# On the Preservation of Monotonicity by Extended Mappings

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## Abstract

Images of fuzzy relations provide powerful access to fuzzifications of properties of and/or relationships between fuzzy sets. As an important example, images of fuzzy orderings canonically lead to a concept of ordering of fuzzy sets. This contribution studies in which way the partial (i.e. componentwise) monotonicity of an *n*-ary mapping transfers to its extension to fuzzy sets.

**Keywords:** extension principle, fuzzy ordering, fuzzy relation, image operator.

## 1 Introduction

Zadeh's famous extension principle [14, 15, 16], as a general methodology for extending crisp concepts to fuzzy sets, has served as the basis for the inception of new disciplines like fuzzy analysis, fuzzy algebra, fuzzy topology, and several others. Most importantly, this fundamental principle allows to extend crisp mappings to fuzzy sets. Another well-known application—which is particularly important in fuzzy decision analysis [12, 13] and fuzzy control [9, 10]—is the possibility to define ordering relations for fuzzy sets.

This contribution is devoted to links between these two fields: we study in which way the monotonicity of a mapping is preserved by its extension to fuzzy sets. However, we do not restrict to the well-known methodology of extending crisp orderings to fuzzy sets, but we start from the more general case that the domain under consideration is equipped with a fuzzy ordering [3] induced by some non-trivial fuzzy concept of indistinguishability. Even in such a case, it is possible to define orderings of fuzzy sets in a way similar to the extension principle [4].

The investigations in this paper are not only of theoretical interest. The preservation of monotonicity has emerged as a significant criterion for assessing ordering/ranking procedures of fuzzy sets [12, 13]. This paper clarifies this topic for the general framework of orderings of fuzzy sets proposed in [4].

We proceed in the following way: after necessary preliminaries and the basic problem statement, we introduce a general theorem that clarifies in which way the extension of an *n*-ary mapping to fuzzy sets preserves inclusion relations between images with respect to fuzzy preorderings. This result is then used to answer the question about preservation of monotonicity by extensions.

We will use the terms monotonicity and non-decreasingness synonymously in this paper. The reason is not to cause terminological confusion, but the fact that the same results can be proved with only minor modifications for non-increasing mappings too.

## 2 Preliminaries

Throughout the whole paper, assume that the symbols $T$ and $\tilde{T}$ denote left-continuous triangular norms.

**Definition 1.** A t-norm $T$ *dominates* another t-norm $\tilde{T}$ if and only if, for any quadruple $(x,y,u,v) \in [0,1]^4$, the following holds:

$$\tilde{T}\big(T(x,u),T(y,v)\big) \leq T\big(\tilde{T}(x,y),\tilde{T}(u,v)\big)$$

**Definition 2 (Extension Principle for Mappings).** Given a mapping $\varphi : X_1 \times \cdots \times X_n \to Y$, the $\tilde{T}$-*extension* of $\varphi$ is a $\mathcal{F}(X_1) \times \cdots \times \mathcal{F}(X_n) \to \mathcal{F}(Y)$ mapping that is defined as (with the additional convention $\sup \emptyset = 0$)

$$\hat{\varphi}_{\tilde{T}}(A_1, \ldots, A_n)(y) = \sup \left\{ \tilde{T}\left(A_1(x_1), \ldots, A_n(x_n)\right) \mid y = \varphi(x_1, \ldots, x_n) \right\}.$$

In the one-dimensional case, i.e. if $\varphi$ is an $X \to Y$ mapping, the extension is given as (independent of the choice of a t-norm)

$$\hat{\varphi}(A)(y) = \sup \left\{ A(x) \mid y = \varphi(x) \right\}.$$

**Definition 3.** A binary fuzzy relation $R : X^2 \to [0,1]$ is called

1. *reflexive* if and only if $R(x,x) = 1$ for all $x \in X$;
2. *symmetric* if and only if $R(x,y) = R(y,x)$ for all $x, y \in X$;
3. $T$-*transitive* if and only if $T\left(R(x,y), R(y,z)\right) \leq R(x,z)$ for all $x, y \in X$;

**Definition 4.** A reflexive and $T$-transitive fuzzy relation is called *fuzzy preordering* with respect to $T$, short $T$-*preordering*. A symmetric $T$-preordering is called *fuzzy equivalence relation* with respect to $T$, short $T$-*equivalence*.

