Embedding Complete Binary Trees into Star Networks *

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Abstract. Star networks have been proposed as a possible interconnection network for massively parallel computers. In this paper we investigate embeddings of complete binary trees into star networks. Let G and H be two networks represented by simple undirected graphs. An *embedding* of G into H is an injective mapping f from the vertices of G into the vertices of H. The *dilation* of the embedding is the maximum distance between $f(u), f(v)$ taken over all edges (u, v) of G.
Low dilation embeddings of binary trees into star graphs correspond to efficient simulations of parallel algorithms that use the binary tree topology, on parallel computers interconnected with star networks.
First, we give a construction of embeddings of dilation 1 of complete binary trees into n-dimensional star graphs. These trees are subgraphs of star graphs. Their height is $\Omega(n \log n)$, which is asymptotically optimal. Constructions of embeddings of complete binary trees of dilation 2δ and $2\delta + 1$, for $\delta \geq 1$, into star graphs are then given. The use of larger dilation allows embeddings of trees of greater height into star graphs. For example, the difference of the heights of the trees embedded with dilation 2 and 1 is greater than $n/2$. All these constructions can be modified to yield embeddings of dilation 1, and 2δ, for $\delta \geq 1$, of complete binary trees into pancake graphs.
Our results show that massively parallel computers interconnected with star networks are well suited for efficient simulations of parallel algorithms with complete binary tree topology.

1 Introduction

Several large-scale processor networks of different topologies have been implemented or are being considered for implementation. Users of these networks might wish to use a parallel algorithm which is designed for a different topology. It is therefore necessary to develop methods which would enable the user to simulate efficiently one network topology, say G, on a different topology, say H. Usually, different processors of G would be mapped on different processors of H. In case when a processor in network H can communicate directly only with

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those processors to which it is directly connected (the store and forward communication mode), an efficient simulation would require that the processors which are adjacent in network G would be mapped either onto adjacent processors of H or onto processors that can communicate with a few intermediate hops. Since a topology of a network can be represented by a graph in which the vertices represent the processors and the edges represent the communication channels, the problem of efficient network simulation can be formulated in graph-theoretical terms as that of finding a graph embedding of G into H with a low dilation.

Let G and H denote two simple, undirected graphs. In general, an *embedding* of the graph G into the graph H is an injective mapping f of the vertices of G into the vertices of H together with a mapping P_f which assigns to each edge (u,v) of G a path between $f(u)$ and $f(v)$ in H.

The *dilation* of a given embedding f, denoted by $dil(f)$, is defined to be the maximum of $\{length(P_f(u,v)) : (u,v) \in E(G)\}$. Since our goal is to construct embeddings of low dilation, we will take P_f to be a mapping that assigns to each edge (u,v) of G a shortest path between the vertices $f(u)$ and $f(v)$ of H. Thus, in this paper $dil(f) = max\{d_H(f(u), f(v)) : (u,v) \in E(G)\}$, where $d_H(x,y)$ denotes the distance between x and y in the graph H. The *minimum dilation* of an embedding of G into H, denoted $dil(G,H)$, is the minimum of $dil(f)$ taken over all embeddings of G into H.

The *expansion* of an embedding f is the ratio of the number of vertices of H to the number of vertices of G. Since we use injective mappings in this paper, the expansion of all embeddings will be at least one. A number of papers has been published in the last ten years on embeddings of a given network into another one for networks such as grids, hypercubes, trees (see [10]).

The *star graphs* were proposed in [1] as a topology for interconnecting processors in large scale parallel computers. These graphs belong to the family of Cayley graphs [3], a family of graphs obtained from representations of groups, and they have very many interesting properties [1].

Let n be a positive integer. The *star graph* S_n *of dimension* n is a graph whose vertex set consists of all permutations of $\{1, 2, \ldots, n\}$. The ith position of a vertex $x_1 x_2 \ldots x_n$ of a star graph will be referred to as the *ith coordinate* of the vertex. A vertex $x_1 x_2 \ldots x_n$ is adjacent to the vertices $x_i x_2 \ldots x_{i-1} x_1 x_{i+1} \ldots x_n$, for $2 \leq i \leq n$, i.e., vertices obtained by a transposition of the symbol in the first coordinate with the symbol in the ith coordinate of the vertex for $2 \leq i \leq n$. Thus, the star graph of dimension n has $n!$ vertices and each of its vertices is adjacent to $n-1$ other vertices. The diameter of S_n is equal to $\lfloor \frac{3}{2}(n-1) \rfloor$ ([1]).

