Continuous dependence for impulsive functional dynamic equations involving variable time scales

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A B S T R A C T

Using a known correspondence between the solutions of impulsive measure functional differential equations and the solutions of impulsive functional dynamic equations on time scales, we prove that the limit of solutions of impulsive functional dynamic equations over a convergent sequence of time scales converges to a solution of an impulsive functional dynamic equation over the limiting time scale.

1. Introduction

The fact that solutions of dynamic equations on time scales depend continuously on time scales is a problem that has been investigated by several researchers. See [1,5,10], for instance. In these papers, the authors prove that the sequence of solutions of the problem

$$\begin{cases} x^\Delta(t) = f(x,t), & t \in \mathbb{T}_n, \\ x(t_0) = x_0, & t_0 \in \mathbb{T}_n \end{cases} \tag{1.1}$$

converges uniformly to the solution of the problem

$$\begin{cases} x^\Delta(t) = f(x,t), & t \in \mathbb{T}, \\ x(t_0) = x_0, & t_0 \in \mathbb{T}, \end{cases} \tag{1.2}$$

whenever $d(\mathbb{T}_n, \mathbb{T}) \to 0$ as $n \to \infty$, where $d(\mathbb{T}_n, \mathbb{T})$ denotes the Hausdorff metric or the induced metric from the Fell topology. To obtain such results, the following conditions on the function f are usually assumed:

- There exists a constant $M > 0$ such that $\|f(x,t)\| \leqslant M$ for every x in a certain subset of the phase space and every $t \in [t_0, t_0 + \eta]_{\mathbb{T}}$.
- There exists a constant $L > 0$ such that $\|f(x,t) - f(y,t)\| \leqslant L\|x - y\|$ for every x and y in a certain subset of the phase space and every $t \in [t_0, t_0 + \eta]_{\mathbb{T}}$.

Here, our goal is to investigate the behavior of solutions of the same initial value problems over different time scales for impulsive functional dynamic equations; that is, we prove that, under certain conditions, the sequence of solutions of the system

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$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)\,\Delta s + \sum_{\substack{k \in \{1,\dots,m\}, \\ t_k < t}} I_k(x(t_k)), & t \in [t_0, t_0 + \eta]_{\mathbb{T}_n}, \\ x(t) = \phi(t), & t \in [t_0 - r, t_0]_{\mathbb{T}_n} \end{cases} \tag{1.3}$$

converges uniformly to the solution of the problem

$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)\,\Delta s + \sum_{\substack{k \in \{1,\dots,m\}, \\ t_k < t}} I_k(x(t_k)), & t \in [t_0, t_0 + \eta]_{\mathbb{T}}, \\ x(t) = \phi(t), & t \in [t_0 - r, t_0]_{\mathbb{T}}, \end{cases} \tag{1.4}$$

whenever $d(\mathbb{T}_n, \mathbb{T}) \to 0$ as $n \to \infty$. Here, $d(\mathbb{T}_n, \mathbb{T})$ denotes the Hausdorff metric. Our results apply to the Fell topology as well.

We also consider the following conditions on the function f:

- There exists a constant $M > 0$ such that

$$\|f(x_t, t)\| \leqslant M$$

for all $t \in [t_0, t_0 + \eta]_{\mathbb{T}}$ and all x in a certain subset of the phase space.

- There exists a constant $L > 0$ such that

$$\left\| \int_{u_1}^{u_2} (f(x_t, t) - f(y_t, t))\mathrm{d}g(t) \right\| \leqslant L \int_{u_1}^{u_2} \|x_t - y_t\|_{\infty}\,\mathrm{d}g(t)$$

for all $u_1, u_2 \in [t_0, t_0 + \eta]_{\mathbb{T}}$ and all x, y in a certain subset of the phase space.

Here, we consider the integral in the sense of Henstock–Kurzweil which is known to integrate highly oscillating functions (see [9], for instance). Thus, the second condition on the indefinite integral of f allows the function f to behave "badly", e.g., f may have many discontinuities or be of unbounded variation, and yet good results can be obtained, as long as its indefinite behaves well enough. Alternatively, one could consider the Riemann or Lebesgue integral.

In order to obtain the continuous dependence result for impulsive functional dynamic equations on time scales involving variable time scales with these conditions, we use a known correspondence between the solutions of impulsive functional dynamic equations on time scales and the solutions of impulsive measure functional differential equations. We also use a correspondence between these solutions and the solutions of measure functional differential equations. For details about these correspondences, see [7].

Further, in order to ensure the convergence of solutions, we suppose some convergence over a operator sequence defined in Section 3. This hypothesis cannot be suppressed as shown by Examples 5.1 and 5.2 in Section 5.

2. Impulsive measure functional differential equations

Let $r, \eta > 0$ be given numbers and $t_0 \in \mathbb{R}$. The theory of functional differential equations (see e.g., [8]) deals with problems as

$$\dot{x} = f(x_t, t), \quad t \in [t_0, t_0 + \eta], \tag{2.1}$$

where $f : \Omega \times [t_0, t_0 + \eta] \to \mathbb{R}^n$, $\Omega \subset C([-r, 0], \mathbb{R}^n)$ and x_t is given by $x_t(\theta) = x(t + \theta), \theta \in [-r, 0]$, for every $t \in [t_0, t_0 + \eta]$. The integral form of (2.1) is given by

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)\,\mathrm{d}s, \quad t \in [t_0, t_0 + \eta],$$

where the integral is in the sense of Henstock–Kurzweil.

The theory of measure functional differential equations deals with problems as

$$\mathrm{D}x = f(x_t, t)\mathrm{D}g,$$

where $\mathrm{D}x$ and $\mathrm{D}g$ denote the distributional derivatives in the sense of L. Schwartz of the functions x and g, respectively. The integral form is given by

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0 + \eta], \tag{2.2}$$

where we consider the integral on the right-hand side to be Henstock–Kurzweil–Stieltjes (we write H–K–S, for short) integrable with respect to a nondecreasing function g. See [6,7].

We assume that g is a left-continuous and nondecreasing function and consider the possibility of adding impulses at pre-assigned times $t_1, \dots, t_m$, where $t_0 \leqslant t_1 < \cdots < t_m < t_0 + \eta$. For every $k \in \{1, \dots, m\}$, the impulse at t_k is described by the operator $I_k : \mathbb{R}^n \to \mathbb{R}^n$. In other words, the solution x should satisfy $\Delta^+ x(t_k) = I_k(x(t_k))$, where $\Delta^+ x(t_k) = x(t_k+) - x(t_k)$ and $x(t_k+) = \lim_{t \to t_k+} x(t)$. This leads us to the problem

$$\begin{cases} x(v) - x(u) = \int_u^v f(x_s,s)\,dg(s), & \text{whenever } u, v \in J_k \text{ for some } k \in \{0,\ldots,m\}, \\ \Delta^+ x(t_k) = I_k(x(t_k)), & k \in \{1,\ldots,m\}, \\ x_{t_0} = \phi, \end{cases} \tag{2.3}$$

where $J_0 = [t_0, t_1], J_k = (t_k, t_{k+1}]$ for $k \in \{1,\ldots,m-1\}$, and $J_m = (t_m, t_0 + \eta]$.