**Definition 5.** Consider an arbitrary fuzzy set $A \in \mathcal{F}(X)$. The *full image* of $A$ under $R$, denoted $R(A)$ is defined as

$$R(A)(x) = \sup\{T\left(A(y), R(y,x)\right) \mid y \in X\}.$$

Note that $R(A)$ has sometimes also been called *direct image* [8] or *conditioned fuzzy set* [1]. We adopt the terminology of [7], where these operators are studied in detail. For a thorough study of the full image operators of fuzzy preorderings, we refer to [6].

## 3 Fuzzy Orderings and Orderings of Fuzzy Sets

We briefly introduce the general framework of fuzzy orderings. For a detailed study of this general concept, we refer to [3, 5].

**Definition 6.** Let $L : X^2 \to [0,1]$ be a binary fuzzy relation. $L$ is called *fuzzy ordering* with respect to $T$ and a $T$-equivalence $E : X^2 \to [0,1]$, for brevity $T$-$E$-*ordering*, if and only if it is $T$-transitive and additionally fulfills the following two axioms for all $x, y \in X$:

1. $E$-Reflexivity: $E(x,y) \leq L(x,y)$
2. $T$-$E$-antisymmetry: $T\left(L(x,y), L(y,x)\right) \leq E(x,y)$

Note that, given a fuzzy set $A \in \mathcal{F}(X)$, the full image $L(A)$ can be interpreted as "at least $A$". Analogously, the full image of the inverse $L^{-1}(A)$ can be interpreted as "at most $A$" (with $L^{-1}(x,y) = L(y,x)$) [6]. These two ordering-based modifiers allow to define a general framework of orderings of fuzzy sets with respect to a fuzzy ordering $L$ that does not make any assumption about the fuzzy ordering $L$, the domain $X$, or the fuzzy sets under consideration [4].

**Definition 7.** Let $L$ be a fuzzy ordering on $X$. Then the relation $\preceq_L$ on $\mathcal{F}(X)$ is defined in the following way:[1]

$$A \preceq_L B \text{ iff } \left(L(A) \supseteq L(B) \text{ and } L^{-1}(A) \subseteq L^{-1}(B)\right)$$

Note that in the case that $L$ is a crisp ordering, the usual ordering of fuzzy sets that can be defined by means of the extension principle [9, 10] is obtained.

It can be shown that the relation $\preceq_L$ is a preordering on $\mathcal{F}(X)$ which is also antisymmetric on a specific subclass of fuzzy sets that fulfill a certain kind of generalized convexity (for details, see [4]).

## 4 Monotonicity of Extensions

To approach the problem of monotonicity preservation, assume that we are given the following:

(a) Two left-continuous t-norms $T$ and $\tilde{T}$;

(b) Non-empty sets $X_1, \ldots, X_n$ and $Y$;

(c) A mapping $\varphi : X_1 \times \cdots \times X_n \to Y$;

(d) For each $i = 1, \ldots, n$, a $T$-equivalence $E_i$ on $X_i$ and a $T$-$E_i$-ordering on $X_i$;

(e) A $T$-equivalence $E_y$ on $Y$ and a $T$-$E_y$-ordering on $Y$;

Componentwise monotonicity of the extension $\hat{\varphi}_{\tilde{T}}$ would mean that the implication

$$A'_i \preceq_{L_i} A''_i \Rightarrow \hat{\varphi}_{\tilde{T}}(A_1, \ldots, A'_i, \ldots, A_n)) \preceq_{L_y} \hat{\varphi}_{\tilde{T}}(A_1, \ldots, A''_i, \ldots, A_n) \qquad (1)$$

[1] The inclusion relation is defined in the usual way, i.e. $A \subseteq B$ iff $A(x) \leq B(X)$ for all $x \in X$;

holds for all $i = 1, \ldots, n$, all fuzzy sets $A'_i, A''_i \in \mathcal{F}(X_i)$, and any choice of fuzzy sets $A_j \in \mathcal{F}(X_j)$ (for $j \neq i$).

It is clear that an extension can only fulfill the above monotonicity if the mapping itself fulfills a certain kind of monotonicity. The following general theorem about preservation of inclusions of images with respect to fuzzy preorderings provides the key to characterize monotonicity of extensions.

