Multipartite W-type state is determined by its single-particle reduced density matrices among all W-type states

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It is known that the n-qubit W-type state is determined by its bipartite reduced density matrices. In this paper, we show that the multipartite W-type state is uniquely determined by its reduced density matrices among W-type states in the sense of local unitary equivalence.

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I. INTRODUCTION

With entanglement being a proven asset to information processing and computational tasks, much effort has been spent on quantifying it as a resource and numerous results have been obtained. It is extremely difficult to give a perfect description of the entanglement in the multipartite case, even though it is well understood for the bipartite pure state case.

One important approach is to consider their interconvertibility through manipulations that do not require quantum communication, which is to determine whether one can interconvert between two given states $|\Psi\rangle$ and $|\Phi\rangle$ by local operation. A widely studied equivalence relation of multipartite state space is stochastic local operations and classical communication (SLOCC) [1–3]: $|\Psi\rangle$ and $|\Phi\rangle$ are considered to be SLOCC equivalent if they can be reversibly converted from one to the other by operations belonging to the class of stochastic local operations and classical communication. On the other hand, if only local unitaries are allowed, the problem become very interesting and significant: local unitaries (LU) do not change entanglement, and LU equivalent states have the same entanglement (both for type and amount). Thus, a LU equivalent relationship can be considered as one key solution of the characterization of multipartite entanglement. Two multipartite states $|\Psi\rangle$ and $|\Phi\rangle$ are called LU equivalent if there exist unitaries $U_1,\dots,U_n$ such that $|\Psi\rangle = (U_1\otimes\cdots\otimes U_n)|\Phi\rangle$. Generally, it is difficult to determine whether two given multipartite states are LU equivalent, while it might be easier in the SLOCC case.

In this paper, we study a special case of the following interesting problem: how to check if two pure quantum states are LU equivalent, provided they lie in the same SLOCC class. In particular, we are most interested in the SLOCC class of the $|W\rangle$ state, whose entanglement has attracted a lot of attention [4–6]. For the n-qubit pure quantum state, it was proved that the maximum degree of entanglement (measured in terms of the concurrence) between any pair is equal to $2/n$ [4] and the maximum bound is achieved when the state is $|W\rangle_n = \frac{1}{\sqrt{n}}(|0\cdots01\rangle+\cdots+|10\cdots0\rangle)$. For two n-partite pure states which are SLOCC equivalent to $|W\rangle_n$, i.e., the W-type state, it was shown that the n-qubit W-type state is determined by its bipartite reduced density matrices in [5]. Here, we generalize this result to show that the W-type state is determined by its single-particle reduced density matrices. More precisely, we show that two W-type states are LU equivalent if and only if they share the spectra of single-particle reduced density matrices.

II. MAIN RESULT

The following lemma gives a very nice characterization of the W-type state up to local unitaries [7,8]. Moreover, it relates the W-type pure state to a $n+1$-dimensional real vector.

Lemma. Any W-type pure state is LU equivalent to

$$\sqrt{x}|0\cdots0\rangle+\sqrt{c_1}|0\cdots01\rangle+\cdots+\sqrt{c_n}|10\cdots0\rangle,$$

with some $c_k>0$ and $x\geqslant 0$.

Proof. Suppose $|\psi\rangle=(A_1\otimes A_2\otimes\cdots\otimes A_n)|W\rangle_n$, with A_k being an all nonsingular 2×2 matrix. For any A_k, there is a unitary V_k such that $A_k=V_kB_k$, with B_k being an upper triangle matrix. Thus, $|\psi\rangle$ is LU equivalent to $(B_1\otimes B_2\otimes\cdots\otimes B_n)|W\rangle_n$, that is,

$$d|0\cdots0\rangle+d_1|0\cdots01\rangle+\cdots+d_n|10\cdots0\rangle,$$

for complex d,d_k. One can find the diagonal local unitaries to transform it into the wanted formalism.

Now we can present our main result as follows:

Theorem 1. The multipartite W state is uniquely determined by its single-particle reduced density matrices. In other words, for two given W-type states $|\varphi\rangle,|\psi\rangle\in H_1\otimes H_2\otimes\cdots\otimes H_n$, if their reduced density matrices enjoys the same spectra, then $|\varphi\rangle,|\psi\rangle$ are LU equivalent.