The value of the integral $\int_u^v f(x_s,s)\,dg(s)$, where $u, v \in J_k$, does not change if we replace g by a function $\tilde{g}$ such that $g - \tilde{g}$ is a constant function on J_k. This follows easily from the definition of the H-K-S integral (see [12], for instance). Thus, without loss of generality, we can assume that g is such that $\Delta^+ g(t_k) = 0$ for every $k \in \{1,\ldots,m\}$. Since g is a left-continuous function, it follows that g is continuous at $t_1,\ldots,t_m$. Under this assumption, our problem (2.3) can be rewritten as

$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^t f(x_s,s)\,dg(s) + \sum_{\substack{k \in \{1,\ldots,m\}, \\ t_k < t}} I_k(x(t_k)), & t \in [t_0, t_0 + \eta], \\ x_{t_0} = \phi. \end{cases} \tag{2.4}$$

Alternatively, the sum on the right-hand side of (2.4) can be written in the form

$$\sum_{\substack{k \in \{1,\ldots,m\}, \\ t_k < t}} I_k(x(t_k)) = \sum_{k=1}^m I_k(x(t_k)) H_{t_k}(t),$$

where H_v denotes the Heaviside function of (v, ∞) given by

$$H_v(t) = \begin{cases} 0 & \text{for } t \leqslant v, \\ 1 & \text{for } t > v. \end{cases} \tag{2.5}$$

Thus, (2.4) becomes

$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^t f(x_s,s)\,dg(s) + \sum_{k=1}^m I_k(x(t_k)) H_{t_k}(t), & t \in [t_0, t_0 + \eta], \\ x_{t_0} = \phi. \end{cases} \tag{2.6}$$

Now, we will define regulated functions, since they are a good framework for dealing with equations having discontinuous right-hand sides. A function $f : [a,b] \to X$, where X is a Banach space, is called *regulated*, if the lateral limits

$$\lim_{s \to t-} f(s) = f(t-) \in X, \quad t \in (a,b], \quad \text{and} \quad \lim_{s \to t+} f(s) = f(t+) \in X, \quad t \in [a,b)$$

exist. The space of all regulated functions $f : [a,b] \to X$ will be denoted by $G([a,b], X)$ and it is a Banach space under the usual supremum norm $\|f\|_\infty = \sup_{a \leqslant b} \|f(t)\|$. The subspace of all continuous functions $f : [a,b] \to X$ will be denoted by $C([a,b], X)$.

The following theorem represents an analogue of Gronwall's inequality for the H-K-S integral. A proof of it can be found in [[15], Corollary 1.43]. This result and the next one will be essentials to prove our auxiliary results.

Theorem 2.1. *Let* $h: [a,b] \to [0,\infty)$ *be a nondecreasing left-continuous function,* $k > 0, l \geqslant 0$. *Assume that* $\psi: [a,b] \to [0,\infty)$ *is bounded and satisfies*

$$\psi(\xi) \leqslant k + l \int_a^\xi \psi(\tau)\,dh(\tau), \quad \xi \in [a,b].$$

Then $\psi(\xi) \leqslant k e^{l(h(\xi) - h(a))}$ *for every* $\xi \in [a,b]$.

For a proof of the next result, see [15, Corollary 1.34]. The inequality below follows directly from the definition of the H-K-S integral.

Theorem 2.2. *If* $f : [a,b] \to \mathbb{R}^n$ *is a regulated function and* $g : [a,b] \to \mathbb{R}$ *is a nondecreasing function, then the integral* $\int_a^b f\,dg$ *exists and*

$$\left\| \int_a^b f(s)\,dg(s) \right\| \leqslant \|f\|_\infty (g(b) - g(a)).$$

3. Dynamic equations on time scales

In this section, we present some basic concepts about the theory of dynamic equations on time scales. For more details about it, the reader may consult [2,3,14].

A time scale is a closed and nonempty subset of $\mathbb{R}$. Throughout this paper, we will denote it by $\mathbb{T}$. For every $t \in \mathbb{T}$, we define the *forward* and *backward jump operators* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition, we use the conventions $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. If $\sigma(t) > t$, then we say that t is *right-scattered*. Otherwise, t is called *right-dense*. Analogously, if $\rho(t) < t$, then t is called *left-scattered* whereas if $\rho(t) = t$, then t is called *left-dense*. We also define the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}^+$ by

$$\mu(t) = \sigma(t) - t.$$

Given a pair of numbers $a, b \in \mathbb{T}$, the symbol $[a, b]_{\mathbb{T}}$ will be used to denote a closed interval in $\mathbb{T}$, that is, $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leqslant t \leqslant b\}$. On the other hand, $[a, b]$ is the usual closed interval on the real line, that is, $[a, b] = \{t \in \mathbb{R} : a \leqslant t \leqslant b\}$. We define the set $\mathbb{T}^\kappa$ which is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 3.1. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, we define the *delta-derivative* of f to be the number (if it exists) with the following property: given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|f(\sigma(t)) - f(t) - f^\Delta(t)[\sigma(t) - s]| < \varepsilon |\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

We say $\delta = (\delta_L, \delta_R)$ is a Δ-*gauge* for $[a, b]_{\mathbb{T}}$ provided $\delta_L(t) > 0$ on $(a, b]_{\mathbb{T}}, \delta_R(t) > 0$ on $([a, b)_{\mathbb{T}}, \delta_L(a) \geqslant 0, \delta_R(b) \geqslant 0$, and $\delta_R(t) \geqslant \mu(t)$ for all $(t \in [a, b)_{\mathbb{T}}$. A *partition* $\mathcal{P}$ for $[a, b]_{\mathbb{T}}$ is a division of $[a, b]_{\mathbb{T}}$ denoted by

$$\mathcal{P} = \{a = t_0 \leqslant \xi_1 \leqslant t_1 \leqslant \ldots \leqslant t_{n-1} \leqslant \xi_n \leqslant t_n = b\},$$

with $t_i > t_{i-1}$ for $1 \leqslant i \leqslant n$ and $t_i, \xi_i \in \mathbb{T}$. We call the points ξ_i *tag points* and the points t_i *end points*. If δ is a Δ-gauge for $[a, b]_{\mathbb{T}}$, then we say a partition $\mathcal{P}$ is δ-*fine* if

$$\xi_i - \delta_L(\xi_i) \leqslant t_{i-1} < t_i \leqslant \xi_i + \delta_R(\xi_i) \quad \text{for} \quad 1 \leqslant i \leqslant n.$$

In what follows, we give a definition of Henstock–Kurzweil delta integrable functions.

