Stability of equilibria of randomly perturbed maps

Paweł Hitczenko and Georgi S. Medvedev*

March 23, 2015

Abstract

We derive a sufficient condition for stability in probability of an equilibrium of a randomly perturbed map in $\mathbb{R}^d$. This condition can be used to stabilize weakly unstable equilibria by random forcing. Analytical results on stabilization are illustrated with numerical examples of randomly perturbed linear and nonlinear maps in one- and two-dimensional spaces.

1 Introduction

The idea of stabilizing unstable equilibria of dynamical systems by noise originates from the pioneering work of Khasminskii on stochastic stability in the nineteen-sixties [22]. Stochastic stabilization has important implications for control theory [7, 26, 5, 6] and for numerical methods for stochastic differential equations [28, 29, 17, 18, 12]. Furthermore, the interplay of stability and noise is important for understanding many dynamical phenomena in applied science including stochastic synchronization [1, 14, 27, 16], stochastic resonance [25, 24, 15], and noise-induced dynamics [8, 13, 19].

In this paper, we study the following difference equation in $\mathbb{R}^d$

$$x_{n+1} = (A + B)x_n + q(x_n), \tag{1.1}$$

where $q(x) = O(|x|^2)$ is a smooth function, A and B are deterministic and stochastic $d \times d$ matrices respectively. We assume that the spectral radius of A is slightly greater than 1, $\rho(A) = 1 + \epsilon$, $0 < \epsilon \ll 1$ and ask how to choose mean-zero matrix $B = B(\epsilon)$ to stabilize the equilibrium at the origin. Our motivation for considering (3.1) is two-fold. On one hand, we want to understand how to tame weak instability in general d-dimensional maps by noise. Eventually, we want to apply these results to stabilize periodic orbits of randomly perturbed stochastic ordinary differential equations in $\mathbb{R}^{d+1}$. In this case, (3.1) represents a Poincare map [20]. Stochastic stabilization of period orbits remains largely unexplored area of research with many promissing applications.

*Department of Mathematics, Drexel University, 3141 Chestnut Street, Philadelphia, PA 19104, phitczen@math.drexel.edu, medvedev@drexel.edu

For scalar difference equations, stabilization was studied by Appleby, Mao, and Rodkina [6] and by Appleby, Berkolaiko, and Rodkina [4] (see also [2, 11, 3, 9]). Certain higher-dimensional models similar to (3.1) were analyzed in the context of stability of finite-difference schemes (see [12] and references therein). In this paper, we show that one can achieve stability with high probability in a general d-dimensional model (3.1) under fairly general assumptions on B. The key requirement for stabilization is that matrix $A^{-1}B$ must be diagonally dominant in the mean square sense.

The organization of this paper is as follows. In the next section, we prove a sufficient condition for stability (in probability) of an equilibrium in a d-dimensional map (cf. Theorem 2.4). To prove this theorem, we use the Strong Law of Large Numbers to show that the Lyapunov exponent of a typical trajectory is negative. The rest of the proof follows an argument developed for deterministic dynamical systems [23]. In §3 we apply Theorem 2.4 to the problem of stabilization. In §4, we illustrate our results with several numerical examples using one- and two-dimensional systems.

2 Stochastic stability

Consider an initial value problem for the following difference equation

$$x_n = M_n x_{n-1} + q(x_{n-1}),\ n \geq 1. \tag{2.2}$$

where (M_n) are independent copies of a $d \times d$ random matrix M; $q : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous function such that

$$|q(x)| \leq C_1 |x|^2,\ x \in B_\delta = \{x : |x| < \delta\} \tag{2.3}$$

for some $C_1, \delta > 0$. The initial condition x_0 is assumed to be deterministic.

Definition 2.1. *[22] The equilibrium at the origin of (2.2) is said to be stable in probability if for any $\epsilon > 0$*

$$\lim_{x_0 \to 0} \mathbb{P}\{\sup_{n \geq 1} |x_n| > \epsilon\} = 0.$$

Theorem 2.2. *Suppose*

$$0 < \lambda = -\mathbb{E} \log \|M\| < \infty. \tag{2.4}$$

Then the equilibrium at the origin of (2.2) is stable in probability.

Remark 2.3. In (2.4), $\|\cdot\|$ is an arbitrary matrix norm. The same matrix norm is used throughout this section.