Proof. Without loss of generality, assume that $|\varphi\rangle,|\psi\rangle$ are given as

$$|\varphi\rangle=\sqrt{u}|0\cdots0\rangle+\sqrt{a_1}|0\cdots01\rangle+\cdots+\sqrt{a_n}|10\cdots0\rangle,$$
$$|\psi\rangle=\sqrt{v}|0\cdots0\rangle+\sqrt{b_1}|0\cdots01\rangle+\cdots+\sqrt{b_n}|10\cdots0\rangle.$$

If their reduced density matrices satisfy $\det\rho_k=\det\sigma_k$ for all k, then the following holds for all $1\leqslant k\leqslant n$:

$$a_k\sum_{j\neq k}^{n}a_j=\det\rho_k=\det\sigma_k=b_k\sum_{j\neq k}^{n}b_j.$$

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One can obtain that $a_k = b_k$ for any $1 \leqslant k \leqslant n$ by proving the following lemma. Thus, $u = 1 - \sum a_k = 1 - \sum b_k = v$, which means that $|\varphi\rangle = |\psi\rangle$.

The following interesting lemma completes the proof of Theorem 1.

Lemma 2. $\{a_k : 1 \leqslant k \leqslant n\}$ and $\{b_k : 1 \leqslant k \leqslant n\}$ are two sets of positive numbers with $n \geqslant 3$; if

$$a_k \sum_{j\neq k}^{n} a_j = b_k \sum_{j\neq k}^{n} b_j \tag{1}$$

is true for any $1 \leqslant k \leqslant n$, then $a_k = b_k$ holds for any $1 \leqslant k \leqslant n$.

Proof. Consider (1) as equations of $\{a_k : 1 \leqslant k \leqslant n\}$; it is sufficient to show that

$$a_k \sum_{j\neq k}^{n} a_j = x_k/4 \tag{2}$$

has at most one positive root, where $x_k = 4b_k \sum_{j\neq k}^{n} b_j$.

Let $A = \sum_{j=1}^{n} a_j$; we have

$$a_k = \frac{A \pm \sqrt{A^2 - x_k}}{2},$$

where

$$x_k = 4b_k \sum_{j\neq k}^{n} b_j = 4a_k(A - a_k) \leqslant (a_k + A - a_k)^2 = A^2.$$

There is at most one k such that $a_k \geqslant A/2$, that is, $a_k = \frac{A+\sqrt{A^2-x_k}}{2}$. Without loss of generality, suppose the largest element of $\{a_k : 1 \leqslant k \leqslant n\}$ is a_1, then for any $k \geqslant 2$, $a_k = \frac{A-\sqrt{A^2-x_k}}{2}$ and $a_1 = A - \sum_{j=2}^{n} a_j$.

We only need to show that there is at most one solution, which satisfies Eq. (3) or Eq. (4), for given $x_1, x_2, \ldots, x_n > 0$:

$$\sum_{k=1}^{n} \left(\frac{A - \sqrt{A^2 - x_k}}{2} \right) = A, \tag{3}$$

$$\frac{A + \sqrt{A^2 - x_1}}{2} + \sum_{k=2}^{n} \left(\frac{A - \sqrt{A^2 - x_k}}{2} \right) = A. \tag{4}$$

Equations (3) and (4) are just $f(A) = 0$ and $g(A) = 0$, respectively, where

$$f(y) = -2y + \sum_{k=1}^{n} \left(\frac{x_k}{y + \sqrt{y^2 - x_k}} \right),$$

$$g(y) = \sum_{k=2}^{n} (\sqrt{y^2 - x_k}) - [\sqrt{y^2 - x_1} + (n-2)y].$$

Case 1: Suppose there is some $r > 0$ such that $f(r) = 0$. It is direct to verify that $f(y)$ is a strictly monotone decreasing function on $[0, +\infty)$, which implies that $f(y)$ has at most one root.

Assume $g(s) = 0$ holds for some $s > 0$. Then, $x_1 \geqslant x_k$ for any k, otherwise $g(s) < 0$ by noticing $s > \sqrt{s^2 - x_k}$. Let $z = \sqrt{s^2 - x_1}$, and for $k > 1$, $z_k = \sqrt{x_1 - x_k}$, then $0 \leqslant z_k \leqslant \sqrt{x_1}$.

