Positive Solutions for a System of Discrete Boundary Value Problem

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Abstract. This paper deals with the existence and multiplicity of positive solutions for a system of second-order discrete boundary value problem. The main results are obtained via Jensen's inequalities, properties of concave and convex functions and the Krasnosel'skii-Zabreiko fixed point theorem. Furthermore, concave and convex functions are employed to emphasize the coupling behaviors of nonlinear terms f and g and we provide two explicit examples to illustrate our main results and the coupling behaviors.

2010 Mathematics Subject Classification: 34B18; 39A12; 47H10

Keywords: boundary value problem; positive solution; fixed point theorem; discrete systems; concave and convex functions.

1 Introduction

The study of discrete boundary value problems has captured special attention in the last years. We refer reader to the recent results [1–9] and the references therein. The studies regarding such type of problems can be placed at the interface of certain mathematical fields such as nonlinear partial differential equations and numerical analysis. On the other hand, they are strongly motivated by their applicability in mathematical physics.

Besides, we note that some systems of discrete boundary value problems (include integer-order and fractional order) are investigated by several authors in recent years, for example, see [10–13].

Sun and Li in [10] studied the following boundary value problem of discrete systems:

$$\begin{cases} \triangle^2 u_1(k) + f_1(k, u_1(k), u_2(k)) = 0, \ k \in \{1, 2, ..., T\}, \\ \triangle^2 u_2(k) + f_2(k, u_1(k), u_2(k)) = 0, \end{cases} \tag{1.1}$$

subject to the boundary value conditions

$$u_1(0) = u_1(T+2) = u_2(0) = u_2(T+2) = 0.$$

They obtained, under some assumptions on f_1, f_2, some sufficient conditions for the existence of one or two positive solutions to the system by using nonlinear alternative of Leray-Schauder type and Krasnosel'skii fixed point theorem in a cone.

Henderson [11] considered the following systems of three-point discrete boundary value problems, as a generalization of [10] :

$$\begin{cases} \triangle^2 u(n-1) + \lambda a(n) f(u(n), v(n)) = 0, \ n \in \{1, 2, ..., N-1\}, \ N \geq 4, \\ \triangle^2 v(n-1) + \mu b(n) g(u(n), v(n)) = 0, \end{cases} \tag{1.2}$$

subject to the boundary value conditions

$$u(0) = \beta u(\eta), u(N) = \alpha u(\eta), v(0) = \beta v(\eta), v(N) = \alpha v(\eta),$$

where $\eta \in \{1, 2, ..., N-1\}, \alpha > 0, \beta > 0, \lambda, \mu > 0$ and f, g, a, b are nonnegative. They deduced the existence of the eigenvalues λ and μ yielding at least one positive solutions to the systems (1.2) under some assumptions on f, g, a and b with weakly coupling behaviors. Their main tools is the Guo-Krasnosel'skii fixed point theorem in cones.

In [12], Goodrich generalized (1.2) to the following discrete fractional difference boundary value problem with more general boundary conditions:

$$\begin{cases} -\triangle^{v_1} y_1(t) + \lambda_1 a_1(t + v_1 - 1) f_1(y_1(t + v_1 - 1), y_2(t + v_2 - 1)) = 0, \\ -\triangle^{v_2} y_2(t) + \lambda_2 a_2(t + v_2 - 1) f_2(y_1(t + v_1 - 1), y_2(t + v_2 - 1)) = 0, \end{cases} \tag{1.3}$$

for $t \in [0, b]_{\mathbb{N}_0}$, subject to the nonlocal boundary value conditions

$$y_1(v_1 - 2) = \psi_1(y_1), \ y_2(v_2 - 2) = \psi_2(y_2), \ y_1(v_1 + b) = \phi_1(y_1), \ y_2(v_2 + b) = \phi 2(y_2).$$

It should be noted that the paper generalizes some results both on discrete fractional boundary value problems and on discrete integer-order boundary value problems. We also note that the conditions on f, g and processing methods as well as the fixed point theorem, employed in the paper, are similar to those given by Henderson in [11].

Motivated by the above, we shall also investigate the existence of positive solution to the following second-order Dirichlet boundary value problem of discrete system:

$$\begin{cases} \triangle^2 u(k-1) + f(k, u(k), v(k)) = 0, \ k \in \{1, 2, ..., T\}, \\ \triangle^2 v(k-1) + g(k, u(k), v(k)) = 0, \\ u(0) = u(T+1) = v(0) = v(T+1) = 0, \end{cases} \tag{1.4}$$

where $T > 2$ is a fixed positive integer number, $\triangle u(k) = u(k+1) - u(k), \triangle^2 u(k) = \triangle(\triangle u(k))$, $f, g : \{1, 2, ..., T\} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ (\mathbb{R}^+ := [0, +\infty))$ are continuous. By positive solutions of the problem (1.4), we mean that a pair of (u, v) solves (1.4) and is nonnegative and nontrivial.

