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Symmetric and Asymmetric Matching of Joint Presentations

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The global psychophysical theory of summation and magnitude production of R. D. Luce (2002) had joint presentations of pairs of intensities (measured above threshold) being matched asymmetrically, with 1 component being 0 intensity and the other the matching intensity. For loudness, an intensity pair to the 2 ears is matched by an intensity in just 1 ear. Realizing this experimentally has been difficult, and so, this article extends the theory to the use of symmetric matches with the same intensity being used in both components. In addition, the representational aspect is much improved; a new formulation of the results of the earlier theory is presented; the theory for symmetric matches is outlined; and it is shown that if 1 form of segregation, right or left, holds for asymmetric matches and 1 for symmetric ones, then all forms of segregation hold.

In experiments motivated by and designed to test Luce's (2002) psychophysical theory of magnitude productions and joint presentations (Steingrimsson & Luce, 2003a, 2003b), a difficulty has been encountered in some experimental realizations. This article attempts to deal with this difficulty.

The motivation included in the earlier article is not repeated, but the notation and underlying assumptions are summarized. That article, as is this one, was based on stimuli of the form (x, u), where x and u are physical intensities measured as an intensity difference between the presented physical intensity and the relevant threshold intensity. (The thresholds of the two components of the presentation may differ.) Using intensity differences is a bit unusual and must be distinguished from using the more usual ratio of the two intensities, the latter of which becomes a difference in decibels. Assuming that subjective matches are described by an equivalence relation $\sim$, let z_l and z_r be solutions to the subjective matches

$$(x, u) \sim (z_l, 0) \sim (0, z_r).$$

The theory was developed in terms of these solutions using the operator notation

$$x \oplus_l u := z_l, \quad x \oplus_r u := z_r.^1$$

In an empirical realization using earphones with intensity x to the left ear and u to the right ear, Steingrimsson (2002) and Steingrimsson and Luce (2003a) were plagued by the relatively severe localization differences between (x, u) and either of the asymmetric matches $(z_l, 0)$ or $(0, z_r)$, which respondents found difficult to ignore. So, a symmetric match of the following form was needed: For each $x \geq 0$ and $u \geq 0$, there exists $z_s \geq 0$ such that

$$(x, u) \sim (z_s, z_s).$$

Define $\oplus_s$ by

$$x \oplus_s u := z_s.$$

This called for developing a theory for symmetric matches, which I do here.

Karin Zimmer and Wolfgang Ellermeier (personal communication, September 28, 2001) interpreted (x, u) to mean that intensity x is presented to both ears for a brief (50-ms) duration followed immediately by u to both ears for 50 ms. People seem automatically to "sum" these immediately successive brief signals to form a single sensation of loudness. The difficulty was that with a 500-ms break between the presentation of such a stimulus pair and the matching one, respondents could not distinguish between $(z, 0)$ and $(0, z)$, in part because there were no markers for the intervals. So, Zimmer and Ellermeier requested a theory that involved symmetric matches because they really could not apply the original theory. This was a further call for a theory of symmetric matches.

In addition, the original theory was stated entirely in terms of properties of either $\oplus_l$ or $\oplus_r$. I failed to ask what would happen if both sets of assumptions held at the same time, and so, that is also taken up.

Finally, I consider what happens when one of the asymmetric segregation conditions, right or left, holds as does one of the symmetric ones. I found that all conditions hold, reducing the results to one version of the symmetric theory.

Asymmetric Matching: Recapitulation and Reformulation

Luce (2002) assumed that in the pair (x, u) each signal is the physical measure of intensity minus the respondent's threshold intensity. In that case, 0 denotes the threshold or any signal less than that. Such a discontinuous signal threshold clearly is an idealization, which means that the model is not quite correct near

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[1] The notation $X := Y$ means that X is defined to be equal to Y.

threshold. This is mainly because the theory is not probabilistic. In particular, the fact that subthreshold signals may sum to super-threshold sounds cannot be accommodated. Alternatively, we may let 0 literally mean 0 intensity, in which case the subjective measures are positive at the behavioral threshold. As a practical matter, it little matters which we use because for intensities corresponding to 30-dB sound pressure level or greater, there is only a tiny decibel difference between the two measures. Equally, there is no room in this theory for differential thresholds.

Therefore, the space of signal-pair presentations is $\mathbb{R}_+ \times \mathbb{R}_+$, where $\mathbb{R}_+$ denotes nonnegative real numbers. Over these pairs, there is a relation $\succ$ of greater subjective intensity (e.g., loudness) and a relation $\sim$ of subjective equality. The union of $\sim$ and $\succ$ is denoted $\succsim$. In terms of signals, $\geq$ on $\mathbb{R}_+$ is the physical order of greater intensity.

Basic Assumptions

Assumption 1: Equivalence relation. The relation $\sim$ on $\mathbb{R}_+ \times \mathbb{R}_+$ is an equivalence relation; that is, for all $(x, u), (y, v), (z, w) \in \mathbb{R}_+ \times \mathbb{R}_+$, it is *transitive,*

$$(x, u) \sim (y, v) \text{ and } (y, v) \sim (z, w) \Rightarrow (x, u) \sim (z, w), \quad (1)$$

symmetric,

$$(x, u) \sim (y, v) \Leftrightarrow (y, v) \sim (x, u), \quad (2)$$

and *reflexive,*

$$(x, u) \sim (x, u). \quad (3)$$

Assumption 2: Compatibility of $\succsim$ and $\geq$. For all $x, y, u, v \in \mathbb{R}_+$,

$$(x, u) \succsim (y, u) \Leftrightarrow x \geq y, \quad (4)$$

and

$$(x, u) \succsim (x, v) \Leftrightarrow u \geq v. \quad (5)$$

Assumption 3: Solvability. For each $x, u \in \mathbb{R}_+$, there exist intensities—denoted $x \oplus_l u$, $x \oplus_r u$, and $x \oplus_s u \in \mathbb{R}_+$—such that the three induced operations exist, that is,

$$(x, u) \sim (x \oplus_l u, 0) \sim (0, x \oplus_r u) \sim (x \oplus_s u, x \oplus_s u). \quad (6)$$

It is not difficult to show that each of $\oplus_l$, $\oplus_r$, and $\oplus_s$ is a binary operation on $\mathbb{R}_+$ (closed under repeated applications) and that $\succsim$ is a weak order on $\mathbb{R}_+ \times \mathbb{R}_+$ (Proposition 1 of Luce, 2002). Each operation encodes in the intensity dimension the information contained in the ordering of pairs, that is, in the conjoint structure $\langle \mathbb{R}_+ \times \mathbb{R}_+, \succsim \rangle$. Luce (2002) studied only the first two operations.