**Theorem 8.** *Suppose the assumptions (a)–(c) from above are fulfilled and that $T$ dominates $\tilde{T}$. Furthermore assume that we are given a $T$-preordering $L_i$ on each $X_i$ ($i = 1, \ldots, n$) and a $T$-preordering $L_y$ on $Y$. Then the following two statements are equivalent for all $i = 1, \ldots, n$:*

*(i) The implication*

$$L_i(A'_i) \subseteq L_i(A''_i) \;\Rightarrow L_y\big(\hat{\varphi}_{\tilde{T}}(A_1, \ldots, A'_i, \ldots, A_n)\big) \subseteq L_y\big(\hat{\varphi}_{\tilde{T}}(A_1, \ldots, A''_i, \ldots, A_n)\big)$$

*holds for all fuzzy sets $A'_i, A''_i \in \mathcal{F}(X_i)$ and any choice of fuzzy sets $A_j \in \mathcal{F}(X_j)$ (for $j \neq i$).*

*(ii) The inequality*

$$L_i(x'_i, x''_i) \leq L_y\big(\varphi(x_1, \ldots, x'_i, \ldots, x_n), \varphi(x_1, \ldots, x''_i, \ldots, x_n)\big) \qquad (2)$$

*holds for all $x'_i, x''_i \in X_i$ and any choice of values $x_j \in X_j$ (for $j \neq i$).*

From this general result, we can immediately deduce a characterization of monotonicity of extensions.

**Theorem 9.** *Suppose the assumptions (a)–(f) from above are fulfilled and that $T$ dominates $\tilde{T}$. Then the following two statements are equivalent for all $i = 1, \ldots, n$:*

*(i) The extension $\hat{\varphi}_{\tilde{T}}$ is monotonic in the i-th component in the sense of* (1).

*(ii) The inequality* (2) *holds for all $x'_i, x''_i \in X_i$ and any choice of values $x_j \in X_j$ (for $j \neq i$).*

The question remains how the inequality (2) can be interpreted.

If both $L_i$ and $L_y$ are crisp orderings, then (2) is nothing else but the classical crisp partial monotonicity of the mapping $\varphi$. In this case, Theorem 9 implies that the extension $\hat{\varphi}_{\tilde{T}}$ of a monotonic mapping $\varphi$ is monotonic with respect to the preordering of fuzzy sets that is obtained by extending crisp orderings to fuzzy sets by means of the extension principle.

If nothing about the specific structure of the fuzzy orderings $L_i$ and $L_y$ is assumed, no further characterization of inequality (2) is possible. In any case, however, this inequality may be intuitively interpreted as a kind of generalized monotonicity.

Finally, let us consider an important sub-class of fuzzy orderings—so-called direct fuzzifications, i.e. fuzzy orderings that can be split up into a crisp ordering and a fuzzy equivalence relation.

**Definition 10.** A $T$-$E$-ordering $L$ is called a *direct fuzzification* of a crisp ordering $\preceq$ if and only if it admits the following resolution:

$$L(x,y) = \begin{cases} 1 & \text{if } x \preceq y \\ E(x,y) & \text{otherwise} \end{cases}$$

It is worth to mention that there is a one-to-one correspondence between direct fuzzifications of crisp linear orderings and so-called fuzzy weak orderings, i.e. fuzzy preorderings fulfilling strong completeness [2, 3].

**Proposition 11.** *With the assumptions of Theorem 9 and the additional assumption that $L_i$ and $L_y$ are direct fuzzifications of crisp orderings $\preceq_i$ and $\preceq_y$, respectively, the following holds: If $\varphi$ is partially monotonic, i.e.*

$$x'_i \preceq_i x''_i \;\Rightarrow \varphi(x_1, \ldots, x'_i, \ldots, x_n)) \preceq_y \varphi(x_1, \ldots, x''_i, \ldots, x_n),$$

*and $\varphi$ fulfills*

$$E(x'_i, x''_i) \leq E\big(\varphi(x_1, \ldots, x'_i, \ldots, x_n)\big), \varphi(x_1, \ldots, x''_i, \ldots, x_n)\big), \qquad (3)$$

*then $\varphi$ fulfills* (2) *for all $x'_i, x''_i \in X_i$ and any choice of values $x_j \in X_j$ (for $j \neq i$).*

Note that the property (3) is well-known as the *extensionality* of a mapping [11].

This finally implies, for the special case of direct fuzzifications, that extensions of monotonic and extensional mappings are monotonic.

## 5 Conclusions and Future Work

As the main result of this paper, we have obtained that the (generalized) monotonicity of mappings carries over to its extension, also in the $n$-ary case. This has been achieved by means of a general theorem about the preservation of inclusion relations between images with respect to fuzzy preorderings (cf. Theorem 8). All results in this paper trivially hold also for the case $n = 1$. In this case, however, a t-norm $\tilde{T}$ is not necessary.

Theorem 8, although having a rather theoretical character, could have applications beyond the study of monotonicity in this paper, which is left for future investigations. Moreover, it is remarkable that the domination property appears in a rather unusual context here—a fact which should also be investigated in more detail in the future.

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