Different to [11, 12], we will not study the problem of eigenvalue yielding a positive solution but lay emphasis on the conditions of coupling behaviors of f and g those yield at least one positive solution. Although the papers have considered some problems similar to the problem (1.4), the nonlinearities f and g were weakly coupled. So [10–12] can not include ours. Our paper has the following characteristics: in our assumptions on the nonlinearities f and g, they have a stronger coupling behaviors, which are characterized by convex and concave functions. We establish existence of one or two positive solutions for boundary value problem (1.4) via the well-known Krasnosel'skii-Zabreiko fixed point theorem in a cone. Moreover, a priori estimates achieved by using Jensen's inequality and the first eigenvalue of relevant operator are applied in our calculation.

The rest of the paper is organized as follows. In Section 2, we introduce some lemmas which are used in main results. In Section 3, Criteria for the existence of one or two positive solutions to boundary value the problem (1.4) are established. In section 4, we offer two examples to illustrate our main results.

2 Preliminaries

Denote

$$\mathbb{T}_1 := \{1, 2, \ldots, T\}, \ \mathbb{T}_2 := \{0, 1, 2, \ldots, T+1\}.$$

Let E be the Banach space of real valued functions defined on the discrete interval $\mathbb{T}_2$ with the norm $\|u\| = \max_{k\in\mathbb{T}_2} |u(k)|$, and therefore $\mathcal{X} = E \times E$ with norm $\|(u, v)\| = \max\{\|u\|, \|v\|\}$ is also a Banach space. Let

$$P = \left\{ u \in E : u(k) \geq 0, \ \forall k \in \mathbb{T}_2, \text{ and } \min_{k\in\mathbb{T}_1} u(k) \geq \frac{1}{T}\|u\| \right\} \tag{2.1}$$

and $B_r = \{x \in \mathcal{X} : \|x\| \leq r\}$ for $r > 0$. It is easy to see $\mathcal{P} = P \times P$ is a cone in $\mathcal{X}$, and the partial ordering $\leq$ in $\mathcal{X}$ is induced by $\mathcal{P}$. We call $(u, v) \leq (x, y)$, if $(x, y) - (u, v) \in \mathcal{P}$.

Lemma 2.1 *(see [11]) Let $h(k) \in C(\mathbb{T}_1, \mathbb{R}^+)$. Then the following Dirichlet boundary value problem of discrete system*

$$\begin{cases} \triangle^2 u(k-1) + h(k) = 0, \ k \in \mathbb{T}_1, \\ u(0) = u(T+1) = 0, \end{cases} \tag{2.2}$$

is equivalent to

$$u(k) = \sum_{l=1}^{T} G(k, l)h(l), \ k \in \mathbb{T}_2, \tag{2.3}$$

where

$$G(k, l) = \frac{1}{T+1} \begin{cases} l(T+1-k), \ 1 \leq l \leq k-1 \leq T, \\ k(T+1-l), \ 0 \leq k \leq l \leq T. \end{cases} \tag{2.4}$$

Moreover, we easily obtain that $G(k, l)$ has the following properties (see [13, Lemma 2]):
(1) $G(k, l) > 0$ and $G(k, l) = G(l, k)$, for $(k, l) \in \mathbb{T}_1 \times \mathbb{T}_1$;
(2) $G(l, l)/T \leq G(k, l) \leq G(l, l)$, for $(k, l) \in \mathbb{T}_1 \times \mathbb{T}_1$.

From Lemma 2.1, it is clear that the discrete system (1.4) is equivalent to

$$\begin{cases} u(k) = \sum_{l=1}^{T} G(k, l)f(l, u(l), v(l)), \ k \in \mathbb{T}_2, \\ v(k) = \sum_{l=1}^{T} G(k, l)g(l, u(l), v(l)), \ k \in \mathbb{T}_2, \end{cases} \tag{2.5}$$

and by the above properties of $G(k, l)$, it is easy to obtain $u(k) \geq \|u\|/T, v(k) \geq \|v\|/T$ for $k \in \mathbb{T}_1$, i.e., $u, v \in P$.

For $u, v \in P, k \in \mathbb{T}_2$, define operators $\mathcal{T}, \mathcal{S} : \mathcal{P} \to E$ by

$$\mathcal{T}(u, v)(k) = \sum_{l=1}^{T} G(k, l)f(l, u(l), v(l)), \ \mathcal{S}(u, v)(k) = \sum_{l=1}^{T} G(k, l)g(l, u(l), v(l))$$

and operator $\mathcal{A} : \mathcal{P} \to \mathcal{X}$ by

$$\mathcal{A}(u, v)(k) = (\mathcal{T}(u, v)(k), \mathcal{S}(u, v)(k)).$$

Lemma 2.2 *The operators $\mathcal{T}, \mathcal{S} : \mathcal{P} \to P$ are completely continuous, and then $\mathcal{A} : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof. From the non-negativity of $G(k,l)$ and f, it follows that $\mathcal{T}(u,v)(k) \geq 0,\ \mathcal{S}(u,v)(k) \geq 0$, for $k \in \mathbb{T}_2$. Moreover, by (2) in Lemma 2.1, one gets