Note that by Proposition 1 of Luce (2002), (x, u) is strictly increasing in each variable, and therefore, so is $x \oplus_s y$. Applying this fact to $(x, x) \sim (x \oplus_s x, x \oplus_s x)$ shows that $\oplus_s$ is idempotent in the sense that

$$x \oplus_s x = x. \quad (7)$$

Moreover, the intensity 0 is neither a left nor a right identity of the operation $\oplus_s$. This is quite different from the earlier theory, where 0 is the right identity of $\oplus_l$ and the left identity of $\oplus_r$.

Subjective Proportions

A key aspect of the theory is existence of a psychophysical measure of subjective intensity and a weight function that permits one to formulate a possible representation of magnitude and ratio productions in a generalized sense. Suppose that x, y are intensities and $x \geq y$. Let p be any positive number. And let $z > y$ denote the signal that the respondent judges to have the following property: The subjective "distance" from (y, y) to (z, z) stands in the ratio p to the subjective distance from (y, y) to (x, x). It is convenient to denote the respondent's choice by

$$(x, x) \circ_p (y, y) := (z, z). \quad (8)$$

This form can be thought of as a possible generalization of Stevens's (1975) procedure of *magnitude production,* which is the case where $y = 0$. He felt that naive people performed better using that simpler form than the more complex one used here. I do not know how to get the following theoretical results under that limitation.

If $x > y$ and $z > y$, then the respondent can also be asked to state the number $p = p(x, y, z)$ such that the distance from (y, y) to (z, z) stands in the ratio p to the distance from (y, y) to (x, x). This is a *ratio estimation.* It is closely related to Stevens's (1975) method of *magnitude estimation,* which also assumed $y = 0$ and, in his favorite version, does not even state x explicitly, letting the respondent establish his or her own "standard." These issues are explored more thoroughly by Steingrimsson and Luce (2004).

By Assumption 3 and Proposition 1 of Luce (2002), it follows immediately that if (x, u) and (y, v) are signal pairs such that $(x, u) \succ (y, v)$ and $p > 0$, the respondent can produce a stimulus of the form (z, w) such that

$$(x, u) \circ_p (y, v) := (z, w).$$

Note that if $(y, v) \sim (y', v')$, then because $(z, z) \sim (y, v) \sim (y', v') \sim (z', z')$ implies $z = z'$, we see that $(x, u) \circ_p (y, v) \sim (x, u) \circ_p (y', v')$.

Luce (2002) considered only the special cases $u = v = w = 0$, in which everything is mapped to the left signal, and $x = y = z = 0$, in which everything is mapped to the right signal. This amounted to working with the following identifications:

$$(x \circ_{p,l} y, 0) = (x, 0) \circ_p (y, 0) \quad (9)$$

and

$$(0, u \circ_{p,r} v) = (0, u) \circ_p (0, v). \quad (10)$$

The following assumptions are made about Equation 8.

Assumption 4: Properties of $\circ_p$. The operation $\circ_p$ satisfies the following properties.

1. Left strictly increasing monotonicity:
$$x \geq x' \Leftrightarrow (x, x) \circ_p (y, y) \succsim (x', x') \circ_p (y, y).$$
2. Nonconstancy: For each fixed x, $\circ_p$ is not constant on a nontrivial y interval.[2]
3. Continuity: For each fixed y, $\circ_p$ is a continuous function

[2] In the functional equation literature, this is called *right philandering.*

of $x > 0$; for each fixed x, $\circ_p$ is a continuous function of $y > 0$.

Note that if we assume that for each $z > y$ there exists a p such that $(x, x) \circ_p (y, y) = (z, z)$, then because the domain is positive real numbers and monotonicity holds, the continuity of $(x, x) \circ_p (y, y)$ with x for fixed y follows.

Assumption 5: Idempotence of $\circ_p$.

$$(x, x) \circ_p (x, x) = (x, x).$$

Dzhafarov's Reformulation of the Representations

The theory of Luce (2002) was formulated just in terms of $\oplus_i$, $\circ_{p,i}$, and two functions $\psi_i : \mathbb{R}_+ \xrightarrow{onto} \mathbb{R}_+$, $i = l, r$, were constructed and shown to have certain properties. No common function $\Psi : \mathbb{R}_+ \times \mathbb{R}_+ \xrightarrow{onto} \mathbb{R}_+$ was assumed to lie behind these induced functions. As a result, I did not (and could not) prove some of the additional results that are given below. This more general formulation, which both Ehtibar Dzhafarov[3] and, independently, A. A. J. Marley (personal communications, 2001) strongly recommended, turns out to have considerable advantages. Many of these advantages Dzhafarov pointed out to me.

To that end, let us define

$$\Psi(x, 0) := \psi_l(x), \tag{11}$$

and

$$\Psi(0, u) := \psi_r(u). \tag{12}$$

Using solvability, Equation 6, we see that Ψ can be extended to the full domain $\mathbb{R}_+ \times \mathbb{R}_+$. Note that

$$\Psi(0, 0) = 0. \tag{13}$$

A common property of many nonprobabilistic theories (e.g., in classical physics) is that the representation is decomposable in terms of the representations of its components. In the present situation, this may be formulated as follows.