Definition 3.2. A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is called *Henstock–Kurzweil delta integrable* on $[a, b]_{\mathbb{T}}$ with value $I = HK \int_a^b f(t)\Delta t$ provided given any $\varepsilon > 0$, there exists a Δ-gauge δ for $[a, b]_{\mathbb{T}}$ such that

$$\left| I - \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}) \right| < \varepsilon$$

for all δ-fine partitions $\mathcal{P}$ of $[a, b]_{\mathbb{T}}$.

Now, we present some definitions which will be essential to our purposes. They were introduced in [16] and here, we use the same notation as in [11]: Let

$$\tilde{\sigma}(t) = \inf\{s \in \mathbb{T} : s \geqslant t\} \quad \text{for} \quad t \in \mathbb{R}.$$

It is clear that $\tilde{\sigma}(t)$ can be different from $\sigma(t)$ depending on $\mathbb{T}$. Since $\mathbb{T}$ is a closed set, we have $\tilde{\sigma}(t) \in \mathbb{T}$. Further, let

$$\mathbb{T}^* = \begin{cases} (-\infty, \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty, \\ (-\infty, \infty) & \text{otherwise}. \end{cases}$$

Given a function $f : \mathbb{T} \to \mathbb{R}^n$, we consider its extension $f^{\tilde{\sigma}} : \mathbb{T}^* \to \mathbb{R}^n$ given by

$$f^{\tilde{\sigma}}(t) = f(\tilde{\sigma}(t)), \quad t \in \mathbb{T}^*.$$

4. Impulsive measure functional differential equations and impulsive functional dynamic equations on time scales

It is a known fact that there exists a correspondence between impulsive measure functional differential equations and impulsive functional dynamic equation on time scales (see [7]).

An impulsive functional dynamic equation on time scales can be described by the system

$$\begin{cases} x^\Delta(t) = f(x_t^{\tilde{\sigma}}, t), & t \in [t_0, t_0 + \eta]_{\mathbb{T}} \setminus \{t_1, \ldots, t_m\}, \\ \Delta^+ x(t_k) = I_k(x(t_k)), & k \in \{1, \ldots, m\}, \\ x(t) = \phi(t), & t \in [t_0 - r, t_0]_{\mathbb{T}}, \end{cases} \tag{4.1}$$

where $t_1, \ldots, t_m \in \mathbb{T}$ are points of impulses, $t_0 \leqslant t_1 < t_2 < \cdots < t_m < t_0 + \eta$, and $I_1, \ldots, I_m : \mathbb{R}^n \to \mathbb{R}^n$. The solution is assumed to be left-continuous. The symbol $x_t^{\tilde{\sigma}}$ should be understood as $(x^{\tilde{\sigma}})_t$; as explained in [6], that is, $(x^{\tilde{\sigma}})_t = x^{\tilde{\sigma}}(t + \theta) = x(\tilde{\sigma}(t + \theta))$, for $\theta \in [-r, 0]$. Also, the advantage of using $x_t^{\tilde{\sigma}}$ rather than x_t stems from the fact that $x_t^{\tilde{\sigma}}$ is always defined on the whole interval $[-r, 0]$, while x_t is defined only on a subset of $[-r, 0]$. Alternatively, the above problem can be written more compactly in the form

$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^t f(x_s^{\tilde{\sigma}}, s)\Delta s + \sum_{\substack{k \in \{1,\dots,m\}, \\ t_k < t}} I_k(x(t_k)), & t \in [t_0, t_0 + \eta]_{\mathbb{T}}, \\ x(t) = \phi(t), & t \in [t_0 - r, t_0]_{\mathbb{T}}. \end{cases} \tag{4.2}$$

The next result describes the correspondence between measure functional differential equations with impulses and impulsive functional dynamic equations on time scales. It can be found in [7].

Theorem 4.1. *Let* $[t_0 - r, t_0 + \eta]_{\mathbb{T}}$ *be a time scale interval,* $t_0 \in \mathbb{T}, B \subset \mathbb{R}^n$, $f : G([-r,0], B) \times [t_0, t_0 + \eta]_{\mathbb{T}} \to \mathbb{R}^n, \phi \in G([t_0 - r, t_0]_{\mathbb{T}}, B)$. *If* $x : [t_0 - r, t_0 + \eta]_{\mathbb{T}} \to B$ *is a solution of the impulsive functional dynamic equation*

$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^t f(x_s^{\tilde{\sigma}}, s)\Delta s + \sum_{\substack{k \in \{1,\dots,m\}, \\ t_k < t}} I_k(x(t_k)), & t \in [t_0, t_0 + \eta]_{\mathbb{T}}, \\ x(t) = \phi(t), & t \in [t_0 - r, t_0]_{\mathbb{T}}, \end{cases} \tag{4.3}$$

then $x^{\tilde{\sigma}} : [t_0 - r, t_0 + \sigma] \to B$ *is a solution of the impulsive measure functional differential equation*

$$\begin{cases} y(t) = y(t_0) + \int_{t_0}^t f(y_s, \tilde{\sigma}(s))\,\mathrm{d}\tilde{\sigma}(s) + \sum_{\substack{k \in \{1,\dots,m\}, \\ t_k < t}} I_k(y(t_k)), & t \in [t_0, t_0 + \eta], \\ y_{t_0} = \phi^{\tilde{\sigma}}. \end{cases} \tag{4.4}$$

Conversely, if $y : [t_0 - r, t_0 + \eta] \to B$ *satisfies (4.4), then it must have the form* $y = x^{\tilde{\sigma}}$, *where* $x : [t_0 - r, t_0 + \eta]_{\mathbb{T}} \to B$ *is a solution of (4.3).*

5. Continuous dependence for impulsive functional dynamic equations on time scales

In this section, we present a continuous dependence result involving variable time scales for impulsive functional dynamic equations on time scales.

Our idea to prove a continuous dependence result for impulsive functional dynamic equations on time scales is to use the correspondence between the solutions of these equations and the solutions of impulsive measure functional differential equations (see Theorem 4.1) and and the correspondence between the solutions of impulsive measure functional differential equations and the solutions of measure functional differential equations, which is given below in Theorem 5.2.