$$\mathcal{T}(u,v)(k) \geq \frac{1}{T}\sum_{l=1}^{T} G(l,l)f(l,u(l),v(l)), \quad \|\mathcal{T}(u,v)\| \leq \sum_{l=1}^{T} G(l,l)f(l,u(l),v(l)),$$

for $k \in \mathbb{T}_1$. Therefore, $\mathcal{T}(u,v)(k) \geq \|\mathcal{T}(u,v)\|/T$. This indicates that $\mathcal{T} : \mathcal{P} \to P$. Similarly, we may also prove that $\mathcal{S} : \mathcal{P} \to P$. Thus, from the definition of operator $\mathcal{A}$ we conclude that $\mathcal{A} : \mathcal{P} \to \mathcal{P}$.

A relatively straightforward application of the Arzela-Ascoil theorem reveals that both operators $\mathcal{T}$ and $\mathcal{S}$ are completely continuous. Hence $\mathcal{A}$ is a completely continuous operator. This completes the proof. ■

From the definition of operator $\mathcal{A}$, we note that the problem (1.4) has a pair of positive solutions if and only if the operator $\mathcal{A}$ has a fixed point in $\mathcal{P}$.

Lemma 2.3 *Let* $\phi(k) = \sin(k\pi)/(T+1),\ k \in \mathbb{T}_2,\ \lambda = 4\sin^2(\pi/(2T+2))$. *Then*

$$\lambda \sum_{k=1}^{T} G(k,l)\phi(k) = \phi(l), \quad \forall l \in \mathbb{T}_1. \tag{2.6}$$

Proof. From [6, Lemma 2.2], we acquire that $\lambda\sum_{l=1}^{T} G(k,l)\phi(l) = \phi(k), k \in \mathbb{T}_2$. Since $\phi(k)$ vanishes at $k = 0,\ T+1$, we have $\lambda\sum_{l=1}^{T} G(k,l)\phi(l) = \phi(k), k \in \mathbb{T}_1$. By (1) in Lemma 2.1, we see $G(k,l)$ is a symmetric function about $k, l \in \mathbb{T}_1$, that is, $G(k,l) = G(l,k)$. Thus $\lambda\sum_{l=1}^{T} G(l,k)\phi(l) = \phi(k)$ for $k \in \mathbb{T}_1$. Hence, (2.6) is true. This completes the proof. ■

Lemma 2.4 *(see [14]) Let E be a real Banach space and P a cone of E. If $A : (\overline{B}_R \backslash B_r) \cap P \to P$ is a completely continuous operator with $0 < r < R$. If either (1) $Av \nleq v$ for each $P \cap \partial B_r$ and $Av \ngeq v$ for each $P \cap \partial B_R$ or (2) $Av \ngeq v$ for each $P \cap \partial B_r$ and $Av \nleq v$ for each $P \cap \partial B_R$, then A has at least one fixed point on $(B_R \backslash \overline{B}_r) \cap P$.*

3 Main results

In this section, we set

$$K = \max_{k \in \mathbb{T}_1} \sum_{l=1}^{T} G(k,l)$$

and λ is defined by Lemma 2.3. Now we list our assumptions.

(H1) There exist $p, q \in C(\mathbb{R}^+, \mathbb{R}^+)$ and constant $c > 0$ such that p is concave on $\mathbb{R}^+$,

$$f(k,u,v) \geq p(v) - c,\ g(k,u,v) \geq q(u) - c,\ \forall (k,u,v) \in \mathbb{T}_1 \times \mathbb{R}^+ \times \mathbb{R}^+ \tag{3.1}$$

and

$$p(Kq(u)) \geq \gamma_1 \lambda^2 K u - c,\ \gamma_1 > 1, \forall u \in \mathbb{R}^+. \tag{3.2}$$

(H2) There exist $\alpha, \beta \in C(\mathbb{R}^+, \mathbb{R}^+)$ and a sufficiently small constant $r > 0$ such that α is convex and strictly increasing on $\mathbb{R}^+$,

$$f(k,u,v) \leq \alpha(v),\ g(k,u,v) \leq \beta(u),\ \forall (k,u,v) \in \mathbb{T}_1 \times [0,r] \times [0,r] \tag{3.3}$$

and

$$\alpha(K(\beta(u))) \leq \gamma_2 K \lambda^2 u,\ 0 < \gamma_2 < 1, \forall u \in [0,r]. \tag{3.4}$$

(H3) There exist $p, q \in C(\mathbb{R}^+, \mathbb{R}^+)$ and a sufficiently small constant $r > 0$ such that p is concave and nondecreasing on $\mathbb{R}^+$,

$$f(k, u, v) \geq p(v),\ g(k, u, v) \geq q(u),\ \forall (k, u, v) \in \mathbb{T}_1 \times [0, r] \times [0, r] \tag{3.5}$$

and

$$p(Kq(u)) \geq \gamma_3 \lambda^2 K u,\ \gamma_3 > 1,\ \forall u \in [0, r]. \tag{3.6}$$