Assumption 6: Decomposability. There exist functions F, $G_p : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\Psi(x, u) = F[\Psi(x, 0), \Psi(0, u)]$$

and

$$\Psi[(x, u) \circ_p (y, v)] = G_p[\Psi(x, u), \Psi(y, v)].$$

Because $\Psi(x, 0)$, $\Psi(0, x)$, and $\Psi(x, x)$ all preserve the order $\gtrsim$, they are related by strictly increasing functions f, g such that

$$\Psi(x, 0) = f[\Psi(x, x)],\ \Psi(0, x) = g[\Psi(x, x)].$$

Thus,

$$\begin{aligned}\Psi(x, u) &= F[\Psi(x, 0), \Psi(0, u)] \\ &= F(f[\Psi(x, x)], g[\Psi(u, u)]) \\ &= F^*[\Psi(x, x), \Psi(u, u)],\end{aligned}$$

where $F^*(X, U) = F[f(X), g(U)]$. Thus, $\Psi(x, u)$ is equally well decomposable in the symmetric terms $\Psi(x, x)$ and $\Psi(u, u)$.

The following is Dzhafarov's[4] reformulation of the representations obtained in Theorem 1 of Luce (2002).

Theorem 1. Suppose that $\langle \mathbb{R}_+ \times \mathbb{R}_+, \gtrsim, \circ_p \rangle$, $p > 0$, satisfies Assumptions 1–6 above using only asymmetric matches. Then, the representation of either ψ_l or ψ_r of Theorem 1 of Luce (2002) is equivalent to the existence of functions $\Psi : \mathbb{R}_+ \times \mathbb{R}_+ \xrightarrow{onto} \mathbb{R}_+$ and $W : \mathbb{R}_+ \xrightarrow{onto} \mathbb{R}_+$, with $W(1) = 1$, that are strictly increasing in each argument, of either continuous $\sigma_l(x)$ or continuous $\sigma_r(u) : \mathbb{R}_+ \xrightarrow{into} \mathbb{R}_{++}$, respectively, and of a constant $\gamma > 0$ such that, for all $x, y, z \in \mathbb{R}_+$, with $x > y$, either

$$\Psi(x, u) = \Psi(x, 0) + \sigma_l(x)\Psi(u, 0) \tag{14}$$

or

$$\Psi(x, u) = \sigma_r(u)\Psi(0, x) + \Psi(0, u), \tag{15}$$

respectively, and

$$\Psi(x, 0) = \gamma\Psi(0, x) \tag{16}$$

and

$$W(p) = \frac{\Psi[(x, x) \circ_p (y, y)] - \Psi(y, y)}{\Psi(x, x) - \Psi(y, y)}. \tag{17}$$

If both ψ_l and ψ_r hold, then there exists a constant $\delta \geq 0$ such that

$$\Psi(x, u) = \Psi(x, 0) + \Psi(0, u) + \delta\Psi(x, 0)\Psi(0, u). \tag{18}$$

See the Appendix for proofs.

I comment on each of these three representations. Although the first two representations, Equations 14 and 15, are not by themselves of great interest, they do each force Equation 16, which is called the *constant-ratio condition.* This is a very strong prediction of the theory of asymmetric matches. It says that there is a consistent bias in the following sense: For all $x > 0$,

$$\text{left bias, } (x, 0) \succ (0, x) \Leftrightarrow \gamma > 1;$$

$$\text{no bias, } (x, 0) \sim (0, x) \Leftrightarrow \gamma = 1;$$

and

$$\text{right bias, } (x, 0) \prec (0, x) \Leftrightarrow \gamma < 1.$$

Moreover and somewhat startling, the bias is constant in the sense of being independent of signal level. Luce (2002) assumed a consistent direction to the bias independent of signal level, not realizing that it and much more were provable.

Steingrimsson and Luce (2003a) reported 2 of 6 respondents with opposite biases depending on whether right or left matches were used, which is not consistent with the theory of a common Ψ unless the theory is somehow modified. In the unmodified form, we would have to assume that when using left matches, we have Ψ_l and γ_l, and when using right ones, Ψ_r and γ_r. An alternative account with a common Ψ is given by Steingrimsson and Luce (2004). It is based on the idea that the asymmetric matches induce a constant attenuation of all signal intensities in the opposite ear. The existence of this behavior was additional motivation for de-

[3] He first recommended using Ψ in March 2001 in his review of Luce (2002). I did not appreciate the power of doing so until he recommended it again in a further review of this article. There, he pointed out what is formulated below as Theorem 1.

[4] From his 2003 review of this article.

veloping the theory for symmetric matches (see Theorems 2 and 3, below) in which such shifts, which may have to do with localization interfering with judgments, are not as much an issue.

The constant-ratio condition means that the theory applies only to individuals whose ears differ subjectively above thresholds only by a constant factor. Steingrimsson and Luce (2004) give an argument based on Equation 17 that leads to the representation

$$\Psi(x, u) = ax^{\beta_l} + bu^{\beta_r}.$$

The constant-ratio condition excludes this except for $\beta_l = \beta_r = \beta$ and $a = \gamma b$.

It would be desirable to verify empirically the constant-ratio condition, but I do not really know how to do so very effectively. No purely behavioral condition equivalent to it has yet been found. At present, it is difficult to see how to test it without doing one of two things. First, we can attempt to estimate the functions $\Psi(x, 0)$ and $\Psi(0, x)$ as functions of x and then determine if $\Psi(x, 0)/\Psi(0, x)$ is independent of x. If the functions are power ones and $\varphi(x)$ is defined by the solution $(0, \varphi(x)) \sim (x, 0)$, then the constant-ratio property is equivalent to, in dB measures,

$$\varphi(x)_{\text{dB}} = x_{\text{dB}} + \frac{10 \log \gamma}{\beta},$$

independent of intensity level. That prediction is easily checked experimentally, although so far it has not been. Second, we can ask the respondent to judge whether the subjective ratio of $(x, 0)$ to $(0, x)$ is the same for several values of x. Steingrimsson (2002) did a version of the latter with somewhat inconclusive results. Should one of these two approaches, or some other, lead to a rejection of the constancy, Equation 16, then neither the left nor the right asymmetric matching theory holds. Fortunately, the symmetric matching theory developed below does not force the constant-ratio condition.