Condition (2.4) guarantees that the largest Lyapunov exponent of a generic trajectory is negative. This implies stability of $x_n \equiv 0$ with high probability. Theorem 2.2 is a stochastic counterpart of the result of Koçak and Palmer for deterministic maps [23, Theorem 4]. It follows immediately from the proof of the following lemma, which also shows that the rate of convergence of (x_n) to the origin is exponential.

Lemma 2.4. *Under the assumptions of Theorem 2.2, the following holds: for any $0 < \epsilon < 1$ there exists $0 < \eta \leq \delta$ such that*

$$|x_i| \leq \eta \mu^i,\ i = 0, 1, 2, \ldots \tag{2.5}$$

with probability at least $1 - \epsilon$ provided $|x_0|$ is sufficiently small.

Proof: Denote $\lambda_k := \log \|M_k\|$. By the Strong Law of Large Numbers [10, Theorem 22.1],

$$\frac{1}{n}\sum_{k=1}^{n}\lambda_k \xrightarrow{a.s.} \mathbb{E}\log\|M\| = -\lambda < 0 \text{ as } n \to \infty.$$

Thus, for every $\epsilon > 0$ there exists n_0 such that

$$\mathbb{P}\left(\bigcup_{n\geq n_0}\left\{\left|\frac{1}{n}\sum_{k=1}^{n}\lambda_k - (-\lambda)\right| > \epsilon\right\}\right) < \frac{\epsilon}{2}. \tag{2.6}$$

In the remainder of the proof, we assume that $0 < \epsilon < \min\{1, \lambda\}$ is arbitrary but fixed. By (2.6),

$$-\lambda - \epsilon \leq \frac{1}{n}\sum_{k=1}^{n}\lambda_k \leq -\lambda + \epsilon \tag{2.7}$$

holds for all $n > n_0$ on the set of probability at least $1 - \epsilon/2$. From now on, we consider realizations (M_k) for which (2.7) holds.

Using (2.7), for any $n > k \geq n_0$, we have

$$\prod_{j=k+1}^{n}\|M_j\| = \frac{\prod_{j=1}^{n}\|M_j\|}{\prod_{j=1}^{k}\|M_j\|} \leq \exp\{n(-\lambda+\epsilon) - k(-\lambda-\epsilon)\} = \mu^{n-k}e^{2k\epsilon}.$$

Further, for every $0 \leq j < n_0$

$$\begin{aligned}\prod_{i=j+1}^{n}\|M_i\| = \prod_{i=j+1}^{n_0}\|M_i\|\prod_{i=n_0+1}^{n}\|M_i\| &\leq \left(\prod_{i=j+1}^{n_0}\|M_i\|\right)\mu^{n-n_0}e^{2n_0\epsilon} \\ &\leq \mu^{n-j}e^{2n_0\epsilon}\max_{j<n_0}\left\{\prod_{i=j+1}^{n_0}\|M_i\|\right\}.\end{aligned}$$

Since $\max_{j<n_0}\left\{\prod_{i=j+1}^{n_0}\|M_i\|\right\}$ is an integrable random variable, there exists $C_2 = C_2(\epsilon)$ such that

$$\mathbb{P}\left(e^{2n_0\epsilon}\max_{j<n_0}\left\{\prod_{i=j+1}^{n_0}\|M_i\|\right\} > C_2\right) \leq \frac{e^{2n_0\epsilon}}{C_2}\mathbb{E}\left(\max_{j<n_0}\prod_{i=j+1}^{n}\|M_i\|\right) \leq \frac{\epsilon}{2}.$$

It then follows that on the set of probability at least $1 - \epsilon$, for all $n \geq k \geq 1$ we have

$$\prod_{i=k}^{n}\|M_i\| \leq C_2\mu^{n-k+1}. \tag{2.8}$$

Let $0 < \eta \leq \delta$ be arbitrary but fixed. Choose $\delta_1 > 0$ such that

$$C_2 \mu \delta_1 + C_1 \delta_1^2 \leq \eta, \tag{2.9}$$

$$\delta_1 \exp\{C_1 C_2 \eta / \mu^2 (1-\mu)\} \leq \eta. \tag{2.10}$$

Next, we prove that

$$|x_i| \leq \eta \mu^i, \ i = 1, 2, \ldots, \tag{2.11}$$

provided $|x_0| \leq \delta_1$.

We prove (2.11) by induction. The statement in (2.11) clearly holds for $i = 1$ (cf. (2.2), (2.8), (2.9)). Assume that (2.11) holds for $0 \leq i < p$ for $p \geq 1$. We need to show that in this case (2.11) holds for $i = p$ as well. To this effect, we note that

$$x_p = \left(\prod_{k=0}^{p-1} M_{p-k} \right) x_0 + \sum_{j=2}^{p+1} \left(\prod_{k=0}^{p-j} M_{p-k} \right) q(x_{j-2}), \tag{2.12}$$

as follows by iterating (2.2).

