X, \|p-q\| < \delta \Rightarrow \|Tp - Tq\| < \varepsilon$. Let $x \in R(T)$. Then there exists periodic $\delta-$pseudo orbit $\{x_i\}_{-\infty}^{\infty}$ by point x. Since $\|Tx_i - x_{i+1}\| < \delta \Rightarrow \|T(Tx_i) - Tx_{i+1}\| < \varepsilon$, we have $\{Tx_i^{'}\}_{-\infty}^{\infty}$ is periodic $\varepsilon-$pseudo orbit by point Tx. So $T(R(T)) \subset R(T)$. On the other hand, we have $T^{-1}(R(T)) = T^{-1}(R(T^{-1})) \subset R(T^{-1}) = R(T)$. Therefore $T(R(T)) = R(T)$. ■

3.2 Non-wandering operator on recurrent set

Lemma 6 *(see[11] Theorem 3.4) Let T be an invertible non-wandering operator relative to closed subspace $E \subset X$, then T has pseudo orbit tracing property.*

Remark 7 *The proof of this lemmas is not relative to the second condition of non-wandering operator–" $Per(T)$ is dense in E", so we can use this lemmas here.*

Lemma 8 *Let T is invertible, then for any $x \in R(T)$, there exists periodic pseudo orbit $\{x_i\}$ by this point, and $x_i \in R(T)$.*

Proof. Let $x_0 \in R(T)$. Since $R(T)$ is invariant set, $T(x_0) \in T(R(T)) = R(T)$. From Proposition 3.1,we know: for $\forall\varepsilon > 0$, there exists $x_1 \in U(T(x_0, \varepsilon))$ such that $\|T(x_0) - x_1\| < \varepsilon$. For x_1, we also have $x_1 \in R(T)$, thus there exists $x_2 \in U(T(x_1, \varepsilon))$ such that $\|T(x_1) - x_2\| < \varepsilon$. Using the same method, we can get $\{x_i\}_0^{\infty}$. Now we will proof $\{x_i\}_0^{\infty}$ is convergent.

In fact, since T is bounded, we only consider $\|T\| < 1$. From above, we know $\|Tx_{i-1} - x_i\| < \varepsilon, (i =$

$1, 2, \cdots$). Thus

$$\|T^n x_0 - x_n\| < (1 + \|T\| + \cdots + \|T\|^{n-1})\varepsilon, (n = 1, 2, \cdots).$$

So for N large enough , when $n > m > N$, we have

$$\|x_n - x_m\| < (1 + \|T\| + \cdots + \|T\|^n + 1 + \|T\| + \cdots + \|T\|^m)\varepsilon + \|T\|^m \|T^{n-m}x_0 - x_0\|.$$

For the first part, when N is large enough,

$$1 + \|T\| + \cdots + \|T\|^n + 1 + \|T\| + \cdots + \|T\|^m \leq M_1.$$

For the second part, when N is large enough,

$$\|T\|^m < \varepsilon, \|T^{n-m}x_0 - x_0\| \leq M_2.$$

Hence

$$\|x_n - x_m\| < (M_1 + M_2)\varepsilon.$$

Set $\eta = (M_1 + M_2)\varepsilon$. Then there exists N, when $n > m > N$, $\|x_n - x_m\| < \eta$ holds. Therefore $\{x_i\}_0^\infty$ is convergent. Since $R(T)$ is closed, there exists $x^* \in R(T)$ such that $x_n \to x^*$. For ε above, there exists N, when $n \geq N$, $\|x_n - x^*\| <\frac{\varepsilon}{\|T\|}$ holds. Particularly, set $n = N$, we have

$$\|Tx_N - Tx^*\| \leq \|T\|\|x_N - x^*\| < \varepsilon.$$

For point x_0, Tx^*, from the closed invariant property of $R(T)$, there exists an orbit which approaches this two point. Thus there exists $y \in R(T), s, t \in N, (s < t)$, such that

$$\|T^s y - x_0\| < \varepsilon \ , \|T^t y - Tx^*\| <\frac{\varepsilon}{\|T\|}.$$

So the periodic $\varepsilon-$pseudo orbit from $x_0, x_1, \cdots, x_N, Tx^*, T^{t+1}y, \cdots, T^{s+1}y$ passes by point x_0. ■