For any nonnegative integer h, the *complete binary tree* of height h, denoted T_h, is the binary tree where each internal vertex has exactly two children and all the leaves are at distance h from the root of the tree. For a complete binary tree T_h, the *level* i, $0 \leq i \leq h$, is defined as the set of all vertices of T_h at distance i from the root of the tree. The tree T_h has $h+1$ levels and level i, $0 \leq i \leq h$, contains 2^i vertices.

The problem of embedding a graph into star graphs has been already studied for some families of graphs. Nigam et al. [11] showed that the star graph S_n

contains a Hamiltonian cycle for every n, $n > 2$, and presented an embedding of hypercubes into the star graphs (see also [2], [9]). Jwo et al. [7] considered embeddings of cycles and grids into star graphs.

Since a complete binary tree is a common topology of many parallel algorithms, it is important to determine how well the star networks can simulate it. Thus, in this paper we consider the problem of embedding complete binary trees into the star graph and give constructions of embeddings of low dilation. In these constructions there is a trade-off between the dilation and expansion i.e., the use of a larger dilation produces a smaller expansion. Notice that embeddings of complete binary trees into star graphs obtained by a composition of the known embeddings of binary trees into hypercubes [8] and hypercubes into star graphs [11] gives embeddings of dilation at least 2 whose expansion is larger than that of the dilation 1 embeddings from Theorem 1.

Our constructions will use the following property of star graphs. Let α, $1 \leq \alpha \leq n$, be an integer and let V_i^α be the set of all vertices of S_n in which the symbol in the ith coordinate is equal to α. For every α, the subgraph of S_n induced by V_i^α is isomorphic to S_{n-1}. Furthermore, the substars induced by V_i^α and V_j^β are vertex disjoint if either $i = j$ and $\alpha \neq \beta$ or $i \neq j$ and $\alpha = \beta$.

We denote by $h(n)$ the height of the largest complete binary tree whose number of nodes is at most equal to the number of nodes of the n-dimensional star graph i.e., $h(n)$ is the largest integer k such that $2^{k+1} - 1 \leq n!$, and therefore $h(n)$ is $O(n \log n)$. We denote by $h_\delta(n)$ the maximum height of a complete binary tree that we can embed into S_n with dilation δ. Clearly, $h_1(n) \leq h_2(n) \leq \ldots h_{\lfloor 3(n-1)/2 \rfloor}(n) = h(n)$ since the diameter of the star graph is $\lfloor \frac{3}{2}(n-1) \rfloor$ ([1]).

We will describe an embedding of a tree into a star graph by giving a labeling of the vertices of the tree with vertices of the star graph on which they are mapped. The label of the root of the tree in our constructions will be $12 \ldots n$.

We first consider dilation 1 embeddings of complete binary trees into the star graphs. This actually produces complete binary subtrees of the star graphs. Our construction gives dilation 1 embeddings of complete binary trees into S_n whose height is $\Omega(n \log n)$, which is asymptotically the best possible.

We then give constructions of embeddings of dilation 2, and discuss embeddings of dilation $2\delta - 1$ and 2δ, for $\delta \geq 2$, of complete binary trees into the star graphs. All constructions for dilations at least 2 follow the same general idea, which is different from the one used for dilation 1. The use of larger dilation allows us to reduce the expansion of the embeddings by a non-constant factor.

Our results show that the star networks are very suitable for efficient simulation of algorithms that are using complete binary tree topology.

In this paper we give only outlines of the main proofs. All proofs can be found in our research report [5].

2 Embeddings of dilation 1

We begin with a simple result from [4], which will be used in the proof of our main theorem on the dilation 1 embeddings of complete binary trees.

Theorem 1. *For every n, $n \geq 2$, there is a dilation 1 embedding of the complete binary tree T_{n-2} into the star graph S_n.*

Note that the height of the embedded tree is only proportional to the dimension n of the star graph which is far from the upper bound $O(n \log n)$. The next theorem improves this result by showing the existence of a complete binary tree which is a subgraph of S_n and has height $\Omega(n \log n)$, the result being therefore asymptotically optimal.