Using the triangle inequality, submultiplicativity of the operator norm, and (2.3), from (2.13) we obtain

$$|x_p| \leq \left(\prod_{k=0}^{p-1} \|M_{p-k}\| \right) |x_0| + C_1 \sum_{j=2}^{p+1} \left(\prod_{k=0}^{p-j} \|M_{p-k}\| \right) |x_{j-2}|^2. \tag{2.13}$$

Here, we are also using the induction hypothesis, which implies that $x_j \in B_\delta$, $j = 0, 1, \ldots, p-1$ so that (2.3) is applicable. Using (2.8) , we further derive

$$|x_p| \leq \mu^p |x_0| + C_3 \sum_{j=2}^{p+1} \mu^{p-j+1} |x_{j-2}|^2, \ C_3 = C_1 C_2. \tag{2.14}$$

By applying the induction hypothesis to the quadratic term on the right-hand side of (2.14), for

$$z_i = \mu^i |x_i|, \ i = 0, 1, 2, \ldots, \tag{2.15}$$

we have

$$z_p \leq z_0 + C_3 \eta \sum_{j=2}^{p+1} \mu^{j-3} z_{j-2} = z_0 + \frac{C_3 \eta}{\mu^2} \sum_{j=1}^{p} \mu^j z_{j-1}. \tag{2.16}$$

Using the discrete Gronwall's inequality (cf. Lemma 2.5), from (2.16) we have

$$z_p \leq z_0 \exp\{C_3 \eta / \mu^2 \sum_{k=1}^{p} \mu^k\} \leq z_0 \exp\{C_3 \eta / \mu^2 (1-\mu)\} \leq \eta,$$

where we used (2.10) to derive the last inequality. Thus, $|x_p| \leq \eta \mu^p$ (cf. (2.15)).
□

Lemma 2.5. *(cf. [23]) Let $\{z_k\}_{k=0}^{\infty}$and $\{\mu_k\}_{k=1}^{\infty}$ be two nonnegative sequences such that*

$$z_k \le B + \sum_{j=1}^{k} \mu_j z_{j-1},\ k \in [p], \tag{2.17}$$

for some $p \in \mathbb{N}$. Then for $k \in [p]$

$$z_k \le B \exp\left\{\sum_{j=1}^{k} \mu_j\right\}.$$

3 Stabilization

Consider the following difference equation in $\mathbb{R}^d$:

$$x_{n+1} = (A + B(\epsilon))x_n + f(x_n),\ n = 0, 1, 2, \ldots, \tag{3.1}$$

where $f(x) = O(|x|^2)$, $A \in \mathbb{R}^{d\times d}$ is an invertible matrix with the spectral radius

$$\rho(A) := \max\{|\lambda| :\ \lambda \text{ is an eigenvalue of } A\} = 1 + \epsilon, \quad 0 < \epsilon \ll 1. \tag{3.2}$$

$(B_n(\epsilon))$ are independent copies of a random matrix $B(\epsilon) \in \mathbb{R}^{d\times d}$ depending on a small parameter ϵ. We assume that the entries of B are mean zero (possibly dependent) random variables (RVs). Specifically, we let $B = AG$, where the entries (g_{ij}) of G are arbitrarily dependent, mean zero, non–degenerate RVs with finite third moments. In particular, we have

$$\exists K > 0 \quad (E|g_{ij}|^3)^{1/3} \le K\sigma_{ij}, \quad 1 \le i, j \le d, \tag{3.3}$$

where we have set $\sigma_{ij}^2 := \mathrm{var}(g_{ij}),\ (i, j) \in [n]^2$. We also let $\sigma := (\sigma_{11}, \sigma_{22}, \ldots, \sigma_{nn})$ and assume that $\lim_{\epsilon\to 0} |\sigma(\epsilon)| = 0$.

We want to identify conditions on B that would stabilize the weakly unstable equilibrium in (3.1) with high probability. In the light of Theorem 2.4, stabilization will be achieved if

$$\mathbb{E} \log \|A + B\| < 0. \tag{3.4}$$

Recall that it is sufficient to establish (3.4) in any matrix norm (see Remark 2.3). In the remainder of this section, we will use a matrix norm that satisfies

$$\rho(A) \le \|A\| \le \rho(A) + \kappa, \tag{3.5}$$

where $\kappa > 0$ is arbitrary but fixed. Such norm always exists (cf. [21, Lemma 5.6.10]).