The form for general $\Psi(x, u)$ of Equation 18 is called *p-additive*, where p is short for polynomial.[5] The term *additive* arises for the following reason. For $\delta = 0$, it simply is additive; for $\delta > 0$, $\Theta(x, u) = \ln[1 + \delta\Psi(x, u)]$ is additive in the sense that

$$\Theta(x, u) = \Theta(x, 0) + \Theta(0, u). \tag{19}$$

Recall that Equation 19 is the representation of additive conjoint measurement and that in addition to monotonicity of $(\cdot, \cdot)$, its most important necessary property is the Thomsen condition: For all $x, y, z, u, v, w \in \mathbb{R}_+$,

$$\left.\begin{matrix}(x, v) \sim (y, w)\\ (y, u) \sim (z, v)\end{matrix}\right\} \Rightarrow (x, u) \sim (z, w). \tag{20}$$

Indeed, under our structural conditions, these properties are sufficient to establish Equation 19 (see Krantz, Luce, Suppes, & Tversky, 1971).

Clearly, it is important to verify Equation 20 experimentally. The literature is quite inconsistent in that Falmagne, Iverson, and Marcovici (1979) and Levelt, Riemersma, and Bunt (1972) supported it, whereas Falmagne (1976), with but 1 respondent, and Gigerenzer and Strube (1983), with 12 respondents, rejected it. In all cases, the domain was auditory pairs. Because of this inconsistency, Steingrimsson and Luce (2003a) have studied it again and found supportive results.

An examination of Luce's (2002) proof, which of course involves ψ_l and ψ_r, shows that δ can be either nonnegative or negative. In the latter case, Ψ is bounded by $1/|\delta|$ (see Luce, 2000). For that case, subjective proportions, Equation 17, are not defined generally but only for those p that maintain

$$0 < \Psi((x, x) \circ_p (y, y), (x, x) \circ_p (y, y)) < 1/|\delta|.$$

Because I do not know of a principled way to formulate that constraint, the statements here are limited to $\delta \geq 0$. It would be desirable to overcome this restriction because a bounded psychophysical function has considerable intuitive appeal.

The third part of the representation, Equation 17, is called a *subjective-proportion representation.* Note that it with Equation 6 implies that

$$\frac{\Psi[(x, u) \circ_p (y, v)] - \Psi(y, v)}{\Psi(x, u) - \Psi(y, v)} = W(p).$$

When Equation 17 holds, we can easily show from it that the operation $\circ_p$ defined by Equation 8 satisfies the important behavioral property

$$((x, x) \circ_p (y, y)) \circ_q (y, y) \sim ((x, x) \circ_q (y, y)) \circ_p (y, y), \tag{21}$$

which is called *proportion commutativity.* Ellermeier and Faulhammer (2000) explored this property empirically, using median responses, for the case of $y = 0$. It was not rejected for the (p, q), $p > 1$, $q > 1$, pairs that they used. This property in the present context is sufficient to derive the subjective-proportion representation. Steingrimsson and Luce (2003a) also studied it and also did not reject it.

Behavioral Properties Equivalent to the Representation

Although, within the framework of these assumptions, the Thomsen condition (Equation 20) is sufficient to construct a p-additive representation, and proportion commutativity is sufficient to construct a subjective-proportion representation (Equation 17), so far there is no reason to expect the two representations to be the same function. For that to be true, some linking properties must also be satisfied. These are formulated in the following.

Corollary 1 to Theorem 1. In the presence of Assumptions 1–6, the representations of Theorem 1 are equivalent to the following four behavioral conditions holding for all $x, y, u, v, p \in \mathbb{R}_+$, $p > 0$.

1. The Thomsen condition (Equation 20).

2. Proportional commutativity (Equation 21).

3. Left segregation defined by

[5] Note that for $\delta > 0$, if we define $\Psi^* := \delta\Psi$, which is dimensionless, then Equation 18 takes the form

$$\Psi^*(x, u) = \Psi^*(x, 0) + \Psi^*(0, u) + \Psi^*(x, 0)\Psi^*(0, u).$$

Probably, it is unwise to use this normalized form because it conceals the dimensional constant δ and the important fact that it may be 0.

$$\begin{aligned} &y \oplus_l (x \circ_{p,l} 0) \sim (y \oplus_l x) \circ_{p,l} (y \oplus_l 0) \\ \Leftrightarrow\ &(y \oplus_l (x \circ_{p,l} 0), 0) \sim (x \oplus_l y, 0) \circ_p (0 \oplus_l y, 0) \\ \Leftrightarrow\ &(y, x \circ_{p,l} 0) \sim (y, x) \circ_p (y, 0); \end{aligned} \tag{22}$$

and/or right segregation defined by

$$\begin{aligned} &(x \circ_{p,r} 0) \oplus_r y \sim (x \oplus_r y) \circ_{p,r} (0 \oplus_r y) \\ \Leftrightarrow\ &(0, (x \circ_{p,r} 0) \oplus_r y) \sim (0, x \oplus_r y) \circ_p (0, 0 \oplus_r y) \\ \Leftrightarrow\ &(x \circ_{p,r} 0, y) \sim (x, y) \circ_p (0, y). \end{aligned} \tag{23}$$

4. Left joint-presentation decomposition defined by the following: For any $(x, u) \succsim (y, v)$, $p > 0$, there exists $q = q(x, y, p)$ such that

$$(x, u) \circ_p (y, v) \sim (x \circ_{p,l} y, u \circ_{q,r} v); \tag{24}$$

and/or right joint-presentation decomposition defined by the following: For any $(x, u) \succsim (y, v)$, $p > 0$, there exists $t = t(u, v, p)$ such that

$$(x, u) \circ_p (y, v) \sim (x \circ_{t,l} y, u \circ_{p,r} v). \tag{25}$$

The Thomsen condition and proportional commutativity have already been discussed. Supportive empirical tests of segregation are in Steingrimsson and Luce (2003b).