(H4) There exist four nonnegative constants $a,\ b,\ c,\ d$ and $e > 0$ such that $\lambda > a,\ \lambda > d$ and $(\lambda - a)(\lambda - d) > bc$,

$$f(k, u, v) \leq au + bv + e,\ g(k, u, v) \leq cu + dv + e,\ \forall (k, u, v) \in \mathbb{T}_1 \times \mathbb{R}^+ \times \mathbb{R}^+. \tag{3.7}$$

(H5) There exists $M > 0$ such that

$$f(k, u, v) < \frac{M}{K},\ g(k, u, v) < \frac{M}{K}, \forall (k, u, v) \in \mathbb{T}_1 \times [0, M] \times [0, M].$$

(H6) There exists $N > 0$ such that

$$f(k, u, v) > N\eta,\ g(t, u, v) > N\eta,\ \forall (k, u, v) \in \mathbb{T}_1 \times [0, N] \times [0, N],$$

where $\eta := T / \sum_{l=1}^{T} G(l, l)$.

Remark 3.1 *Equation* (3.2) *implies that* $\lim_{t \to +\infty} p(t) = +\infty$, *and* p *is concave on* $\mathbb{R}^+$, *therefore,* p *is strictly increasing on* $\mathbb{R}^+$.

Theorem 3.1 *Suppose (H1) and (H2) hold. Then problem* (1.4) *has at least one positive solution.*

Proof. If (H2) holds, we claim that

$$(u, v) \not\leq \mathcal{A}(u, v),\ \forall (u, v) \in \mathcal{P} \cap \partial B_r. \tag{3.8}$$

For contradiction we assume that there exists $(u, v) \in \mathcal{P} \cap \partial \overline{B}_r$ such that $(u, v) \leq \mathcal{A}(u, v)$, that is, $u \leq \mathcal{T}(u, v), v \leq \mathcal{S}(u, v)$. By (3.3), it follows that

$$\mathcal{T}(u, v)(k) \leq \sum_{l=1}^{T} G(k, l) \alpha(v(l)),\quad \mathcal{S}(u, v)(k) \leq \sum_{l=1}^{T} G(k, l) \beta(u(l)), \tag{3.9}$$

$\forall (k, u, v) \in \mathbb{T}_1 \times [0, r] \times [0, r]$, and then

$$u(k) \leq \sum_{l=1}^{T} G(k, l) \alpha(v(l)),\quad v(k) \leq \sum_{l=1}^{T} G(k, l) \beta(u(l)). \tag{3.10}$$

It follows from (3.4) that $\alpha(K\beta(0)) \leq 0$. Note that $\beta \in C(\mathbb{R}^+, \mathbb{R}^+)$ and α is strictly increasing, we get $\alpha(0) \leq \alpha(K\beta(0)) \leq 0$, and then $\alpha(0) = 0$ for $\alpha \in C(\mathbb{R}^+, \mathbb{R}^+)$. Furthermore, we can get

$$\alpha(v(k)) \leq \alpha \left(\sum_{l=1}^{T} G(k, l) \beta(u(l)) \right) \leq K^{-1} \sum_{l=1}^{T} G(k, l) \alpha(K\beta(u(l))), \tag{3.11}$$

by convex nature of α and Jensen's inequality. Substituting (3.11) into the first inequality in (3.10), we acquire

$$u(k) \le K^{-1} \sum_{l=1}^{T} G(k,l) \sum_{s=1}^{T} G(l,s)\alpha(K\beta(u(s))).$$

Multiply both sides of the above inequality by $\phi(k)$ and sum for $k = 1$ to T, and by (2.6) and (3.4) to obtain

$$\sum_{k=1}^{T} u(k)\phi(k) \le K^{-1}\lambda^{-2} \sum_{k=1}^{T} \phi(k)\alpha(K\beta(u(k))) \le \gamma_2 \sum_{k=1}^{T} \phi(k)u(k).$$

Which implies $\sum_{k=1}^{T} \phi(k)u(k) = 0$ since $\gamma_2 \in (0,1)$, therefore $u \equiv 0$. Equations (3.4) and (3.11) lead to

$$\alpha(v(k)) \le K^{-1} \sum_{l=1}^{T} G(k,l)\alpha(K\beta(u(l))) \le \gamma_2\lambda^2 \sum_{l=1}^{T} G(k,l)u(l) = 0.$$

Since α is strictly increasing, then $v \equiv 0$, which contradict $(u,v) \in \mathcal{P} \cap \partial B_r$. Hence, (3.8) is true.