Let $\mathbb{T}_n$ be time scales for each $n \in \mathbb{N}$ with corresponding forward jumps σ_n and $\tilde{\sigma}_n$. Assume that the distance $d(\mathbb{T}_n, \mathbb{T}) \to 0$ as $n \to \infty$. Here, we are considering the Hausdorff topology and Hausdorff metric in which the distance between two sets is defined by

$$d(A, B) = \max\{\sup\{\inf\{|a - b| : b \in B\} : a \in A\}, \sup\{\inf\{|a - b| : a \in A\} : b \in B\}\}.$$

Now, our goal is to prove, under certain conditions, a result that guarantees that the sequence of solutions of the problem

$$\begin{cases} x_n^{\tilde{\sigma}_n}(t) = x_n^{\tilde{\sigma}_n}(t_0) + \int_{t_0}^t f((x_n^{\tilde{\sigma}_n})_s, s)\,\mathrm{d}\tilde{\sigma}_n(s), & t \in \mathbb{T}_n^*, \\ (x_n^{\tilde{\sigma}_n})_{t_0} = \phi^{\tilde{\sigma}_n} \end{cases} \tag{5.1}$$

converges uniformly to the solution of the problem

$$\begin{cases} x^{\tilde{\sigma}}(t) = x^{\tilde{\sigma}}(t_0) + \int_{t_0}^t f(x_s^{\tilde{\sigma}}, s)\,\mathrm{d}\tilde{\sigma}(s), & t \in \mathbb{T}^*, \\ x_{t_0}^{\tilde{\sigma}} = \phi^{\tilde{\sigma}}. \end{cases} \tag{5.2}$$

Thus, after proving this result, using the correspondence between impulsive measure functional differential equations and measure functional differential equations, we obtain an analogous result for measure functional differential equations with impulses and therefore, using the other correspondence (Theorem 4.1), our main theorem concerning continuous dependence for impulsive functional dynamic equations on time scales follows as well.

Now, assume $O \subset G([t_0 - r, t_0 + \eta], \mathbb{R}^n)$ is open, $P = \{y_t; y \in O, t \in [t_0, t_0 + \eta]\}, f : P \times [t_0, t_0 + \eta] \to \mathbb{R}^n$, and $g : [t_0, t_0 + \eta] \to \mathbb{R}$ is nondecreasing and left-continuous function. We assume the following three conditions on the function $f : P \times [t_0, t_0 + \eta] \to \mathbb{R}^n$:

(A) The H-K-S integral $\int_{t_0}^{t_0+\eta} f(y_t, t)\,\mathrm{d}g(t)$ exists for every $y \in O$.
(B) There exists a constant $M > 0$ such that

$$\|f(y_t, t)\| \leqslant M,$$

whenever $t_0 \leqslant t \leqslant t_0 + \eta$ and $y \in O$.
(C) There exists a constant $L > 0$ such that

$$\left\| \int_{u_1}^{u_2} (f(y_t, t) - f(z_t, t)) \mathrm{d}g(t) \right\| \leqslant L \int_{u_1}^{u_2} \|y_t - z_t\|_\infty \, \mathrm{d}g(t)$$

whenever $t_0 \leqslant u_1 \leqslant u_2 \leqslant t_0 + \eta$ and $y, z \in O$.

Theorem 5.1. *Suppose f satisfies conditions* (A), (B) *and* (C), *and $x_n^{\tilde{\sigma}_n}$ is a solution of the system*

$$\begin{cases} x_n^{\tilde{\sigma}_n}(t) = x_n^{\tilde{\sigma}_n}(t_0) + \int_{t_0}^t f((x_n^{\tilde{\sigma}_n})_s, s) \, \mathrm{d}\tilde{\sigma}_n(s), \quad t \in \mathbb{T}_n^*, \\ (x_n^{\tilde{\sigma}_n})_{t_0} = \phi^{\tilde{\sigma}_n} \end{cases} \tag{5.3}$$

and $x^{\tilde{\sigma}}$ is a solution of the measure functional differential equation given by

$$\begin{cases} x^{\tilde{\sigma}}(t) = x^{\tilde{\sigma}}(t_0) + \int_{t_0}^t f(x_s^{\tilde{\sigma}}, s) \, \mathrm{d}\tilde{\sigma}(s), \quad t \in \mathbb{T}^*, \\ x_{t_0}^{\tilde{\sigma}} = \phi^{\tilde{\sigma}}. \end{cases} \tag{5.4}$$

Moreover, suppose $d(\mathbb{T}_n, \mathbb{T}) \to 0$ as $n \to \infty$ and the sequence of functions $\{\tilde{\sigma}_n\}_{n=1}^\infty$ converges uniformly to $\tilde{\sigma}$ as $n \to \infty$. Also, suppose the sequence of initial conditions $\{\phi^{\tilde{\sigma}_n}\}_{n=1}^\infty$ converges uniformly to $\phi^{\tilde{\sigma}}$ as $n \to \infty$. Then, for every $\varepsilon > 0$, there exists $N > 0$ sufficiently large such that, for $n > N$, we have

$$\|x_n^{\tilde{\sigma}_n}(t) - x^{\tilde{\sigma}}(t)\| < \varepsilon \quad \text{for} \quad t \in \mathbb{T}_n^* \cap \mathbb{T}^*. \tag{5.5}$$

Proof. Given $\varepsilon > 0$ and since the sequence of functions $\{\tilde{\sigma}_n\}$ converges uniformly to $\tilde{\sigma}$, there exists $N_1 > 0$ sufficiently large such that for every $n > N_1$, we obtain

$$\|\tilde{\sigma}_n(t) - \tilde{\sigma}(t)\| < \varepsilon \quad \text{for every} \quad t \in \mathbb{T}_n^* \cap \mathbb{T}^*. \tag{5.6}$$

Moreover, since the sequence of functions $\phi^{\tilde{\sigma}_n}$ converges uniformly to $\phi^{\tilde{\sigma}}$, there exists $N_2 > 0$ sufficiently large such that for every $n > N_2$, we have

$$\|\phi^{\tilde{\sigma}_n} - \phi^{\tilde{\sigma}}\| < \varepsilon \quad \text{for every} \quad t \in \mathbb{T}_n^* \cap \mathbb{T}^*. \tag{5.7}$$