Theorem 2. *For $n = 5$ or 6 there exists a dilation 1 embedding of the complete binary tree of height $2n - 5$ into the star graph S_n. For $n \geq 7$, there exists a dilation 1 embedding of the complete binary tree of height $h_1(n) = p(n - 2^p) + 3$ into the star graph S_n, where p is defined to be the integer such that $(p+1)2^{p-1} < n \leq (p+2)2^p$.*

The proof of the theorem is omitted here and can be found in [5]. The idea is to first construct embeddings directly for the cases $n = 5, 6$ or 7. And then, for $n > 7$, $(p+1)2^{p-1} < n \leq (p+2)2^p$, to proceed by induction on p, and, for a given p, by induction on n, using extensively the next lemma, for which we include below the main idea of the construction.

We first introduce some definitions. The graph formed by a path $c_0c_1 \cdots c_{p+1}$ of length $p+1$ in which each vertex c_i, $0 \leq i \leq p$, is adjacent to a pendant vertex, will be called the *b-comb* of length $p+1$ (a short for a "broken comb"). The path $c_0c_1 \cdots c_{p+1}$ will be called the main path of the comb. The vertex c_0 will be called the initial vertex of the b-comb.

We define T'_{2p} to be a binary tree of height $2p$ having the following shape. The first $p - 1$ levels (from 0 to $p - 2$) of the tree T'_{2p} form a complete binary tree. Each vertex of level $p - 2$ has two children, each of them being the initial vertex of a b-comb of length $p + 1$.

A complete binary tree $T_{h_1(n)}$, $h_1(n) \geq 2p$, can be obtained from T'_{2p} by identifying each leaf at level $p+i$, $0 \leq i \leq p$, of T'_{2p} with the root of a complete binary tree of height $h_1(n) - p - i$. Thus we construct a dilation 1 embedding of $T_{h_1(n)}$ into S_n by giving a dilation 1 embedding of T'_{2p} into S_n and by using embeddings of T_l, $l \leq h_1(n) - p$, into S_n (which exist by the induction hypothesis).

Lemma 3. *For any integer n, define p to be the integer such that $(p+1)2^{p-1} < n \leq (p+2)2^p$. Then, for any $n \geq 8$,*

$$h_1(n) = p + \min(h_1(n-1), p + h_1(n-2)).$$

Outline of proof. We first construct a labeling of the vertices of the binary tree T'_{2p} with vertices of S_n so that
(i) adjacent vertices of the tree are labeled with adjacent vertices of S_n,
(ii) the labels of all vertices of the complete binary subtree on the first p levels (obtained by using Theorem 1) and all vertices of the main paths of the 2^{p-1} combs, are vertices of the substar of S_n having n in the last coordinate. Moreover the last vertices of these paths (which are at level $2p$) are labeled with vertices

of different substars of S_n of dimension $n-2$ whose symbols in the last two coordinates are equal to αn for different symbols α which are not used in the levels strictly less than $2p$ in coordinate $n-1$.
(iii) the labels of the pendant vertices of all the combs (the number of which is equal to $2^{p-1}(p+1)$) end with different symbols excluding n (this is possible since $n > 2^{p-1}(p+1)$) and therefore they belong to different $(n-1)$ dimensional substars of S_n.
An example of such a labeling of the tree T'_{2p} for $p=3$ is given in Figure 1. The details of the construction are omitted here and can be found in [5].

We now use T'_{2p} and its labeling to obtain an embedding of a complete binary tree T into S_n. Identify each leaf of T'_{2p} at level $p+i$, $0 \le i \le p$, having a label $a_1a_2\cdots a_n$, $a_n \neq n$, with the root of a complete binary tree of height $h_1(n-1)-i$. The labeling of this complete binary subtree is obtained from a dilation 1 embedding of $T_{h_1(n-1)}$ into the star graph S_{n-1} by applying the permutation

$$\begin{pmatrix} 1 & 2 & \ldots n-1 \\ a_1 & a_2 & \ldots a_{n-1} \end{pmatrix}$$