The functions q and t of the joint-presentation decomposition condition can be expressed in terms of the representation as

$$W(q) = \frac{W(p)[1 + \delta\Psi(x, 0)]}{1 + \delta(\Psi(x, 0)W(p) + \Psi(y, 0)[1 - W(p)])} \tag{26}$$

and

$$W(t) = \frac{W(p)[1 + \delta\Psi(0, u)]}{1 + \delta(\Psi(0, u)W(p) + \Psi(0, v)[1 - W(p)])}. \tag{27}$$

The important aspect of Equation 24 is that the value of q is independent of u and v, and the important aspect of Equation 25 is that t is independent of x and y. Nonetheless, it is a difficult property to test experimentally because one must, for several values of intensity, estimate the functions q and t and determine whether they are independent of (u, v) and (x, y), respectively.

From Part 4 together with the idempotence of $\circ_{p,l}$ and $\circ_{p,r}$ (Assumption 5), the following properties, called *left* and *right distributivity*,[6] respectively, are immediate:

$$(x \circ_{p,l} y, z) = (x, z) \circ_p (y, z) \tag{28}$$

and

$$(z, u \circ_{p,r} v) = (z, u) \circ_p (z, v). \tag{29}$$

Note that segregation is the special case of distributivity with $z = 0$. This property of course affords an additional test of the theory.

Corollary 2 to Theorem 1. Assume the representations of Equations 16 and 18. Then, the property of bisymmetry,

$$(x \oplus_i u) \oplus_i (y \oplus_i v) \sim (x \oplus_i y) \oplus_i (u \oplus_i v) \quad (i = l, r), \tag{30}$$

holds iff either $\gamma = 1$ or $\delta = 0$.

The corollary is of interest for two reasons. The data on matching suggest that $\gamma \neq 1$, and when $\delta = 0$, the issues of experimental testing are considerably simplified (see Steingrimsson & Luce, 2003a, 2003b). Second, the theory of symmetric matching below predicts $\delta = 0$.

Symmetric Matching

The following is the psychophysical analogue of Theorem 2 of Luce (in press), in which Ψ plays a role analogous to utility U and the notation (x, u) is used instead of $x \oplus u$. One has to observe that the proof generalizes for values of $p > 1$. The result below focuses on $\oplus_s$, defined by Equation 6, rather than on $\oplus_l$ or $\oplus_r$. Whereas 0 is a left identity of $\oplus_l$ and a right identity of $\oplus_r$, it is neither for $\oplus_s$, but this operation satisfies idempotence (Equation 7). I introduce the following definition: Let $\oplus_s$ be given by Equation 6, and define $\circ_{p,s}$ in terms of $\circ_p$ by

$$((x \circ_{p,s} y), (x \circ_{p,s} y)) := (x, x) \circ_p (y, y). \tag{31}$$

Theorem 2. Suppose that $\langle \mathbb{R}_+ \times \mathbb{R}_+, \succsim, \circ_p \rangle$, $p > 0$, satisfies Assumptions 1–6 above. Then any two of the following statements imply the third, where $x, y, u, p \in \mathbb{R}_+$, $p > 0$, $\psi_s : \mathbb{R}_+ \xrightarrow{onto} \mathbb{R}_+$, $W : \mathbb{R}_+ \xrightarrow{onto} \mathbb{R}_+$, $W(1) = 1$:

1. Left segregation holds in the sense that

$$y \oplus_s (x \circ_{p,s} 0) \sim (y \oplus_s x) \circ_{p,s} (y \oplus_s 0). \tag{32}$$

2. (ψ_s, W) forms a subjective proportion representation

$$W(p) = \frac{\psi_s(x \circ_{p,s} y) - \psi_s(y)}{\psi_s(x) - \psi_s(y)}. \tag{33}$$

3. There is a continuous function $\tau_l : \mathbb{R}_+ \xrightarrow{into} \mathbb{R}_{++} =]0, \infty[$[7] such that the operation $\oplus_s$ has the representation

$$\psi_s(x \oplus_s u) = \psi_s(x \oplus_s 0) + \psi_s(u)\tau_l[\psi_s(x)]. \tag{34}$$

Let Ψ and ψ_s be related by[8]

$$\Psi(x, u) := \psi_s(x \oplus_s u). \tag{35}$$

Observe that Equation 34 can be rewritten as

$$\Psi(x, u) = \Psi(x, 0) + \Psi(u, u)\tau_l[\Psi(x, x)] \tag{36}$$

$$= \Psi(x, 0) + \Psi(0, u)\tau_l^*[\Psi(x, x)], \tag{37}$$

where $\tau_l^*[\Psi(x, x)] = \tau_l[\Psi(x, x)]/\tau_l(0)$.

A parallel theorem holds for right segregation. For example, Equation 36 is changed to

[6] The second half of Luce (in press) developed the theory on the basis of distributivity for $y > 0$ rather than on segregation. It thereby avoids mixing elements of the form $(x, x) \circ_p (y, y)$ and $(x, x) \circ_p (0, 0)$ within a condition. Steingrimsson's (2002) experience suggests that mixing these may be a problem; the simpler $(x, x) \circ_p (0, 0)$ seems to be dealt with differently from the case where $y > 0$. It definitely seems to be a problem in the utility case where monotonicity is violated (Birnbaum, 1997). Luce (in press) accommodated this possibility.

[7] The notation $]a, b[$ means the open interval from a to b.