Also, for $t \in \mathbb{T}_n^* \cap \mathbb{T}^*$ and $n > \max\{N_1, N_2\}$, we have

$$\begin{aligned} \|x_n^{\tilde{\sigma}_n}(t) - x^{\tilde{\sigma}}(t)\| &= \left\| x_n^{\tilde{\sigma}_n}(t_0) - x^{\tilde{\sigma}}(t_0) + \int_{t_0}^t f((x_n^{\tilde{\sigma}_n})_s, s) d\tilde{\sigma}_n(s) - \int_{t_0}^t f((x^{\tilde{\sigma}})_s, s) d\tilde{\sigma}(s) \right\| \\ &\leqslant \|x_n^{\tilde{\sigma}_n}(t_0) - x^{\tilde{\sigma}}(t_0)\| + \left\| \int_{t_0}^t f((x_n^{\tilde{\sigma}_n})_s, s) d\tilde{\sigma}_n(s) - \int_{t_0}^t f((x^{\tilde{\sigma}})_s, s) d\tilde{\sigma}(s) \right\| \\ &\leqslant \|\phi^{\tilde{\sigma}_n} - \phi^{\tilde{\sigma}}\| + \left\| \int_{t_0}^t f((x_n^{\tilde{\sigma}_n})_s, s) d\tilde{\sigma}_n(s) - \int_{t_0}^t f((x^{\tilde{\sigma}})_s, s) d\tilde{\sigma}(s) \right\| \\ &\leqslant \|\phi^{\tilde{\sigma}_n} - \phi^{\tilde{\sigma}}\| + \left\| \int_{t_0}^t f((x_n^{\tilde{\sigma}_n})_s, s) d\tilde{\sigma}_n(s) - \int_{t_0}^t f((x_n^{\tilde{\sigma}_n})_s, s) d\tilde{\sigma}(s) \right\| \\ &\quad + \left\| \int_{t_0}^t f((x_n^{\tilde{\sigma}_n})_s, s) d\tilde{\sigma}(s) - \int_{t_0}^t f((x^{\tilde{\sigma}})_s, s) d\tilde{\sigma}(s) \right\| \\ &\leqslant \varepsilon + \int_{t_0}^t M d[\tilde{\sigma}_n(s) - \tilde{\sigma}(s)] + \int_{t_0}^t L \|(x_n^{\tilde{\sigma}_n})_s - (x^{\tilde{\sigma}})_s\| d\tilde{\sigma}(s), \end{aligned}$$

where we used (B) and (C) for the last inequality. Thus, by Theorem 2.2, we obtain

$$\|x_n^{\tilde{\sigma}_n}(t) - x^{\tilde{\sigma}}(t)\| \leqslant \varepsilon + 2\varepsilon M + \int_{t_0}^t L \|(x_n^{\tilde{\sigma}_n})_s - (x^{\tilde{\sigma}})_s\| d\tilde{\sigma}(s).$$

Using $(x_n^{\tilde{\sigma}_n})_{t_0} = \phi^{\tilde{\sigma}_n}$ and $(x^{\tilde{\sigma}})_{t_0} = \phi^{\tilde{\sigma}}$ and the uniform convergence $\phi^{\tilde{\sigma}_n} \to \phi^{\tilde{\sigma}}$, we have, for $n > N_2$,

$$\|(x_n^{\tilde{\sigma}_n})_s - x_s^{\tilde{\sigma}}\|_\infty = \sup_{\theta \in [-r, 0]} \|x_n^{\tilde{\sigma}_n}(s + \theta) - x^{\tilde{\sigma}}(s + \theta)\| \leqslant \varepsilon + \sup_{\eta \in [0, s]} \|x_n^{\tilde{\sigma}_n}(\sigma) - x^{\tilde{\sigma}}(\sigma)\|$$

and, therefore,

$$\|x_n^{\tilde{\sigma}_n}(t) - x^{\tilde{\sigma}}(t)\| \leqslant \varepsilon + 2\varepsilon M + \int_{t_0}^t L \left(\varepsilon + \sup_{\eta \in [0, s]} \|x_n^{\tilde{\sigma}_n}(\eta) - x^{\tilde{\sigma}}(\eta)\| \right) d\tilde{\sigma}(s).$$

Then,

$$\|x_n^{\tilde{\sigma}_n}(t) - x^{\tilde{\sigma}}(t)\| \leqslant \varepsilon + 2\varepsilon M + L\varepsilon(\tilde{\sigma}(t) - \tilde{\sigma}(t_0)) + \int_{t_0}^t L \sup_{\eta \in [0,s]} \|x_n^{\tilde{\sigma}_n}(\eta) - x^{\tilde{\sigma}}(\eta)\| \mathrm{d}\tilde{\sigma}(s).$$

By the Gronwall inequality (Theorem 2.1), we get

$$\|x_n^{\tilde{\sigma}_n}(t) - x^{\tilde{\sigma}}(t)\| \leqslant \varepsilon(1 + 2M + (\tilde{\sigma}(t) - \tilde{\sigma}(t_0)))e^{L(g(t) - g(t_0))}$$

and, since $\varepsilon > 0$ is arbitrary, we have the desired result. □

Note that the hypothesis in Theorem 5.1 which guarantees that the sequence of functions $\{\tilde{\sigma}_n\}_{n=1}^\infty$ converges uniformly to $\tilde{\sigma}$ as $n \to \infty$ is necessary, since one cannot expect this to happen only using the fact that $d(\mathbb{T}_n, \mathbb{T}) \to 0$ as $n \to \infty$. Below, we present an example that illustrates this.

Example 5.1. Let $\mathbb{T} = [0, a] \cup [a + 1, b]$ and $\mathbb{T}_n = [0, a + 1/n] \cup [a + 1, b]$, for every $n \in \mathbb{N}$. Then $d(\mathbb{T}, \mathbb{T}_n) = 1/n \to 0$ as $n \to \infty$. However $\tilde{\sigma}_n(a + 1/n) = a + 1/n$, for every $n \in \mathbb{N}$, while $\tilde{\sigma}(a + 1/n) = a + 1$. In other words, for every $n \geqslant 2$, there exists t such that $\tilde{\sigma}(t) - \tilde{\sigma}_n(t) \geqslant 1/2$, which means that the sequence $\{\tilde{\sigma}_n\}_{n=1}^\infty$ does not converge uniformly to $\tilde{\sigma}$.

Even if we consider the Fell topology instead of the Hausdorff topology, the hypothesis of Theorem 5.1 guaranteeing the uniform convergence of the sequence of functions $\{\tilde{\sigma}_n\}_{n=1}^\infty$ is necessary. The next example illustrates this. Here, the notation CL(M) represents the set of all closed, nonempty subsets of M.

Example 5.2. Assume $\mathbb{R}$ with the usual metric and CL($\mathbb{R}$) is endowed with the Fell topology. Then it is known that

$$\mathbb{T}_n = \{z + 1/n : z \in \mathbb{Z}\} \to \mathbb{Z}.$$

For details, see [5, Lemma 4]. Also,

$$\tilde{\sigma}_n(z + 1/n) = z + 1/n,$$

whereas $\tilde{\sigma}(z + 1/n) = z + 1$, which implies that $\tilde{\sigma}_n$ does not converge uniformly to $\tilde{\sigma}$.

Now, the next result describes a correspondence between measure functional differential equations and impulsive measure functional differential equations. A proof of it can be found in [7]. It will be necessary to prove an analogous result to Theorem 5.1 for impulsive measure functional differential equations.