On the other hand, if (H1) holds, by (3.1) and the definitions of $\mathcal{T}$ and $\mathcal{S}$, we see

$$\mathcal{T}(u,v)(k) \ge \sum_{l=1}^{T} G(k,l)p(v(l)) - c_1, \quad \mathcal{S}(u,v)(k) \ge \sum_{l=1}^{T} G(k,l)q(u(l)) - c_1, \tag{3.12}$$

where $c_1 = Kc$. Let

$$\mathcal{M}_1 = \{(u,v) \in \mathcal{P} : (u,v) \ge \mathcal{A}(u,v)\}.$$

Then we shall prove that $\mathcal{M}_1$ is bounded in $\mathcal{P}$. In fact, if $(u,v) \in \mathcal{M}_1$, then $u \ge \mathcal{T}(u,v)$ and $v \ge \mathcal{S}(u,v)$. From (3.12), it follows that

$$u(k) \ge \sum_{l=1}^{T} G(k,l)p(v(l)) - c_1, \quad v(k) \ge \sum_{l=1}^{T} G(k,l)q(u(l)) - c_1. \tag{3.13}$$

By the concavity and increasing nature of p and the second inequality of (3.13), in view of Jensen's inequality and $p(a+b) \le p(a) + p(b)$ for $a, b \ge 0$ (see Lemma 5 in [15]), we obtain

$$\begin{aligned} p(v(k)) &\ge p(v(k) + c_1) - p(c_1) \ge p\left(\sum_{l=1}^{T} G(k,l)q(u(l))\right) - p(c_1) \\ &\ge \sum_{l=1}^{T}(p(G(k,l)q(u(l)))) - p(c_1) \ge K^{-1} \sum_{l=1}^{T} G(k,l)p(Kq(u(l))) - p(c_1). \end{aligned} \tag{3.14}$$

Substitute (3.14) into the first inequality of (3.13), and use (3.2) to obtain

$$\begin{aligned} u(k) &\ge \sum_{l=1}^{T} G(k,l)\left(K^{-1} \sum_{m=1}^{T} G(l,m)(\gamma_1\lambda^2 Ku(m) - c) - p(c_1)\right) - c_1 \\ &\ge \gamma_1\lambda^2 \sum_{l=1}^{T} G(k,l) \sum_{m=1}^{T} G(l,m)u(m) - c_2, \end{aligned} \tag{3.15}$$

where $c_2 = 2Kc + Kp(c_1)$. Multiply both sides of the above by $\phi(k)$ and sum for $k = 1$ to T and use (2.6) to obtain

$$\sum_{k=1}^{T} u(k)\phi(k) \ge \gamma_1 \sum_{k=1}^{T} u(k)\phi(k) - c_2 \sum_{k=1}^{T} \phi(k). \tag{3.16}$$

Consequently, $\sum_{k=1}^T u(k)\phi(k) \le c_2 \sum_{k=1}^T \phi(k)/(\gamma_1 - 1)$. By (2.1), we acquire

$$\sum_{k=1}^T \|u\|\phi(k) \le T\sum_{k=1}^T u(k)\phi(k) \le \frac{Tc_2 \sum_{k=1}^T \phi(k)}{\gamma_1 - 1}, \tag{3.17}$$

therefore

$$\|u\| \le \frac{Tc_2}{\gamma_1 - 1}. \tag{3.18}$$

Multiply the first inequality of (3.13) by $\phi(k)$ and sum for $k = 1$ to T and use (2.6) to obtain

$$\|u\| \sum_{k=1}^T \phi(k) \ge \sum_{k=1}^T u(k)\phi(k) \ge \lambda^{-1} \sum_{k=1}^T p(v(k))\phi(k) - c_1 \sum_{k=1}^T \phi(k),$$

which implies

$$\sum_{k=1}^T p(v(k))\phi(k) \le \lambda(\|u\| + c_1) \sum_{k=1}^T \phi(k). \tag{3.19}$$

For any $(u, v) \in \mathcal{M}_1$, we first assume $v \not\equiv 0$, thus $p(\|v\|) > 0$ by Remark 3.1. And from $v \in P$, it follows that

$$\frac{1}{T}\|v\| \sum_{k=1}^T \phi(k) \le \sum_{k=1}^T v(k)\phi(k) = \frac{\|v\|}{p(\|v\|)} \sum_{k=1}^T \phi(k)\frac{v(k)}{\|v\|}p(\|v\|) \le \frac{\|v\|}{p(\|v\|)} \sum_{k=1}^T \phi(k)p(v(k)).$$

Consequently, by (3.19) we get $p(\|v\|) \le T\lambda(\|u\| + c_1)$.