First, we show that $r = \sqrt{x_1}$. To do so, we suppose $r > \sqrt{x_1}$. Since $f(y)$ is monotone decreasing, we have

$$\begin{aligned} 0 &= f(r) < f(\sqrt{x_1}) \\ &= -2\sqrt{x_1} + \sum_{i=1}^{n} \frac{x_i}{\sqrt{x_1} + \sqrt{x_1 - x_i}} \\ &\Rightarrow \sum_{k=2}^{n} \sqrt{x_1 - x_k} < (n-2)\sqrt{x_1} \\ &\Rightarrow \sum_{k=2}^{n} z_k < (n-2)\sqrt{x_1}. \end{aligned}$$

Define a real function for any $l > 0$:

$$h_l(y_2, y_3, \ldots, y_n) = \sum_{k=2}^{n} \sqrt{l^2 + y_k^2}.$$

Invoking the concavity of function $h_z(y_2, y_3 \ldots, y_n)$, $0 \leqslant z_k \leqslant \sqrt{x_1}$ and $\sum_{k=2}^{n} z_k < (n-2)\sqrt{x_1}$, we have the following:

$$\begin{aligned} \sum_{k=2}^{n} \sqrt{s^2 - x_k} &= \sum_{k=2}^{n} \sqrt{z^2 + z_k^2} = h_z(z_2, z_3, \ldots, z_n) \\ &< g(0, \sqrt{x_1}, \ldots, \sqrt{x_1}) = \sqrt{s^2 - x_1} + (n-2)s. \end{aligned}$$

But $g(s) = 0$. This is a contradiction.

Thus, in this case, we know that $r = \sqrt{x_1}$. One can also obtain that $g(\sqrt{x_1}) = 0$ from $f(\sqrt{x_1}) = 0$.

Now, suppose there are $s_1 > s_0 > 0$ such that $g(s_1) = g(s_0) = 0$. Assume $t = s_1^2 - s_0^2$, and we have

$$\begin{aligned} &(n-2)s_1 + \sqrt{s_1^2 - x_1} \\ &= \sum_{k=2}^{n} \sqrt{s_1^2 - x_k} = \sum_{k=2}^{n} \sqrt{t + s_0^2 - x_k} \\ &= h_{\sqrt{t}}\left(\sqrt{s_0^2 - x_2}, \sqrt{s_0^2 - x_3}, \ldots, \sqrt{s_0^2 - x_n}\right). \end{aligned}$$

According to $g(s_0) = 0$, we have $\sum_{k=2}^{n} \sqrt{s_0^2 - x_k} = (n-2)s_0 + \sqrt{s_0^2 - x_1}$. Noting that $\sqrt{s_0^2 - x_k} < s_0$, we invoke the concavity of $h_z(y_2, y_3, \ldots, y_n)$ again and obtain

$$\begin{aligned} &h_{\sqrt{t}}\left(\sqrt{s_0^2 - x_2}, \sqrt{s_0^2 - x_3}, \ldots, \sqrt{s_0^2 - x_n}\right) \\ &\quad < h_{\sqrt{t}}(0, s_0, \ldots, s_0) = (n-2)s_1 + \sqrt{s_1^2 - x_1}, \end{aligned}$$

which is a contradiction from $g(s_1) = 0$. Therefore, $g(y)$ can have, at most, one root. Thus, $f(y)$ has, at most, one root. If $f(r) = 0$ and $g(s) = 0$, then $r = s = \sqrt{x_1}$.

Case 2: Suppose $f(y)$ has no root. As we discussed above, $g(y)$ has, at most, one root. Thus, there is, at most, one solution, which satisfies Eq. (3) or Eq. (4). Obviously, we can choose $A = \sum_{k=1}^{n} b_k$; then, $a_k = b_k$ is the only possible case.

III. CONCLUSION

In this short paper, we show that the entanglement of W-type states is uniquely determined by their reduced density matrices. It would be quite interesting to generalize this argument to the SLOCC class of the symmetric state, for instance, the Dicke state.

Note added in proof. Recently, we have learned about the independent work by Sawicki *et al.* [9], in which the three-qubit case is studied and they left the conjecture of the n-qubit W-type state in their previous version. Our result provides a positive answer to that conjecture.

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[1] J. Eisert and H. J. Briegel, Phys. Rev. A **64**, 022306 (2001).

[2] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Phys. Rev. A **65**, 052112 (2002).

[3] M. Hein, J. Eisert, and H. J. Briegel, Phys. Rev. A **69**, 062311 (2004).

[4] M. Koashi, V. Buzek, and N. Imoto, Phys. Rev. A **62**, 050302(R) (2000).

[5] P. Parashar and S. Rana, Phys. Rev. A **80**, 012319 (2009).

[6] N. Yu, E. Chitambar, C. Guo, and R. Duan, Phys. Rev. A **81**, 014301 (2010).

[7] W. Cui, E. Chitambar, and H. K. Lo, Phys. Rev. A **84**, 052301 (2011).

[8] S. Rana and P. Parashar, arXiv:1110.4597 [Phys. Rev. A (to be published)].

[9] A. Sawicki, M. Walter, and M. Kus, J. Phys. A: Math. Theor. **46**, 055304 (2013).