Lemma 3.1. *Suppose A and B satisfy the assumptions of this section. In addition, let*

$$\frac{1}{2}|\sigma(\epsilon)|^2 - \epsilon > 0 \tag{3.6}$$

and

$$\lim_{\epsilon\to 0} \frac{\sigma_{ij}}{\sigma_{ii}^2} = 0,\ i \ne j. \tag{3.7}$$

Then (3.4) holds for sufficiently small $\epsilon > 0$.

Proof: Using the submultiplicativity of the matrix norm, (3.2), and (3.5), we have

$$\log\|A+\sigma B\| = \log\|A(I+G)\| \le \ln(1+\epsilon+\kappa) + \ln\|I+G\|. \tag{3.8}$$

We rewrite the second term on the right-hand side of (3.8) as follows

$$\mathrm{diag}(1+g_{ii}) + \tilde{G},$$

where all off–diagonal terms of $I+G$ are collected in $\tilde{G}$. By Gershgorin Theorem (cf. [21]),

$$\|I+G\| \le \rho(I+G) + \kappa \le \max_i \Big(|1+g_{ii}| + \sum_{j\ne i} |g_{ij}|\Big) + \kappa.$$

By the monotonicity of the logarithm,

$$\log\|I+G\| \le \max_i \log(|1+g_{ii}| + \sum_{j\ne i}|g_{ij}| + \kappa) \le \sum_i \log(|1+g_{ii}| + \sum_{j\ne i}|g_{ij}| + \kappa).$$

Taking expectations on both sides we get

$$\mathbb{E}\log\|I+G\| \le \sum_{i=1}^{d} \mathbb{E}\log\Big(|1+g_{ii}| + \sum_{j\ne i}|g_{ij}| + \kappa\Big).$$

For each i

$$\begin{aligned}\mathbb{E}\log\Big(|1+g_{ii}| + \sum_{j\ne i}|g_{ij}| + \kappa\Big) &\le \mathbb{E}\log(1+g_{ii} + \sum_{j\ne i}|g_{ij}| + \kappa) I_{|g_{ii}|<1} \\ &+ \mathbb{E}\log\Big(1+|g_{ii}| + \sum_{j\ne i}|g_{ij}| + \kappa\Big) I_{|g_{ii}|\ge 1}.\end{aligned} \tag{3.9}$$

By expanding the logarithm in the first term and using the fact that $\mathbb{E}g_{ii}=0$ we get

$$\begin{aligned}&\mathbb{E}\left(g_{ii} + \sum_{j\ne i}|g_{ij}| + \kappa - \frac{(g_{ii}+\sum_{j\ne i}|g_{ij}|+\kappa)^2}{2} + O((|g_{ii}| + \sum_{j\ne i}|g_{ij}| + \kappa)^3)\right) I_{|g_{ii}|<1} \\ &= \mathbb{E}\left(g_{ii} + \sum_{j\ne i}|g_{ij}| + \kappa - \frac{1}{2}(g_{ii} + \sum_{j\ne i}|g_{ij}| + \kappa)^2 + O((|g_{ii}| + \sum_{j\ne i}|g_{ij}| + \kappa)^3)\right) \\ &\quad -\mathbb{E}\left(g_{ii} + \sum_{j\ne i}|g_{ij}| + \kappa - \frac{1}{2}(g_{ii} + \sum_{j\ne i}|g_{ij}| + \kappa)^2 + O((|g_{ii}| + \sum_{j\ne i}|g_{ij}| + \kappa)^3)\right) I_{|g_{ii}|\ge 1} \\ &= \sum_{j\ne i}\mathbb{E}|g_{ij}| + \kappa - \frac{1}{2}\mathbb{E}g_{ii}^2 - \sum_{j\ne i}\mathbb{E}g_{ii}|g_{ij}| - \frac{1}{2}\mathbb{E}(\sum_{j\ne i}|g_{ij}| + \kappa)^2 + \mathbb{E}O((|g_{ii}| + \sum_{j\ne i}|g_{ij}| + \kappa)^3) \\ &\quad + O(\sum_{m=1}^{3}\mathbb{E}(|g_{ii}| + \sum_{j\ne i}|g_{ij}| + \kappa)^m I_{|g_{ii}|\ge 1}).\end{aligned}$$

Note that since $\log(1+x) \le x$, a bound on the big 'Oh' term will also give a bound on (3.9).