to its labels, and appending a_n as a suffix. Finally, identify each leaf of level $2p$ of T'_{2p} having a label $a_1a_2\cdots\alpha n$ with the root of a complete binary tree of height $h_1(n-2)$. The labeling of this complete binary subtree is obtained from a dilation 1 embedding of $T_{h_1(n-2)}$ into the star graph S_{n-2} by applying the permutation

$$\begin{pmatrix} 1 & 2 & \ldots n-2 \\ a_1 & a_2 & \ldots a_{n-2} \end{pmatrix}$$

to its labels, and appending αn to them as a suffix. In both cases such embeddings of $T_{h_1(n-1)}$ and $T_{h_1(n-2)}$ exist by the definition of $h_1(n-1)$ and $h_1(n-2)$.

Thus we have obtained a labeling of a tree T of height $p+\min(h_1(n-1), p+h_1(n-2))$, which defines an embedding of the tree into S_n. It is clear from the construction that this embedding has dilation 1. □

Using the expression for $h_1(n)$ from Theorem 2 we can calculate a lower bound on $h_1(n)$ and get the following.

Proposition 4. *$h_1(n) \ge (1/2+\epsilon(n))n\log_2 n$ where $\epsilon(n)$ tends to 0 when n tends to infinity.*

Since the number of vertices of $S(n)$ is $n!$, the largest complete binary tree which could be embedded into S_n irrespective of dilation has height $\lfloor \log_2(n!+1)\rfloor - 1 \le n\log_2 n$ for n large enough. From the previous proposition, $h_1(n) = \Omega(n\log_2 n)$, and thus the star graph has as a subgraph a complete binary tree with asymptotically optimal height. The existence of such complete binary subtrees could be useful for designing efficient parallel algorithms for star networks.

3 Embeddings of dilation 2

By allowing a larger dilation we can embed larger complete binary tree into star graphs. It is simple to verify that T_3 cannot be embedded into S_4 with dilation

1, but it is very easy to obtain a dilation 2 embedding of T_3 into S_4. In this section we give a recursive construction of dilation 2 embeddings of complete binary trees into star graphs. The following lemma, given without proof, will be used as a starting point for the recursion.

Lemma 5. *There exists a dilation 2 embedding of T_8 into S_6.*

The next lemma will be used for the recursive step of the main theorem of this section.

Lemma 6. *For $n \geq 5$,*

$$h_2(n) = h_2(n-1) + \lfloor \log_2(n) \rfloor .$$

Outline of proof. Let n be an integer, $n \geq 5$, and let $j = \lfloor \log_2 n \rfloor$, i.e., $2^j \leq n < 2^{j+1}$. We show that, given dilation 2 embeddings of complete binary tree $T_{h_2(i)}$ into S_i, for $5 \leq i \leq n-1$, we can construct a labeling of the vertices of the complete binary tree $T_{h_2(n-1)+j}$ with vertices of S_n such that any two adjacent labels correspond to vertices at distance at most two in S_n, i.e., a dilation 2 labeling.

Assume that for every i, $5 \leq i \leq n-1$, there is a labeling of $T_{h_2(i)}$ with vertices of S_i, such that the labels of any two adjacent vertices of $T_{h_2(i)}$ are at distance at most 2 in S_i. We may also assume without loss of generality that the label of the root of $T_{h_2(i)}$, $5 \leq i \leq n-1$, is equal to $12\ldots i$.

If n is not a power of 2, i.e., $n > 2^j$, $j \geq 2$, then S_n consists of at least 2^j+1 disjoint substars of dimension $n-1$. We construct a dilation 2 embedding of $T_{h_2(n)}$ into S_n by labeling the first j levels of the tree with vertices of the substar of dimension $n-1$ having n in the last coordinate, and by labeling each of the 2^j subtrees rooted at level j with vertices of a substar of dimension $n-1$ having symbol i in the last coordinate, $1 \leq i \leq 2^j$, as follows.

The labels in the levels $0, 1, \ldots, j-1$ of $T_{h_2(n)}$ are obtained from the labeling of the first j levels of an embedding of $T_{h_1(n-1)}$ into S_{n-1}, by appending n to each label as a suffix. This is possible since $h_1(n-1) \geq j-1$ by Theorem 1.