Theorem 5.2. *Let* $m \in \mathbb{N}, t_0 \leqslant t_1 < \cdots < t_m < t_0 + \eta, B \subset \mathbb{R}^n, I_1, \ldots, I_m : B \to \mathbb{R}^n, P = G([-r, 0], B), f : P \times [t_0, t_0 + \eta] \to \mathbb{R}^n$. *Assume that* $g : [t_0, t_0 + \eta] \to \mathbb{R}$ *is a regulated left-continuous function which is continuous at the points* $t_1, \ldots, t_m$. *For every* $y \in P$, *define*

$$\bar{f}(y, t) = \begin{cases} f(y, t), & t \in [t_0, t_0 + \eta] \setminus \{t_1, \ldots, t_m\}, \\ I_k(y(0)), & t = t_k \text{ for some } k \in \{1, \ldots, m\}. \end{cases}$$

Moreover, let $c_1, \ldots, c_m \in \mathbb{R}$ *be constants such that the function* $\bar{g} : [t_0, t_0 + \eta] \to \mathbb{R}$ *given by*

$$\bar{g}(t) = \begin{cases} g(t), & t \in [t_0, t_1], \\ g(t) + c_k, & t \in (t_k, t_{k+1}] \text{ for some } k \in \{1, \ldots, m - 1\}, \\ g(t) + c_m, & t \in (t_m, t_0 + \eta] \end{cases}$$

satisfies $\Delta^+ \bar{g}(t_k) = 1$ *for every* $k \in \{1, \ldots, m\}$. *Then* $x \in G([t_0 - r, t_0 + \eta], B)$ *is a solution of*

$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^t f(x_s, s) \mathrm{d}g(s) + \sum_{\substack{k \in \{1, \ldots, m\}, \\ t_k < t}} I_k(x(t_k)), & t \in [t_0, t_0 + \eta], \\ x_{t_0} = \phi, \end{cases} \tag{5.8}$$

if and only if x *satisfies*

$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^t \bar{f}(x_s, s) \mathrm{d}\bar{g}(s), & t \in [t_0, t_0 + \eta], \\ x_{t_0} = \phi. \end{cases} \tag{5.9}$$

We also consider the following conditions on the impulse operators $I_k : \mathbb{R}^n \to \mathbb{R}^n$:

(A*) There exists a constant $K_1 > 0$ such that

$$\|I_k(x)\| \leqslant K_1$$

for every $k \in \{1, \ldots, m\}$ and $x \in B$.

(B*) There exists a constant $K_2 > 0$ such that

$$\|I_k(x) - I_k(y)\| \leqslant K_2\|x - y\|$$

for every $k \in \{1, \ldots, m\}$ and $x, y \in B$.

The next lemma can be found in [7] and it describes how the Carathéodory and Lipschitz-type conditions concerning the function f and the Lipschitz and boundedness conditions for the impulse operators can be transferred to $\bar{f}$, when it is defined the same way as described in Theorem 5.2.

Lemma 5.1. *Let* $m \in \mathbb{N}, t_0 \leqslant t_1 < \cdots < t_m < t_0 + \eta$, $B \subset \mathbb{R}^n, I_1, \ldots, I_m : B \to \mathbb{R}^n, P = G([-r, 0], B), O = G([t_0 - r, t_0 + \eta], B)$. *Assume that* $g : [t_0, t_0 + \eta] \to \mathbb{R}$ *is a left-continuous nondecreasing function which is continuous at* $t_1, \ldots, t_m$. *Let* $f : P \times [t_0, t_0 + \eta] \to \mathbb{R}^n$ *be a function such that the integral* $\int_{t_0}^{t_0+\eta} f(y_t, t) \, \mathrm{d}g(t)$ *exists for every* $y \in O$. *For every* $y \in P$, *define*

$$\bar{f}(y, t) = \begin{cases} f(y, t), & t \in [t_0, t_0 + \eta] \setminus \{t_1, \ldots, t_m\}, \\ I_k(y(0)), & t = t_k \text{ for some } k \in \{1, \ldots, m\}. \end{cases}$$

Moreover, let $c_1, \ldots, c_m \in \mathbb{R}$ *be constants such that the function* $\bar{g} : [t_0, t_0 + \eta] \to \mathbb{R}$ *given by*

$$\bar{g}(t) = \begin{cases} g(t), & t \in [t_0, t_1], \\ g(t) + c_k, & t \in (t_k, t_{k+1}] \text{ for some } k \in \{1, \ldots, m-1\}, \\ g(t) + c_m, & t \in (t_m, t_0 + \eta] \end{cases}$$

satisfies $\Delta^+ \bar{g}(t_k) = 1$ *for every* $k \in \{1, \ldots, m\}$.

1. *If conditions (B) and (A*) hold, then*

$$\|\bar{f}(y_t, t)\| \leqslant M + K_1,$$

whenever $t_0 \leqslant t \leqslant t_0 + \eta$ *and* $y \in O$.
2. *If conditions (C) and (B*) hold, then*

$$\left\| \int_{u_1}^{u_2} (\bar{f}(y_t, t) - \bar{f}(z_t, t)) \, \mathrm{d}\bar{g}(t) \right\| \leqslant (L + K_2) \int_{u_1}^{u_2} \|y_t - z_t\|_\infty \, \mathrm{d}\bar{g}(t),$$

whenever $t_0 \leqslant u_1 \leqslant u_2 \leqslant t_0 + \eta$ *and* $y, z \in O$.

The next theorem shows that, under certain conditions, it is possible to obtain a correspondence between the solutions of impulsive measure functional differential equations, depending on the conditions about the functions $\tilde{\sigma}_n$ and $\tilde{\sigma}$ and the corresponding time scales, that is, $\mathbb{T}_n$ and $\mathbb{T}$.

Theorem 5.3. *Suppose f satisfies the conditions (A), (B) and (C), and for each $k = 1, 2, \ldots, m$, the impulse operators $I_k : \mathbb{R}^n \to \mathbb{R}^n$ satisfy conditions (A*) and (B*). Moreover, suppose $x_n^{\tilde{\sigma}_n}$ is a solution of the system*

$$\begin{cases} x_n^{\tilde{\sigma}_n}(t) = x_n^{\tilde{\sigma}_n}(t_0) + \int_{t_0}^t f((x_n^{\tilde{\sigma}_n})_s, s) \, \mathrm{d}\tilde{\sigma}_n(s) + \sum_{\substack{k \in \{1, \ldots, m\}, \\ t_k < t}} I_k(x_n^{\tilde{\sigma}_n}(t_k)), & t \in \mathbb{T}_n^*, \\ (x_n^{\tilde{\sigma}_n})_{t_0} = \phi^{\tilde{\sigma}_n} \end{cases} \tag{5.10}$$

and $x^{\tilde{\sigma}}$ is a solution of the measure functional differential equation given by

$$\begin{cases} x^{\tilde{\sigma}}(t) = x^{\tilde{\sigma}}(t_0) + \int_{t_0}^t f(x_s^{\tilde{\sigma}}, s) \, \mathrm{d}\tilde{\sigma}(s) + \sum_{\substack{k \in \{1, \ldots, m\}, \\ t_k < t}} I_k(x^{\tilde{\sigma}}(t_k)), & t \in \mathbb{T}^*, \\ x_{t_0}^{\tilde{\sigma}} = \phi^{\tilde{\sigma}}. \end{cases} \tag{5.11}$$