[8] Strictly, I should use the notation Ψ^* because no reason has yet been given for assuming that this representation is the same as that of Theorem 1. This is dealt with following Theorem 3, where it is shown that they are, up to a multiplicative constant, identical.

$$\Psi(x, u) = \Psi(x, x)\tau_r[\Psi(u, u)] + \Psi(0, u) \quad (38)$$

$$= \Psi(x, 0)\tau_r^*[\Psi(u, u)] + \Psi(0, u). \quad (39)$$

We next consider what results when both forms of segregation hold. The following corollary translates the Corollary of Theorem 4 of Luce (in press) from utility terms into psychophysical ones.

Corollary. Suppose that $\langle \mathbb{R}_+ \times \mathbb{R}_+, \succsim, \circ_p \rangle$, $p > 0$, satisfies Assumptions 1–6 above. Then, for $x, y, z, u, v, p \in \mathbb{R}_+$, $p > 0$:

1. $\langle \mathbb{R}_+, \oplus_s, \circ_{p,s}, \psi_s, W \rangle$ satisfies Parts 1–3 of Theorem 2 for both left and right segregation iff (Ψ, W) satisfies the proportion representation (Equation 17), and $(\Psi, \oplus)$ satisfies either
   - A. $\tau_l^*[\Psi(x, x)] \equiv \tau_r^*[\Psi(u, u)] \equiv 1$, and so

   $$\Psi(x, u) = \Psi(x, 0) + \Psi(0, u), \quad (40)$$

   or
   - B. there is a constant $\eta \in]0, 1[$ such that

   $$\Psi(x, u) = \eta\Psi(x, x) + (1 - \eta)\Psi(u, u). \quad (41)$$

   - C. Equation 41 holds iff both Equation 40 and the constant-ratio property (Equation 16) hold with $\gamma = \eta/(1 - \eta)$.

2. Given the properties of Part 1:
   - A. $\oplus_s$ is bisymmetric in the sense of Equation 30 with $i = s$.
   - B. For $(x, u) \succ (y, v)$,

   $$(x, u) \circ_p (y, v) \sim (x \circ_{p,s} y, u \circ_{p,s} v). \quad (42)$$

   - C. Right and left distributivity hold in the following sense:

   $$(x, z) \circ_p (y, z) \sim (x \circ_{p,s} y, z) \quad (43)$$

   and

   $$(z, x) \circ_p (z, y) \sim (z, x \circ_{p,s} y). \quad (44)$$

Note that Equations 40 and 41 each imply the Thomsen condition.

That the constant-ratio condition (Equation 16) follows from Equation 41 is easily shown: If we set $u = 0$ and then separately set $x = 0$ and $u = x$, we have

$$\Psi(x, 0) = \eta\Psi(x, x)$$

and

$$\Psi(0, x) = (1 - \eta)\Psi(x, x),$$

hence the constant-ratio condition with $\gamma = \eta/(1 - \eta)$.

An interesting example is $\Psi(x, y) = ax^{\beta_l} + by^{\beta_r}$, which satisfies Equation 40 but for $\beta_l \neq \beta_r$ does not satisfy the constant-ratio condition and so not Equation 41. So, there is room in the symmetric matching theory for an individual with more profound differences between the ears such as in the power case with differential rates of growth. Such an individual will not satisfy the conclusions of Theorem 1 because either asymmetric operator yields the constant-ratio condition.

The result of Equation 42 is similar to joint-presentation decomposition, Equations 24 and 25, but with the simplifying feature that $q(x, y, p) = t(u, v, p) = p$. It is called *simple joint-presentation decomposition.* Clearly, this behavioral prediction is far easier to test than the more general one because one does not need to estimate the functions q or t, which are simply the experimenter-determined p.

Both Symmetric and Asymmetric Segregation

Both Corollary 1 to Theorem 1 and the Corollary to Theorem 2 of this article invoke segregation in the following ways: left for $\oplus_l$ and right for $\oplus_r$ in the first case and both left and right for $\oplus_s$. Because the former leads to p-additivity and the latter to that with $\delta = 0$, it is clear that neither set of segregation assumptions implies the other. But one may ask, what happens if there is one segregation property at each level, asymmetric and symmetric?

A priori, the possibility exists that empirical results will show that for some people only some of these hold. The following results show that under the assumption of one form of segregation for asymmetric matches and one form for symmetric ones, then all forms of segregation prevail, and so the representation $\Psi(x, u)$ is the conjoint additive one, Equation 40, which is p-additive, Equation 18, with $\delta = 0$.

Theorem 3. Suppose that $\langle \mathbb{R}_+ \times \mathbb{R}_+, \sim, \circ_p \rangle$, $p > 0$, satisfies Assumptions 1–6 above and that the subjective proportion representation, Equation 17, holds. Then, if either left segregation is satisfied by $\oplus_l$ (and so Equation 14 holds) or right segregation is satisfied by $\oplus_r$ (and so Equation 15 holds) and if either left or right segregation is satisfied by $\oplus_s$, then all forms of segregation hold for the asymmetric and symmetric operations. Moreover, Equations 16 and 40 hold, that is,

$$\Psi(x, 0) = \gamma\Psi(0, x)$$

and

$$\Psi(x, u) = \Psi(x, 0) + \Psi(0, u).$$

Suppose that Ψ and Ψ^* are the representations corresponding, respectively, to Theorems 1 and 2. Then, given that they each satisfy Equation 17, a well-known uniqueness result implies that $\Psi^* = a\Psi + b$, where $a > 0$. By the fact that $\Psi(0, 0) = \Psi^*(0, 0) = 0$, we have $b = 0$. There is no loss of generality in choosing $a = 1$; so, there is a common representation Ψ.