By Remark 3.1, we know that p is strictly increasing, thus v is bounded and then there exists $c_3 > 0$ such that

$$\|v\| \le c_3, \forall (u, v) \in \mathcal{M}_1. \tag{3.20}$$

(If the $v \equiv 0$, then the inequality $\|v\| \le c_3$ also hold. So we first assume $v \not\equiv 0$.) According to (3.20) and (3.18) we know $\mathcal{M}_1$ is bounded in $\mathcal{P}$. Set $R > \sup \mathcal{M}_1$ and $R > r$, then

$$(u, v) \not\ge \mathcal{A}(u, v), \ \forall (u, v) \in \mathcal{P} \cap \partial B_R. \tag{3.21}$$

Consequently, by (2) in Lemma 2.4, (3.8) and (3.21) indicate that $\mathcal{A}$ has at least one fixed point in $(B_R \backslash \overline{B}_r) \cap \mathcal{P}$. Therefore (1.4) has at least one positive solution. This completes the proof. ■

Theorem 3.2 *Assume (H3) and (H4) hold, then problem* (1.4) *has at least one positive solution.*

Proof. By (3.5), we have

$$\mathcal{T}(u, v)(k) \ge \sum_{l=1}^T G(k, l)p(v(l)), \ \ \mathcal{S}(u, v)(k) \ge \sum_{l=1}^T G(k, l)q(u(l)), \tag{3.22}$$

for $(k, u, v) \in \mathbb{T}_1 \times [0, r] \times [0, r]$. Let

$$\mathcal{M}_2 = \{(u, v) \in \mathcal{P} \cap \overline{B}_r : (u, v) \ge \mathcal{A}(u, v)\}.$$

If $(u, v) \in \mathcal{M}_2$, then $u \ge \mathcal{T}(u, v), v \ge \mathcal{S}(u, v)$, that is,

$$u(k) \ge \sum_{l=1}^T G(k, l)p(v(l)), \ \ v(k) \ge \sum_{l=1}^T G(k, l)q(u(l)). \tag{3.23}$$

By the increasing and concave nature of p together with the second inequality of (3.23), applying Jensen's inequality, it follows that

$$p(v(k)) \geq p\left(\sum_{l=1}^{T} G(k,l)q(u(l))\right) \geq K^{-1}\sum_{l=1}^{T} G(k,l)p(Kq(u(l))). \tag{3.24}$$

Substituting the inequality (3.24) into the first inequality of (3.23), we obtain

$$u(k) \geq K^{-1}\sum_{l=1}^{T} G(k,l)\sum_{s=1}^{T} G(l,s)p(Kq(u(s))).$$

Multiply both sides of the above by $\phi(k)$ and sum for $k = 1$ to T, and use (2.6) and (3.6) to obtain

$$\sum_{k=1}^{T} u(k)\phi(k) \geq \gamma_3 \sum_{k=1}^{T} u(k)\phi(k).$$

Since $\gamma_3 > 1$, it implies $\sum_{k=1}^{T} u(k)\phi(k) = 0$, and then $u \equiv 0$. It follows from the first inequality of (3.23) that $p(v) = 0$, thus $v \equiv 0$ as p is nondecreasing (see Lemma 2.5 in [16]). It proves $\mathcal{M}_2 = \{(0,0)\}$. Hence

$$(u,v) \not\geq \mathcal{A}(u,v), \ \forall (u,v) \in \mathcal{P} \cap \partial B_r. \tag{3.25}$$

On other hand, for all $(k,u,v) \in \mathbb{T}_1 \times \mathbb{R}^+ \times \mathbb{R}^+$, it follows that, by (3.7),

$$\mathcal{T}(u,v)(k) \leq \sum_{l=1}^{T} G(k,l)(au(l)+bv(l)+e), \ \mathcal{S}(u,v)(k) \leq \sum_{l=1}^{T} G(k,l)(cu(l)+dv(l)+e). \tag{3.26}$$

Next, we will prove there exists a number $R > r$ such that set

$$\mathcal{M}_3 = \{(u,v) \in \mathcal{P} \cap \overline{B}_R : (u,v) \leq \mathcal{A}(u,v)\}$$

is bounded. Let $(u,v) \in \mathcal{M}_3$. Then, by (3.26),

$$u(k) \leq \sum_{l=1}^{T} G(k,l)(au(l)+bv(l)+e), \ v(k) \leq \sum_{l=1}^{T} G(k,l)(cu(l)+dv(l)+e).$$

Multiply both sides of the above by $\phi(k)$ and sum for $k = 1$ to T and use (2.6) to obtain

$$\begin{aligned} \sum_{k=1}^{T} u(k)\phi(k) &\leq \lambda^{-1}\sum_{k=1}^{T} \phi(k)(au(k)+bv(k)+e), \\ \sum_{k=1}^{T} v(k)\phi(k) &\leq \lambda^{-1}\sum_{k=1}^{T} \phi(k)(cu(k)+dv(k)+e), \end{aligned} \tag{3.27}$$

which can be written in the form

$$\begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix} \begin{bmatrix} \sum_{k=1}^{T} \phi(k)u(k) \\ \sum_{k=1}^{T} \phi(k)v(k) \end{bmatrix} \leq \begin{bmatrix} e\sum_{k=1}^{T} \phi(k) \\ e\sum_{k=1}^{T} \phi(k) \end{bmatrix}.$$