Moreover, suppose $d(\mathbb{T}_n, \mathbb{T}) \to 0$ as $n \to \infty$ and the sequence of functions $\{\tilde{\sigma}_n\}_{n=1}^\infty$ converges uniformly to $\tilde{\sigma}$ as $n \to \infty$. Also, suppose the sequence of initial conditions $\{\phi^{\tilde{\sigma}_n}\}_{n=1}^\infty$ converges uniformly to $\phi^{\tilde{\sigma}}$ as $n \to \infty$. Then, for every $\varepsilon > 0$, there exists $N > 0$ sufficiently large such that, for $n > N$, we have

$$\|x_n^{\tilde{\sigma}_n}(t) - x^{\tilde{\sigma}}(t)\| < \varepsilon \quad \text{for} \quad t \in \mathbb{T}_n^* \cap \mathbb{T}^*.$$

Proof. Define the functions $\bar{f}, \bar{\tilde{\sigma}}$ and $\bar{\tilde{\sigma}}_n$ as described in the statement of Theorem 5.2. Since the sequence of functions $\{\tilde{\sigma}_n\}_{n=1}^\infty$ converges uniformly to $\tilde{\sigma}$, it follows immediately from the definition that the sequence of functions $\{\bar{\tilde{\sigma}}_n\}_{n=1}^\infty$ converges uniformly to $\bar{\tilde{\sigma}}$. Also, by Lemma 5.1, we obtain that all hypotheses of Theorem 5.1 are satisfied and then, using the correspondence (Theorem 4.1), the desired result follows. $\square$

Now, consider the next result (see [7]) that will be essential to prove our final theorem on continuous dependence.

Lemma 5.2. *Let* $[t_0 - r, t_0 + \eta]_{\mathbb{T}}$ *be a time scale interval,* $t_0 \in \mathbb{T}$, $O = G([t_0 - r, t_0 + \eta], B)$, $P = G([-r, 0], B)$, $f : P \times [t_0, t_0 + \eta]_{\mathbb{T}} \to \mathbb{R}^n$ *be an arbitrary function. Define* $f^{\tilde{\sigma}}(y, t) = f(y, \tilde{\sigma}(t))$ *for every* $y \in P$ *and* $t \in [t_0, t_0 + \eta]$.

1. *If the integral* $\int_{t_0}^{t_0+\eta} f(y_t, t) \Delta t$ *exists for every* $y \in O$, *then* $\int_{t_0}^{t_0+\eta} f^{\tilde{\sigma}}(y_t, t) \, \mathrm{d}\tilde{\sigma}(t)$ *exists for every* $y \in O$.
2. *Assume there exists a constant* $M > 0$ *such that*

$$\|f(y_t, t)\| \leqslant M$$

for every $y \in O$ *and* $t \in [t_0, t_0 + \eta]_{\mathbb{T}}$. *Then*

$$\|f^{\tilde{\sigma}}(y_t, t)\| \leqslant M,$$

whenever $t_0 \leqslant t \leqslant t_0 + \eta$ *and* $y \in O$.
3. *Assume there exists a constant* $L > 0$ *such that*

$$\left\| \int_{u_1}^{u_2} (f(y_t, t) - f(z_t, t)) \Delta t \right\| \leqslant L \int_{u_1}^{u_2} \|y_t - z_t\|_{\infty} \Delta t$$

for every $y, z \in O$ *and* $u_1, u_2 \in [t_0, t_0 + \eta]_{\mathbb{T}}, u_1 \leqslant u_2$. *Then*

$$\left\| \int_{u_1}^{u_2} (f^{\tilde{\sigma}}(y_t, t) - f^{\tilde{\sigma}}(z_t, t)) \, \mathrm{d}g(t) \right\| \leqslant L \int_{u_1}^{u_2} \|y_t - z_t\|_{\infty} \, \mathrm{d}g(t),$$

whenever $t_0 \leqslant u_1 \leqslant u_2 \leqslant t_0 + \eta$ *and* $y, z \in O$.

Now, consider the following conditions concerning the function $f : G([-r, 0], B) \times [t_0, t_0 + \eta]_{\mathbb{T}_n} \to \mathbb{R}^n$:

(A_1) The integral $\int_{t_0}^{t_0+\eta} f(y_t, t) \Delta t$ exists for every $y \in O$.
(B_1) There exists a constant $M > 0$ such that

$$\|f(y_t, t)\| \leqslant M$$

for every $y \in O$ and $t \in [t_0, t_0 + \eta]_{\mathbb{T}}$.
(C_1) There exists a constant $L > 0$ such that

$$\left\| \int_{u_1}^{u_2} (f(y_t, t) - f(z_t, t)) \Delta t \right\| \leqslant L \int_{u_1}^{u_2} \|y_t - z_t\|_{\infty} \Delta t$$

for every $y, z \in O$ and $u_1, u_2 \in [t_0, t_0 + \eta]_{\mathbb{T}}, u_1 \leqslant u_2$.

The next theorem is our main result. It concerns continuous dependence for impulsive functional dynamic equations on time scales involving variable time scales.

Theorem 5.4. *Suppose* $x_n : \mathbb{T}_n \to \mathbb{R}^n$ *is a solution of the impulsive functional dynamic equation on time scales*

$$\begin{cases} x_n(t) = x_n(t_0) + \int_{t_0}^t f_n((x_n^{\tilde{\sigma}_n})_s, s) \Delta s + \sum_{\substack{k \in \{1, \ldots, m\} \\ t_k < t}} I_k(x_n(t_k)), & t \in [t_0, t_0 + \eta]_{\mathbb{T}_n}, \\ x_n(t) = \phi(t), & t \in [t_0 - r, t_0]_{\mathbb{T}_n}, \end{cases} \tag{5.12}$$

where the functions $f_n : G([-r, 0], B) \times [t_0, t_0 + \eta]_{\mathbb{T}_n} \to \mathbb{R}^n$ *satisfy the conditions* (A_1), (B_1) *and* (C_1). *Also, suppose* $x : \mathbb{T} \to \mathbb{R}^n$ *is a solution of the impulsive functional dynamic equation on time scales*