Conclusion

This article extends Luce (2002) in two ways. First, the theory is recast in terms of a general psychophysical representation $\Psi(x, u)$ of the presented signal pair, such as to the left and right ears. Common to all of the results is the form of subjective proportions,

$$\frac{\Psi[(x, x) \circ_p (y, y)] - \Psi(y, y)}{\Psi(x, x) - \Psi(y, y)} = W(p),$$

and left and right forms of segregation relating (x, u) and $\circ_p$. Two types of decomposition of $\Psi(x, u)$ into functions of x and of u separately resulted. As in the earlier article, we can reduce it to asymmetric terms, $\Psi(x, 0)$ and $\Psi(0, u)$, which correspond to asymmetric matching of signals. Under the assumption of subjective proportions and left (right) segregation, this led to Equation 14 (or to Equation 15). When both hold the simple p-additive form,

$$\Psi(x, u) = \Psi(x, 0) + \Psi(0, u) + \delta\Psi(x, 0)\Psi(0, u) \quad (\delta \geq 0)$$

Table 1
Predictions About $\Psi(x, u)$ Under Various Segregation Conditions

Type of $\oplus$	Segregation	$\frac{\Psi(x,0)}{\Psi(0,x)}$	$\Psi(x, u)$	Equation
Asymmetric, $\oplus_i$	Left, $i = l$	γ	$\Psi(x, 0) + \sigma_l(x)\Psi(u, 0)$	14
	Right, $i = r$	γ	$\Psi(0, x)\sigma_r(u) + \Psi(0, u)$	15
	Both, $i = l, r$	γ	$\Psi(x, 0) + \Psi(0, u) + \delta\Psi(x, 0)\Psi(0, u)$	18
Symmetric, $\oplus_s$	Left	—	$\Psi(x, 0) + \Psi(u, u)\tau_l[\Psi(x, x)]$	36
	Right	—	$\Psi(x, x)\tau_r[\Psi(u, u)] + \Psi(0, u)$	38
	Both	—	$\Psi(x, 0) + \Psi(0, u)$	40
			or	
		γ	$\eta\Psi(x, x) + (1 - \eta)\Psi(u, u) = \Psi(x, 0) + \Psi(0, u)$	41
Mixed	One of each type	γ	$\Psi(x, 0) + \Psi(0, u)$	18[a]

Note. Dashes indicate that there are no predictions in these cases.
[a] Where $\delta = 0$.

results. Here, I also explored the decomposition of $\Psi(x, u)$ corresponding to symmetric matching of signals. The study of symmetric matches was strongly motivated by some experimental difficulties that arose using asymmetric matches. When $\Psi(x, u)$ has both right and left generalized weighted average forms, Equations 36 and 38, then we can conclude $\Psi(x, u)$ has the above form but with $\delta = 0$. It may or may not also satisfy the constant-ratio condition, Equation 16. In the theorems and their corollaries, I considered various issues of axiomatization and how the two types of representation are constrained when segregation holds for both symmetric and asymmetric matches. Table 1 summarizes the various forms for $\Psi(x, u)$ as a function of which segregation assumptions hold.

Because both the symmetric and asymmetric matches imply the Thomsen condition (Equation 20) of additive conjoint measurement, this is a crucial empirical test. If satisfied, then it is important to explore bisymmetry (Equation 30) and the four versions of distribution (Equations 28, 29, 43, and 44) in an attempt to decide which is the better model. Also, the invariance of $\Psi(x, 0)/\Psi(0, x)$ as a function of x (the constant-ratio condition) arises in the case of either asymmetric matches or as part of one solution of the two-sided case of symmetric matches. So, it too is important to test. In the case of power functions for $\Psi(x, 0)$ and $\Psi(0, x)$, it is equivalent to one ear having a constant decibel loss compared with the other, independent of intensity level above their respective thresholds.

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Appendix

Proofs

Theorem 1

Proof

Without fully restating Theorem 1 of Luce (2002), the results under Assumptions 1–6 are that if left segregation, Equation 22, holds for $\oplus_l$ and $\circ_{p,l}$, then for some positive continuous function σ_l,

$$\psi_l(x \oplus_l u) = \psi_l(x) + \sigma_l(x)\psi_l(u)$$

$$\Leftrightarrow \Psi(x, u) = \Psi(x \oplus_l u, 0) = \Psi(x, 0) + \sigma_l(x)\Psi(u, 0),$$

which is Equation 14. If we set $x = 0$ in that expression,

$$\Psi(0, u) = \sigma_l(0)\Psi(u, 0).$$

If right segregation, Equation 23, holds for $\oplus_r$ and $\circ_{p,r}$, the proof of Equation 15 is similar. Setting $u = 0$ in that yields

$$\Psi(x, 0) = \sigma_r(0)\Psi(0, x).$$

So if, indeed, there is a common psychophysical function Ψ underlying both left and right matches, then $\sigma_r(0) = 1/\sigma_l(0)$. We call this common constant γ. Thus, we have proved Equation 16 using either right or left segregation.

Assuming both, by Equations 14, 15, and 16,

$$\begin{aligned}\Psi(x, u) &= \Psi(x, 0) + \sigma_l(x)\Psi(u, 0)\\ &= \gamma\Psi(0, x) + \sigma_l(x)\gamma\Psi(0, u)\\ &= \sigma_r(u)\Psi(0, x) + \Psi(0, u).\end{aligned}$$

Rearranging the final equality gives

$$\frac{\gamma - \sigma_r(u)}{\Psi(0, u)} = \frac{1 - \sigma_l(x)\gamma}{\Psi(0, x)} = \rho,$$

where ρ is a constant because x and u are independent. Thus,

$$\begin{aligned}\Psi(x, u) &= \Psi(x, 0) + \sigma_l(x)\Psi(u, 0)\\ &= \Psi(x, 0) + \frac{1 - \rho\Psi(0, x)}{\gamma}\Psi(u, 0)\\ &= \Psi(x, 0) + \Psi(0, u) + \delta\Psi(x, 0)\Psi(0, u),\end{aligned}$$

where $\delta = -\rho/\gamma$. This proves Equation 18.