By (3.7), let $\rho := (\lambda - a)(\lambda - d) - bc$, the above implies

$$\begin{bmatrix} \sum_{k=1}^{T} \phi(k)u(k) \\ \sum_{k=1}^{T} \phi(k)v(k) \end{bmatrix} \leq \rho^{-1} \begin{bmatrix} \lambda - d & b \\ c & \lambda - a \end{bmatrix} \begin{bmatrix} e\sum_{k=1}^{T} \phi(k) \\ e\sum_{k=1}^{T} \phi(k) \end{bmatrix},$$

from which we have

$$\sum_{k=1}^{T} \phi(k)u(k) \leq \rho^{-1}e((\lambda - d) + b)\sum_{k=1}^{T} \phi(k),$$

$$\sum_{k=1}^{T} \phi(k)v(k) \leq \rho^{-1}e(c + (\lambda - a))\sum_{k=1}^{T} \phi(k).$$

Noting that $(u, v) \in \mathcal{P}$, we obtain

$$\sum_{k=1}^{T} \|u\|\phi(k) \leq T\sum_{k=1}^{T} \phi(k)u(k), \ \sum_{k=1}^{T} \|v\|\phi(k) \leq T\sum_{k=1}^{T} \phi(k)v(k)$$

and thus

$$\|u\| \leq T\rho^{-1}e((\lambda - d) + b), \ \|v\| \leq T\rho^{-1}e(c + (\lambda - a)).$$

This proves $\mathcal{M}_3$ is bounded. Taking $R > \max\{\sup \mathcal{M}_3, r\}$, we obtain

$$(u, v) \not\leq \mathcal{A}(u, v), \ \forall (u, v) \in \mathcal{P} \cap \partial\overline{B}_R. \tag{3.28}$$

By (1) in Lemma 2.4, (3.25) and (3.28) indicate $\mathcal{A}$ has at least one fixed point in $(B_R\backslash\overline{B}_r) \cap \mathcal{P}$. Therefore (1.4) has at least one positive solution. This completes the proof. ■

Theorem 3.3 *Assume that (H1), (H3) and (H5) hold. Then problem* (1.4) *has at least two positive solutions.*

Proof. By (H5), we have

$$\mathcal{T}(u, v)(k) < \frac{M}{K}\sum_{l=1}^{T} G(k, l) \leq M, \quad \mathcal{S}(u, v)(k) < \frac{M}{K}\sum_{l=1}^{T} G(k, l) \leq M,$$

for any $(k, u, v) \in \mathbb{T}_1 \times \partial B_M \times \partial B_M$, from which we obtain

$$\|\mathcal{A}(u, v)\| < \|(u, v)\|, \forall (u, v) \in \mathcal{P} \cap \partial B_M.$$

It implies that

$$\mathcal{A}(u, v) \not\geq (u, v), \forall (u, v) \in \mathcal{P} \cap \partial B_M. \tag{3.29}$$

On the other hand, by (H1) and (H3), we take $R > M$ and $0 < r < M$ such that (3.21) and (3.25) hold (see Theorem 3.1 and Theorem 3.2). Combining (3.21), (3.25) and (3.29), by Lemma 2.4, we acquire $\mathcal{A}$ has at least two positive fixed points, one in $(B_R\backslash\overline{B}_M)\cap\mathcal{P}$ and another in $(B_M\backslash\overline{B}_r)\cap\mathcal{P}$. Thus (1.4) has at least two positive solutions. The proof is completed. ■

Theorem 3.4 *Suppose (H2), (H4) and (H6) hold. Then problem* (1.4) *has at least two positive solutions.*

Proof. By (H6) and (2) in Lemma 2.1, we have

$$\mathcal{T}(u,v)(k) > N\eta \sum_{l=1}^{T} G(k,l) \geq \frac{N\eta}{T} \sum_{l=1}^{T} G(l,l) = N,$$

$$\mathcal{S}(u,v)(k) > N\eta \sum_{l=1}^{T} G(k,l) \geq \frac{N\eta}{T} \sum_{l=1}^{T} G(l,l) = N,$$

for any $(k,u,v) \in \mathbb{T}_1 \times \partial B_N \times \partial B_N$, from which we obtain

$$\|\mathcal{A}(u,v)\| > \|(u,v)\|, \forall (u,v) \in \mathcal{P} \cap \partial B_N,$$

which implies that

$$\mathcal{A}(u,v) \not\leq (u,v), \forall (u,v) \in \mathcal{P} \cap \partial B_N. \tag{3.30}$$

On the other hand, by (H2) and (H4), we take $0 < r < N$ and $R > N$ such that (3.8) and (3.28) hold (See Theorem 3.1 and Theorem 3.2). Combining (3.8), (3.28) and (3.30), by Lemma 2.4, we acquire $\mathcal{A}$ has at least two positive fixed points, one in $(B_R \backslash \overline{B}_N) \cap \mathcal{P}$ and another in $(B_N \backslash \overline{B}_r) \cap \mathcal{P}$. Thus (1.4) has at least two positive solutions. The proof is completed. ∎

4 Numerical Examples

We now present two numerical examples illustrating Theorem 3.1 and Theorem 3.2 respectively.