Theorem 9 *Let T has hyperbolic structure on $R(T)$, then $\overline{Per(T)} = \Omega(T) = R(T)$, therefore T is a non-wandering operator on $R(T)$.*

Proof. For any $x \in R(T)$, from Lemma 3.2 : there exists periodic $\alpha-$pseudo orbit $\{x_i\} \subset R(T)$ by point x, and let its period is m. Besides, from Lemma 3.1, there exists orbit $\{f^i y\}_{i=0}^\infty$ by point y $\beta-$traced $\{x_i\}$. Then

$$\|T^i T^m y - T^i y\| \leq \|T^{i+m}y - x_{i+m}\| + \|x_i - T^i y\| \leq 2\beta, \forall i \in Z.$$

And we let β is small enough, then $T^m y = y$, that is to say $y \in Per(T)$. Hence for any $x \in R(T)$ and β small enough , there exists $y \in Per(T)$ such that $\|x - y\| < \beta$. Therefore we proof $\overline{Per(T)} = R(T)$, so T is a non-wandering operator on $R(T)$. ■

Lemma 10 *Let $(X, \|\cdot\|)$ be an infinite dimensional separable Banach space. Suppose $T \in L(X)$ satisfies Axiom A[12], Ω_i is a basic set of T, $y, z \in \Omega_i$, $\zeta > 0$ is a given real number. Then there exists periodic point $p \in \Omega_i$ of T and $k \in N$ such that $\|p - y\| < \zeta, \|T^k p - z\| < \zeta$.*

Proof. Let orbit sending from the point $x \in \Omega_i$ is dense in Ω_i. Then there exists $m, n \in Z$ such that

$$\|T^m x - y\| <\frac{\zeta}{2}, \|T^n x - z\| <\frac{\zeta}{2}.$$

Besides since periodic point is also dense in Ω_i, there exists periodic point $p \in \Omega_i$ of T such that

$$p \in B(T^m x, \frac{\zeta}{2}) \bigcap T^{-(n-m)} B(T^n x, \frac{\zeta}{2}).$$

Let period of p is h and $k \in N$. Set $k = n - m$. Then $\|p - T^m x\| <\frac{\zeta}{2}, \|T^k p - T^n x\| <\frac{\zeta}{2}$. Therefore $\|p - y\| < \zeta, \|T^k p - z\| < \zeta$. ■

Theorem 11 *Let T has hyperbolic structure on $R(T)$, then $\Omega(T) = R(T)$ satisfies no-cycle condition.*

Proof. Let there exists cycle of basic sets $\Omega(T) = R(T) = \cup_{i=1}^\infty \Omega_i$: $\Omega_1 \succ \Omega_2 \succ \cdots \succ \Omega_r \succ \Omega_{r+1} \succ \cdots \succ \Omega_\infty = \Omega_1$. Then we have

$$q_i \in W^u(\Omega_i) \cap W^s(\Omega_{i+1}), (i = 1, 2, 3, \cdots).$$

Thus for any $\varepsilon > 0$, there exists large enough $m \in N$ such that

$$d(T^{-m}q_i, \Omega_i) <\frac{\varepsilon}{2}, d(T^m q_i, \Omega_{i+1}) <\frac{\varepsilon}{2}, (i = 1, 2, 3, \cdots).$$

From Lemma 3.3 : there exists $p_i \in \Omega_i$ and $k_i \in N$ such that

$$\|p_i - T^m q_{i-1}\| <\frac{\varepsilon}{2}, (i = 2, 3, 4, \cdots) \qquad (*), \qquad \|T^{k_i} p_i - T^{-m} q_i\| <\frac{\varepsilon}{2}, (i = 1, 2, 3, \cdots).$$

On the other hand, let $\Omega_n^* = \cup_{i=n}^\infty \Omega_i$, then $\lim_{n\to\infty} \cup_{i=n}^\infty \Omega_i = \Omega_\infty = \Omega_1$. Thus for $\varepsilon > 0$ above, exists N, when $n > N$, $D(\Omega_n^*, \Omega_1) <\frac{\varepsilon}{2}$ holds. Particularly, set $n = N + 1$, we have $D(\Omega_{N+1}^*, \Omega_1) <\frac{\varepsilon}{2}$. And set $i = N + 1$ in $(*)$, we get $\|p_{N+1} - T^m q_N\| <\frac{\varepsilon}{2}$.