Let u_i, for $1 \leq i \leq 2^j$, denote the label of the ith vertex from the left at the level j of $T_{h_2(n)}$. The label of u_i is obtained from the label of its parent, say $p(u_i)$, by a transposition of the symbol i of $p(u_i)$ with the symbol in the first coordinate of $p(u_i)$ and then by transposing the symbol i with the symbol n in the last coordinate. If $na_2\ldots a_{n-1}i$ is the resulting label of u_i then the labels of the vertices in the subtree of this vertex are obtained from the labels of the tree $T_{h_2(n-1)}$ into S_{n-1} by applying the permutation

$$\begin{pmatrix} 1 & 2 & \ldots & n-1 \\ n & a_2 & \ldots & a_{n-1} \end{pmatrix}$$

and by appending i to them as a suffix.

If n is a power of 2, the construction of a labeling of $T_{h_2(n)}$ is essentially similar to the one above, although it involves more work, and will not be given here. □

Using the two previous lemmas it is now not difficult to show the following result by induction on n (the proof is omitted here).

Theorem 7. *For every integer $n \geq 6$, there is a dilation 2 embedding of the complete binary tree $T_{h_2(n)}$ into the star graph S_n, where $h_2(n) = (n+1)\lfloor \log_2 n \rfloor - 2^{\lfloor \log_2 n \rfloor + 1} + 2$.*

The next proposition shows that the difference between the heights of the trees we can embed into S_n with dilation 1 and 2 is at least linear in n.

Proposition 8. *$h_2(n) - h_1(n) \geq n/2$ for $n \geq 12$.*

4 Embeddings of dilation 2δ and $2\delta - 1$ for $\delta \geq 2$

The construction used in Theorem 7 can be generalized to dilations $2\delta - 1$ and 2δ for $\delta \geq 2$. Further increase in the dilation allows us to get closer to the upper bound on the size of the complete binary tree that can be embedded into a star graph.

Lemma 9. *Let n, δ be integers such that $\delta \geq 2$, $n > \delta + 1$, and let j be equal to $\lfloor \log_2 n \rfloor$.*

$$h_{2\delta-1}(n) = h_{2\delta-1}(n-1) + j \qquad \text{if} \quad (n-1)\prod_{l=1}^{\delta-1}(n-\delta+l) < 2^{\delta j+1}$$

$$h_{2\delta-1}(n) = h_{2\delta-1}(n-\delta) + \delta j + i \quad \text{if} \quad (n-1)\prod_{l=1}^{\delta-1}(n-\delta+l) \geq 2^{\delta j+i}$$
$$\text{for some } i > 0 \text{ and } \delta j + i - 1 \leq h_{2\delta-1}(n-\delta).$$

Outline of proof. The construction given in the proof of Lemma 6 is valid for embeddings of any dilation ≥ 2 and, thus, $h_{2\delta-1}(n) \geq h_{2\delta-1}(n-1) + j$ for any $\delta \geq 2$. Therefore, we only need to give a construction in case when $(n-\delta+1)(n-\delta+2)\ldots(n-2)(n-1)^2 \geq 2^{\delta j+i}$ for some $i > 0$, and $\delta j + i - 1 \leq h_{2\delta-1}(n-\delta)$.
Let $k = h_{2\delta-1}(n-\delta)$. We can construct a dilation $2\delta - 1$ labeling of the vertices of a complete binary tree $T_{k+\delta j+i}$ with vertices of S_n such that
(i) the labels of the vertices of the first $\delta j + i$ levels are vertices of the substar of S_n of dimension $n - \delta$ having the symbols of the last δ coordinates equal to $(n-\delta+1)(n-\delta+2)\cdots n$
(ii) the labels of the vertices of the $2^{\delta j+i}$ subtrees of height k rooted at level $\delta j + i$ are vertices of different substars of dimension $n - \delta$. The vertices of each of the substars contain n in a fixed coordinate i between 2 and n, and if $2 \leq i \leq n - \delta$ then they have the symbols in the last $\delta - 1$ coordinates fixed, else if $n - \delta + 1 \leq i \leq n$ then they have the symbols in the last δ coordinates fixed. In either case the symbols in the last coordinates differ in at least one coordinate from $(n-\delta+1)(n-\delta+2)\cdots n$. The number of substars verifying these conditions is $(n-\delta+1)\ldots(n-2)(n-1)^2$ which is at least equal to $2^{\delta j+i}$ by the assumption.
The details of the construction can be found in [5]. □

Similarly we can prove the following lemma.