Example 4.1 *Consider the problem, for* $k \in \mathbb{T}_2 = \{0, 1, 2, ..., 51\}$,

$$\begin{cases} \triangle^2 u(k-1) + \frac{2(u+2)v^{3/2}}{u+1} = 0, \ k \in \{1, 2, ..., 50\}, \\ \triangle^2 v(k-1) + \frac{1}{3250}\frac{u^2(11+10v^2)}{1+v^2} = 0, \\ u(0) = u(51) = v(0) = v(51) = 0. \end{cases} \tag{4.1}$$

In the following, we will check that the example (4.1) fits the conditions (H1) and (H2).

First, we compute $K = 325$ and $\lambda = 0.0038$. We have set

$$f(k,u,v) := \frac{2(u+2)v^{3/2}}{u+1}, \quad g(k,u,v) := \frac{u^2(11+10v^2)}{3250(1+v^2)}.$$

Set $p(v) = v^{4/5}/2 + 10$ and $q(u) = u^2/1300 + 1/5$. Then $p, q : [0, +\infty) \to [0, +\infty)$ are continuous and p is concave. Observe that

$$\lim_{v \to +\infty} \frac{f(k,u,v)}{p(v)} = +\infty, \ \forall u \in \mathbb{R}^+,$$

and that

$$\lim_{u \to +\infty} \frac{g(k,u,v)}{q(u)} \geq 4, \ \forall v \in \mathbb{R}^+.$$

So, there exist constant $c = 10 > 0$ such that (3.1) of (H1) holds.

Moreover, taking $\gamma_1 = 100 > 1$, we get

$$p(Kq(u)) > 0.468u,$$

that is, inequality (3.2) of (H1) holds.

On the other hand, we shall check that conditions (H2) is satisfied. Set $\alpha(v) = 21v^{3/2}/5$ and $\beta(u) = 11u^2/3250$. Then $\alpha, \beta : [0, +\infty) \to [0, +\infty)$ are continuous and α is convex and strictly increasing.

Obviously, for $\|u\| \le 0.03 =: r$, $\|v\| \le 0.03$ we observe that $f(k, u, v) \le \alpha(v)$ and that $g(k, u, v) \le \beta(u)$. Moreover, taking $\gamma_2 = 0.99 < 1$, we obtain $\alpha(K\beta(u)) \le 0.0046u, \forall u \in [0, r]$. Thus, condition (H2) holds.

In summary, the example (4.1) fits the conditions of Theorem 3.1 and has at least one positive solution by the theorem.

Example 4.2 *Consider the problem, for* $k \in \mathbb{T}_2 = \{0, 1, 2, ..., 41\}$,

$$\begin{cases} \triangle^2 u(k-1) + 1 + e^{-(u+v)} = 0, \ k \in \{1, 2, ..., 40\}, \\ \triangle^2 v(k-1) + 1 + \frac{1}{u+v+1} = 0, \\ u(0) = u(41) = v(0) = v(41) = 0. \end{cases} \tag{4.2}$$

In the following, we will check that the example (4.2) fits the conditions (H3) and (H4).

First, we find $K = 210$ and $\lambda = 0.0059$. We have set

$$f(k, u, v) := 1 + e^{-(u+v)}, \quad g(k, u, v) := 1 + \frac{1}{u + v + 1}.$$

Set $p(v) = \ln(v + 1)$ and $q(u) = u^2$. Then $p, q : [0, +\infty) \to [0, +\infty)$ are continuous and p is concave and nondecreasing. We observe that

$$\lim_{v \to 0} \frac{f(k, u, v)}{p(v)} = +\infty, \quad \forall (k, u) \in \mathbb{T}_1 \times [0, +\infty)$$

and

$$\lim_{u \to 0} \frac{g(k, u, v)}{q(u)} = +\infty, \quad \forall (k, v) \in \mathbb{T}_1 \times [0, +\infty).$$

In fact, when $r := 1.2$ the inequality (3.5) holds.

Moreover,

$$p(Kq(u)) = \ln(210u^2 + 1) \ge 0.72u,$$

for $\gamma_3 = 100, \forall u \in [0, r]$. So, (H3) holds.

In the following, we shall check (H4) is satisfied. Set $a = 0.0009 < \lambda, b = 0.004, e = 0.5$ and $c = 0.006, d = 0.0009 < \lambda$, then $(\lambda - a)(\lambda - d) = 2.5 \times 10^{-5} > bc = 2.4 \times 10^{-5}$. We get

$$\lim_{(u,v) \to (+\infty, +\infty)} \frac{f(k, u, v)}{au + bv + e} \le \lim_{(u,v) \to (+\infty, +\infty)} \frac{400}{v + u} = 0$$

and

$$\lim_{(u,v) \to (+\infty, +\infty)} \frac{g(k, u, v)}{cu + dv + e} \le \lim_{(u,v) \to (+\infty, +\infty)} \frac{1000}{3(u + v)} = 0.$$

Thus, (H4) holds. Therefore, we conclude that (4.2) has at least one solution by Theorem 3.2.

Acknowledgments. The authors are grateful to the editor and anonymous referee for their valuable suggestions.

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