The representations

$$W(p) = \frac{\psi_i(x \circ_{p,i} y) - \psi_i(y)}{\psi_i(x) - \psi_i(y)}, \quad (i = l, r)$$

were also established. So, for $i = l$ and using the definition relating $\circ_{p,l}$ and $\circ_p$, Equation 9,

$$\begin{aligned}W(p) &= \frac{\Psi(x \circ_{p,i} y, 0) - \Psi(y, 0)}{\Psi(x, 0) - \Psi(y, 0)}\\ &= \frac{\Psi[(x, 0) \circ_p (y, 0)] - \Psi(y, 0)}{\Psi(x, 0) - \Psi(y, 0)}.\end{aligned}$$

Using solvability, Equation 6 puts this in the form of Equation 17.

Corollary 1 to Theorem 1

Proof

The first three conditions are direct from Luce (2002). The fourth, joint-presentation decomposition, has its counterpart in terms of $\circ_{p,l}$, which we now show. In the left case, consider $y = v = 0$. Observe that by Equation 24 and the definitions of $\oplus_l$ and $\circ_{p,l}$,

$$\begin{aligned}((x \oplus_l u) \circ_{p,l} 0, 0) &\sim (x \oplus_l u, 0) \circ_p (0, 0) \quad (9)\\ &\sim (x, u) \circ_p (0, 0) \quad (6)\\ &\sim (x \circ_{p,l} 0, u \circ_{q,r} 0) \quad (24)\\ &\sim ((x \circ_{p,l} 0) \oplus_l (u \circ_{q,r} 0), 0) \quad (6),\end{aligned}$$

hence, by monotonicity, $(x \oplus_l u) \circ_{p,l} 0 = (x \circ_{p,l} 0) \oplus_l (u \circ_q 0)$, which is the form given in Luce (2002). The right case is similar.

Theorem 2

Proof

See Theorem 2 of Luce (in press).

Corollary to Theorem 2

Proof

Part 1. Corollary to Theorem 4 of Luce (in press).

Part 2. (a) Bisymmetry is immediate from Corollary 2 of Theorem 1. (b) To show Equation 42, consider

$$\begin{aligned}\Psi(x \circ_{p,s} y, u \circ_{p,s} v) &= \Psi(x \circ_{p,s} y, 0) + \Psi(0, u \circ_{p,s} v)\\ &= \Psi[(x, 0) \circ_p (y, 0)] + \Psi[(0, u) \circ_p (0, v)]\\ &= [\Psi(x, 0) - \Psi(y, 0)]W(p) + \Psi(y, 0)\\ &\quad + [\Psi(0, u) - \Psi(0, v)]W(p) + \Psi(0, v)\\ &= [\Psi(x, u) - \Psi(y, v)]W(p) + \Psi(y, v)\\ &= \Psi[(x, u) \circ_p (y, v)].\end{aligned}$$

By the order-preserving property of Ψ, Equation 42 follows.

(c) This is immediate from Part 2 (b) and the idempotence of $\circ_{p,s}$ (Assumption 5).

(*Appendix continues*)

Theorem 3

Proof

It suffices to assume left segregation for both $\oplus_l$ and $\oplus_s$ or right for $\oplus_l$ and left for $\oplus_s$. The other two cases, right–right and left–right, are completely parallel.

First, suppose left segregation holds for both. Thus, both Equations 14 and 36 are satisfied. By Theorem 1, Equation 16 is satisfied. Invoking both Equations 14 and 36 yields

$$\Psi(x, u) = \Psi(x, 0) + \sigma_l(x)\Psi(u, 0) = \Psi(x, 0) + \Psi(u, u)\tau_l[\Psi(x, x)].$$

Solving for $\Psi(u, u)$ and $\Psi(u, 0)$ yields

$$\frac{\Psi(u, u)}{\Psi(u, 0)} = \frac{\sigma_l(x)}{\tau_l[\Psi(x, x)]} = \nu > 0,$$

where ν is a constant because x and u may be chosen independently. Using this and the form of Equation 14 from Theorem 1,

$$\nu\Psi(x, 0) = \Psi(x, x) = \Psi(x, 0) + \sigma_l(x)\Psi(x, 0) = [\sigma_l(x) + 1]\Psi(x, 0),$$

and so, $\sigma_l(x) = \nu - 1$. Therefore,

$$\Psi(x, u) = \Psi(x, 0) + (\upsilon - 1)\Psi(u, 0)$$

$$= \Psi(x, 0) + (\upsilon - 1)\gamma\Psi(0, u).$$

Setting $x = 0$ shows that $(\upsilon - 1)\gamma = 1$, and so, Equation 40 is satisfied. From that and Equation 17, left segregation holds for $\oplus_l$ and for $\oplus_s$.

Next, suppose left segregation of $\oplus_l$ and right segregation of $\oplus_s$. So, we have Equations 14 and 38. As before, by the right version of Corollary 1 of Theorem 1, Equation 16 holds. Thus, in the final equality,

$$\Psi(x, u) = \Psi(x, 0) + \sigma_l(x)\Psi(u, 0) = \Psi(x, x)\tau_r[\Psi(u, u)] + \Psi(0, u).$$

Set $u = x$ to get $\Psi(x, x) = \Psi(x, 0)[1 + \sigma_l(x)]$, and so,

$$\Psi(x, u) = \Psi(x, 0)[1 + \sigma_l(x)]\tau_r[\Psi(u, u)] + \Psi(0, u).$$

Setting $u = 0$, we see $[1 + \sigma_l(x)]\tau_r(0) = 1$, and therefore, $\sigma_l(x) = \rho$, a constant. So, $\Psi(x, u) = \Psi(x, 0) + \rho\gamma\Psi(0, u)$. Setting $x = 0$ shows that $\rho\gamma = 1$, that is, Equation 18. Thus, again, all forms of segregation hold.

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