Lemma 10. *Let n, δ be integers such that $\delta \geq 2$, $n > \delta + 1$, and let j be equal to $\lfloor \log_2 n \rfloor$.*

$$h_{2\delta}(n) = h_{2\delta}(n-1) + j \quad \text{if} \quad (n-\delta+1)(n-\delta+2)\cdots n < 2^{\delta j+1}$$
$$h_{2\delta}(n) = h_{2\delta}(n-\delta) + \delta j + i \quad \text{if} \quad (n-\delta+1)(n-\delta+2)\cdots n \geq 2^{\delta j+i}$$
$$\text{for some } i > 0 \text{ and } \delta j + i - 1 \leq h_{2\delta}(n-\delta).$$

In the case of dilation i, for $i = 3$ or 4, we obtained the explicit formulas of $h_i(n)$ given in the theorem below.

Theorem 11. *For every integer $n \geq 8$, there is a dilation i embedding of the complete binary tree $T_{h_i(n)}$, for $i = 3$ or 4, into the star graph S_n, where*
$h_3(n) = h_4(n-1) + \lfloor log_2 n \rfloor - 1$,
$h_4(n) = (n+1)\lfloor log_2 n \rfloor - 2^{\lfloor log_2 n \rfloor + 1} + 2^{\lfloor log_2 n \rfloor - 2} + 1 \quad for\ 2^p \leq n \leq 2^p + 2^{p-1} - 1$
$h_4(n) = (n+1)\lfloor log_2 n \rfloor - 2^{\lfloor log_2 n \rfloor + 1} - 2^{\lfloor log_2 n \rfloor - 1} + \lfloor \frac{n}{2} \rfloor + 2$
for $2^p + 2^{p-1} \leq n < 2^{p+1}$.

Proposition 12. *For every integer n, $n \geq 8$, $(n-5)/6 \leq h_4(n) - h_2(n) < n/4$.*

5 Table of results and conclusion

The results of the previous sections for star graphs of dimensions 3 to 18 are summarized in Table 1. If our embedding is the best possible with respect to the height of the embedded tree, we print the value in bold.

Table 1.

n	order of $S_n = n!$	$h(n)$	order of $T_{h(n)}$	dilation 1 height $h_1(n)$	2 $h_2(n)$	3 $h_3(n)$	4 $h_4(n)$	5 $h_5(n)$
3	6	1	3	**1**				
4	24	3	15	**2**	**3**			
5	120	5	63	**5**				
6	720	8	511	7	**8**			
7	5040	11	4095	9	10		**11**	
8	40320	14	32767	11	13		**14**	
9	362880	17	261143	13	16		**17**	
10	$\approx 3.6\ 10^6$	20	$\approx 2\ 10^6$	15	19		**20**	
11	$\approx 4\ 10^7$	24	$\approx 3.4\ 10^7$	17	22		23	
12	$\approx 4.8\ 10^8$	27	$\approx 2.7\ 10^8$	19	25		**27**	
13	$\approx 6.2\ 10^9$	31	$\approx 4.3\ 10^9$	21	28	29	30	
14	$\approx 8.7 10^{10}$	35	$\approx 6.9 10^{10}$	23	31	32	34	
15	$\approx 1.3\ 10^{12}$	39	$\approx 1.1\ 10^{12}$	25	34	36	37	38
16	$\approx 2.1\ 10^{13}$	43	$\approx 1.8\ 10^{13}$	27	38	40	41	42
17	$\approx 3.5\ 10^{14}$	47	$\approx 2.8\ 10^{14}$	30	42	44	45	46
18	$\approx 6.4\ 10^{15}$	51	$\approx 4.5\ 10^{15}$	33	46	48	49	50

The low dilation embeddings of complete binary trees into star graphs presented in this paper are asymptotically optimal. In particular, for the range of dimensions of star graphs shown in the table, they approach closely the best possible expansion. Notice that our constructions give embeddings of trees of optimum height with dilation 4 into S_n for n up to 10 and $n = 12$. Since the star graph of dimension 12 has more than 10^8 vertices, our results give low dilation, best expansion embeddings of complete binary trees into star graphs of feasible sizes. Thus we have shown that star networks, similarly as hypercubes and de Bruijn graphs [8], can efficiently simulate any algorithm designed for complete binary trees.