We estimate the terms above as follows

$$\begin{aligned}
\sum_{j\neq i} E|g_{i,j}| &= \sum_{j\neq i} O(\sigma_{i,j}) = o(\sigma_{i,i}^2), \quad \text{(by (3.7))} \\
|\sum_{j\neq i} Eg_{i,i}|g_{i,j}|| &\le \sigma_{i,i}\sum_{j\neq i}\sigma_{i,j} = o(\sigma_{i,i}^2), \quad \text{(by the Cauchy-Schwarz inequality and (3.7))} \\
E(\sum_{j\neq i}|g_{i,j}|+\kappa)^2 &= O(\kappa^2) + \sum_{j\neq i} O(\sigma_{i,j}^2) = o(\kappa) + o(\sigma_{i,i}^2), \\
E(|g_{i,i}|+\sum_{j\neq i}|g_{i,j}|+\kappa)^3 &= O(\mathbb{E}|g_{ii}|^3) + \sum_{j\neq i} O(E|g_{ij}|^3) + O(\kappa^3), \\
E(|g_{i,i}|+\sum_{j\neq i}|g_{i,j}|+\kappa)^m I_{|g_{ii}|>1} &= O(E|g_{i,i}|^m I_{|g_{i,i}|>1}) + \sum_{j\neq i} O(E|g_{i,j}|^m) + O(\kappa^m P(|g_{ii}|>1)).
\end{aligned}$$

For $m = 1, 2$ and $j \neq i$, $E|g_{i,j}|^m = o(\sigma_{ij}^2)$ as verified above. Further, for $1 \le m \le 3$

$$E|g_{ii}|^m I_{|g_{ii}|>1} \le E|g_{ii}|^3 I_{|g_{ii}|>1} \le E|g_{ii}|^3, \qquad \kappa^m P(|g_{ii}|>1) \le \kappa^m E|g_{ii}|^3.$$

Hence, by (3.3) for all $1 \le i, j \le d$,

$$E|g_{ij}|^3 = O(\sigma_{ij}^3) = o(\sigma_{i,j}^2) = o(\sigma_{ii}^2).$$

Plugging all of this into (3.8) and using $\ln(1+\epsilon+\kappa) \le \epsilon+\kappa$ we obtain that

$$\ln\|A+\sigma B\| \le \epsilon + \kappa(1+o(\kappa)) - \frac{1}{2}\sum_{i=1}^{d}(\sigma_{i,i}^2 + o(\sigma_{i,i}^2)).$$

This quantity can be made negative by choosing of $\kappa = \kappa(\epsilon)$ sufficiently small and using (3.6)).
$\square$

4 Examples

In this section, we illustrate our analysis of stabilization with several numerical examples.

4.1 One-dimensional maps

We consider first a scalar difference equation

$$x_{n+1} = f(x_n) + \xi_{n+1}x_n, \quad n = 0, 1, 2, \ldots, \tag{4.1}$$

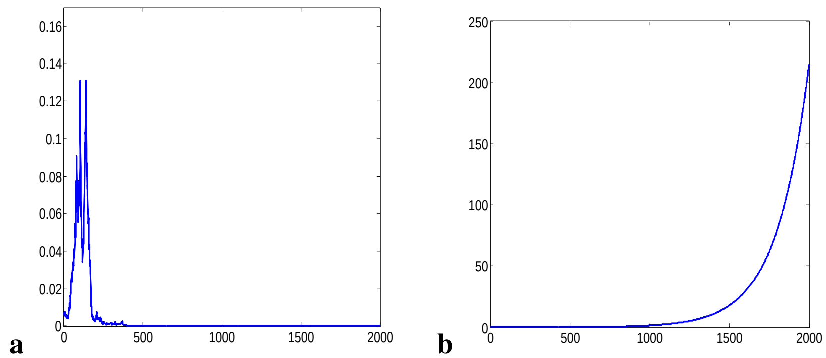

Figure 1: **a**) Timeseries generated by the stochastic one-dimensional system defined in Example 4.1. The values of parameters are $\epsilon = 0.005$, $\rho = 4$. **b**) The timeseries generated by the underlying deterministic system ($\rho = 0$) is included for comparison.

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, $f(0) = 1 + \epsilon$, and (ξ_n) are independent copies of a RV ξ with $\sigma^2 := \mathrm{var}(\xi) < \infty$.