$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^t f(x_s^{\tilde{\sigma}}, s) \Delta s + \sum_{\substack{ck \in \{1, \ldots, m\}, \\ t_k < t}} I_k(x(t_k)), & t \in [t_0, t_0 + \eta]_{\mathbb{T}}, \\ x(t) = \phi(t), & t \in [t_0 - r, t_0]_{\mathbb{T}}, \end{cases} \tag{5.13}$$

where $f : G([-r, 0], B) \times [t_0, t_0 + \eta]_{\mathbb{T}} \to \mathbb{R}^n$ *satisfies the conditions* (A_1), (B_1) *and* (C_1) *and for each* $k = 1, 2, \ldots, m$, *the impulse operators* $I_k : \mathbb{R}^n \to \mathbb{R}^n$ *satisfy conditions* (A^*) *and* (B^*). *Suppose* $d(\mathbb{T}_n, \mathbb{T}) \to 0$ *as* $n \to \infty$ *and the sequence of functions* $\{\tilde{\sigma}_n\}_{n=1}^{\infty}$ *converges uniformly to* $\tilde{\sigma}$ *as* $n \to \infty$. *Also, suppose the sequence* $\{\phi^{\tilde{\sigma}_n}\}_{n=1}^{\infty}$ *converges uniformly to* $\phi^{\tilde{\sigma}}$ *as* $n \to \infty$. *Then, for every* $\varepsilon > 0$, *there exists* $N > 0$ *sufficiently large such that, for* $n > N$, *we have*

$$\|x_n(t) - x(t)\| < \varepsilon \quad \text{for} \quad t \in \mathbb{T}_n \cap \mathbb{T}.$$

Proof. Since the function $f_n : G([-r, 0], B) \times [t_0, t_0 + \eta]_{\mathbb{T}} \to \mathbb{R}^n$ satisfies the conditions (A_1), (B_1) and (C_1), it follows from Lemma 5.2 that the respective ones (conditions (A), (B) and (C)) are satisfied for the extension of f_n and therefore, all hypotheses from Theorem 5.3 are satisfied, and the desired result follows immediately applying the correspondence between impulsive measure functional differential equations and impulsive functional dynamic equations on time scales. □

6. Applications

In this section, our goal is to discuss some applications of our main results.

The results about continuous dependence of solutions of dynamic equations on variable time scales have several applications for numerical approximations. It is a known fact that many differential equations cannot be solved analytically, however, a numerical approximation to the solution is usually good enough to solve a problem described by models in engineering and sciences. In order to do this, it is possible to construct algorithms to compute such an approximation. Therefore, results as the ones presented in this paper are very useful to study the solutions of ordinary differential equations as well as other dynamic equations depending on the chosen time scale without the necessity to solve them analytically.

In what follows, we present some examples to illustrate this fact. For more details about them, the reader may want to consult [4,10,13].

Example 6.1 [4]. Consider a simple autonomous linear dynamic equation given by

$$\begin{cases} x^\Delta(t) = ax(t), \\ x(0) = x_0. \end{cases} \tag{6.1}$$

Solving the Eq. (6.1) for the case $\mathbb{T} = \mathbb{R}$, we get

$$x(t) = x_0 e^{at}.$$

On the other hand, solving the Eq. (6.1) for the case $\mathbb{T} = \frac{1}{n}\mathbb{Z}$, for $n \in \mathbb{N}$, we obtain

$$y_n(t) = x_0\left(1 + \frac{a}{n}\right)^{nt}.$$

It is not difficult to see that

$$\mathbb{T}_n = \frac{1}{n}\mathbb{Z} \to \mathbb{R} \quad \text{as} \quad n \to \infty$$

and

$$\tilde{\sigma}_n = \tilde{\sigma} \quad \text{uniformly} \quad \text{as} \quad n \to \infty.$$

Moreover, we have

$$\lim_{n\to\infty} y_n(t) = x(t).$$

Example 6.2 ([4,13]). Consider a particular (logistic) initial value problem

$$\begin{cases} x^\Delta(t) = 4x\left(\frac{3}{4} - x\right), \\ x(0) = x_0. \end{cases} \tag{6.2}$$

If we take $\mathbb{T}_n = \frac{1}{n}\mathbb{Z}_+$, for $n \in \mathbb{N}$, in Eq. (6.2), we obtain

$$\frac{x\left(t + \frac{1}{n}\right) - x(t)}{\frac{1}{n}} = 4x(t)\left(\frac{3}{4} - x(t)\right),$$

which implies that

$$\begin{aligned} x\left(t + \frac{1}{n}\right) &= \frac{4}{n}x(t)\left(\frac{3}{4} - x(t)\right) + x(t) \\ &= \frac{4}{n}x(t)\left(\frac{\frac{3}{n} + 1}{\frac{4}{n}} - x(t)\right) \\ &= \frac{4}{n}x(t)\left(\frac{3 + n}{4} - x(t)\right). \end{aligned}$$

Notice that the solution is found by iterating the following equation (see [13]):

$$x_n(t) = \frac{4}{n}x(t)\left(\frac{3 + n}{4} - x(t)\right).$$

Then, taking $n \to \infty$, the solutions tend to the solution of the logistic differential equation on $\mathbb{R}_+$ and $\frac{1}{n}\mathbb{Z}_+ \to \mathbb{R}_+$ (see [13]). Also, it is clear that $\tilde{\sigma}_n \to \tilde{\sigma}$ uniformly as $n \to \infty$.

7. Conclusions

The examples presented in the previous section show the importance of the continuous dependence results involving variable time scales proved in this paper, since one can find a good approximation for solutions of differential equations without the necessity to calculate it analytically.

For instance, as described in the last section, taking a sequence of time scales given by $\mathbb{T}_n = \frac{1}{n}\mathbb{Z}$, it is possible to find a good approximation of a solution of a differential equation, just by using a sequence of solutions of the corresponding dynamic equations on $\mathbb{T}_n$, since by applying our results, one can obtain that this sequence converges to the solution of the differential equation. Notice that to get this approximation, one just has to use iteration of solutions of the dynamic equations on $\mathbb{T}_n$, which can be done by using a computational algorithm. Thus, due to this fact, the results presented here turn out to be very useful in numerical approximations.

We point out that our results are general enough to be applied for equations involving retarded arguments and impulsive behavior, which make them helpful for obtaining these approximations for more complicated equations without the necessity to calculate their solutions analytically.

Also, the results presented here can be applied to investigate the stability and asymptotic behavior of the solutions of impulsive functional dynamic equations on time scales. This fact happens since knowing the behavior of the solutions of the dynamic equations on $\mathbb{T}_n$, it is possible by applying our results to investigate the behavior of the solution of dynamic equation on $\mathbb{T}$, whenever $\mathbb{T}_n \to \mathbb{T}$, using the convergence properties.

Acknowledgements

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