Although we did not include the results here, it can be easily obtained that the average dilation of our dilation 2,3 or 4 embeddings into S_n is less than 1.2422, 1.2423, 1.943, respectively. We should also point out that our constructions and results can be easily modified to obtain embeddings of dilation 1,2, and 2δ of complete binary trees into pancake graphs (and more generally into recursively decomposable Cayley graphs), see [6] for the definitions.

Many interesting problems remain open. We conclude our paper by mentioning some of them below.

1. Determine a nontrivial upper bound on the height of a complete binary tree which can be embedded into S_n (or P_n) with dilation δ for $1 \leq \delta$.
2. Construct embeddings of dilation $2i + 1$ into pancake graphs such that $h_{2i}(n) < h_{2i+1}(n)$ for large n.
3. Given n, determine the smallest dilation for which there is an embedding of a complete binary tree into the star graph S_n having the optimum expansion.

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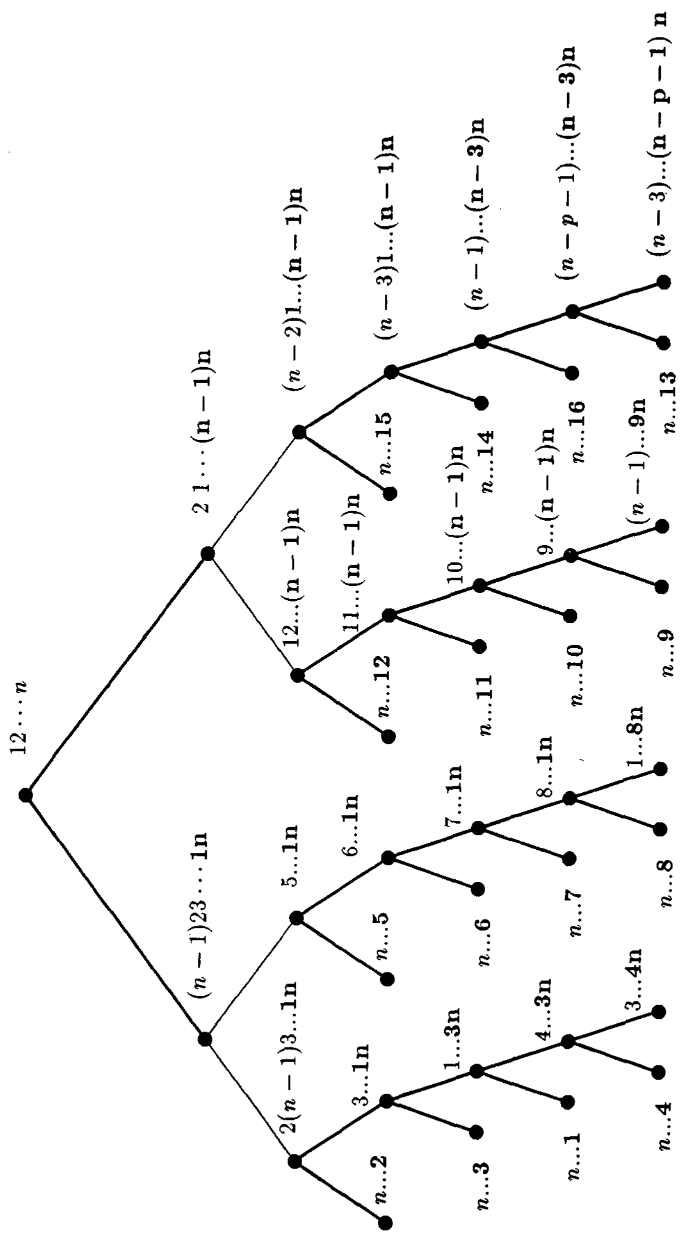

Figure 1. Labeling of the tree T'_{2p}, $p = 3$, embedded into the star graph S_n for $n = 17$