For $p_{N+1} \in \Omega^*_{N+1}$, since $D(\Omega^*_{N+1}, \Omega_1) < \frac{\varepsilon}{2}$, we can find x^* in Ω_1 such that $\|p_{N+1} - x^*\| < \frac{\varepsilon}{2}$. Hence

$$\|T^m q_N - x^*\| \leq \|T^m q_N - p_{N+1}\| + \|p_{N+1} - x^*\| < \varepsilon.$$

And since Ω_1 is invariant set, for $q_1 \in \Omega_1$, then $T^j q_1 \in \Omega_1, (j \in Z)$. Let orbit sending from the point $y \in \Omega_i$ is dense in Ω_i, then there exists $s, t \in N, (s < t)$ such that

$$\|T^s y - x^*\| < \frac{\varepsilon}{\|T\|}, \qquad \|T^t y - T^{-m} q_1\| < \varepsilon.$$

Therefore we get a periodic pseudo orbit from following points:

$x^*, T^{s+1}y, T^{s+2}y, \cdots, T^{t-1}y, T^{-m}q_1, \cdots, q_1, \cdots, T^m q_1; p_2, \cdots, T^{k_2}p_2; T^{-m}q_2, \cdots, q_2, \cdots, T^m q_2;$ $\cdots\cdots; p_N, \cdots, T^{k_N}p_N; T^{-m}q_N, \cdots, q_N, \cdots, T^{m-1}q_N.$

Since N can be large enough, so for any $\varepsilon > 0$ and all N, there exists periodic pseudo orbit by point q_i, that is to say, $q_i \in R(T) = \Omega(T), (i = 1, 2, 3, \cdots)$, but this is contradictive to the choosing method of q_i, so we can get the conclusion. ■

Acknowledgements

Research was supported by the National Nature Science Foundation of China (No: 10771088) and Nature Science Foundation of Jiangsu (No: 2007098) and Outstanding Personnel Program in Six Fields of Jiangsu (No: 6-A-029).

References

[1] C. Kitai: Invariant closed sets for linear operators. *Ph. D. thesis, University of Toronto*. Ontario(1982)

[2] X.-C. Fu, J. Duan: Infinite-dimensional linear dynamical systems with chaoticity. *J. Nonlinear Sci.*. 9(2): 197-211(1999)

[3] P.S. Bourdon: Invariant manifolds of hypercyclic vectors. *Proc. Amer. Math. Soc*. 118(2): 587-595(1993)

[4] C. R. MacCluer: Chaos in linear distributed systems. *J. Dyn. Syst. Meas. Control*. 114(2): 322-324(1992)

[5] Lixin Tian, Jiangbo Zhou, Xun Liu, Guangsheng Zhong: Non-wandering operators in Banach space.*International Journal of Mathematics and Mathematical Sciences*. 24(2005): 3895-3908

[6] Lixin Tian, Dianchen Lu: The property of nonwandering operator. *Appl Math Mech* . 17: 155-161(1996)

[7] Jiangbo Zhou, Lixin Tian, Dianchen Lu: The hereditarily decomposition of non-wandering operators in infinite dimensional Frechet space. *J Jiangsu University*. 22(6): 88 - 91(2001)

[8] Xun Liu, Lixin Tian: The property of non-wandering semigroup. *J Jiangsu University*. 23(5): 9-12(2002)

[9] Lixin Tian, Shaoguang Shi, Lihong Ren: The invariance of nonwandering operator under small perturbation. *International Journal of Nonlinear Science*. 2(28):(2008).

[10] Lixin Tian, Lihong Ren: N-multiple nonwandering unilateral weighted backward shift operators and the property of direct sum operators in Banach space, *International Journal of Nonlinear Science*. 2(2): 104-110(2006)

[11] Lixin Tian, Minggang Wang: Pseudo orbit tracing property of non-wandering operator. *International Journal of Nonlinear Science.* 1(3): 3-7 (2007)

[12] Zhusheng Zhang: Differential dynamical system. *Beijing, Science Press* (2003)