Lemma 3.1 yields

$$\frac{\sigma^2}{2} - \epsilon > 0 \tag{4.2}$$

as a sufficient condition for stabilization provided ϵ and σ are small enough.

Example 4.1. *Let $f(x) = (1+\epsilon)x$, $\sigma^2 = \rho\epsilon$, and $\xi \in \mathcal{N}(0, \sigma^2)$.*

*The results of numerical simulations of (4.1) with the linear map above with small positive initial condition are shown in Figure 1. Plot **a** shows that the trajectory of the random system with noise intensity subject to (4.2) after a brief explosion converges to the origin. The deterministic trajectory in **b** grows exponentially.*

Example 4.2. *Next, we consider a nonlinear map $f(x) = \lambda x(1-x)$. For $\lambda = 1 + \epsilon > 1$, the logistic map f has two fixed points: $\bar{x}_1 = 0$ and $\bar{x}_2 = \epsilon(1+\epsilon)^{-1}$. For $0 < \epsilon \ll 1$, the former is unstable, while the latter is stable. All trajectories of the deterministic map $x \mapsto f(x)$ starting from $x_0 \in (0,1)$ converge to $\bar{x}_2$ (see Fig. 2**b**). In the presence of noise, however, the iterations of (4.1) with high probability converge to $\bar{x}_1$, provided (4.2) holds and ϵ is small enough (see Fig. 2**a**).*

4.2 Two-dimensional maps

We next turn to the 2D case. To this effect, we consider

$$x_{n+1} = (A + B)x_n, \quad n = 0, 1, 2, \ldots, \tag{4.3}$$

where A is a 2×2 deterministic matrix and

$$B = \sigma \begin{pmatrix} \xi_{11} & \epsilon\xi_{12} \\ \epsilon\xi_{12} & \xi_{22} \end{pmatrix}, \quad \xi_{ij} \in \mathcal{N}(0,1), \sigma^2 = \rho\epsilon. \tag{4.4}$$

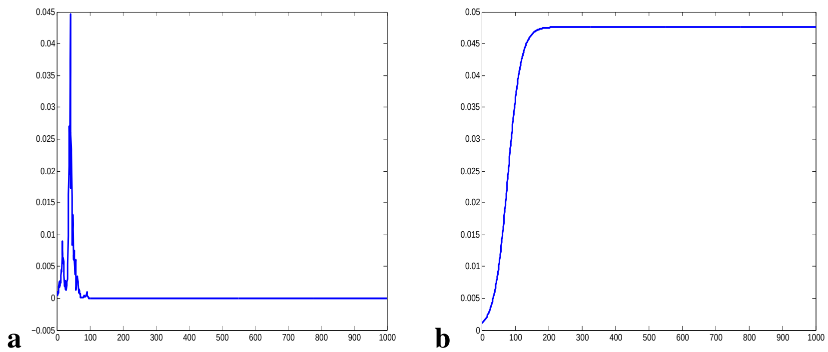

Figure 2: **a**) Timeseries generated by the randomly perturbed logistic map (see Example 4.2). Here, $\xi \in \mathcal{N}(0, \rho\epsilon)$ and the values of parameters are $\epsilon = 0.05$, $\rho = 3$. **b**) The timeseries generated by the underlying deterministic system ($\rho = 0$) is included for comparison.

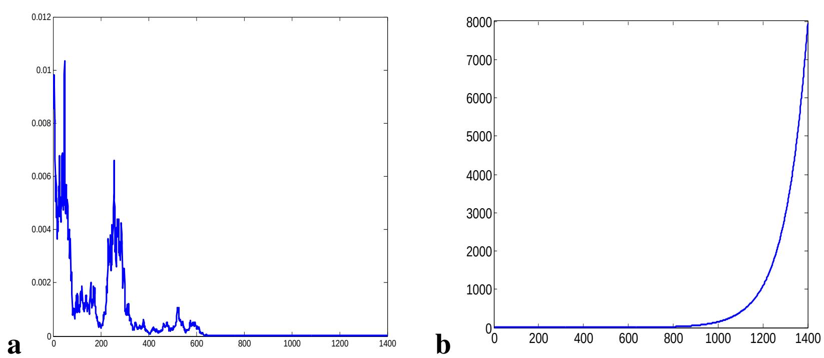

Figure 3: **a**) Timeseries $|x_n|$ generated by the stochastic two-dimensional system defined in Example 4.3. The values of parameters are $\epsilon = 0.01$, $\rho = 5$. **b**) The timeseries generated by the underlying deterministic system ($\sigma = 0$) is included for comparison.

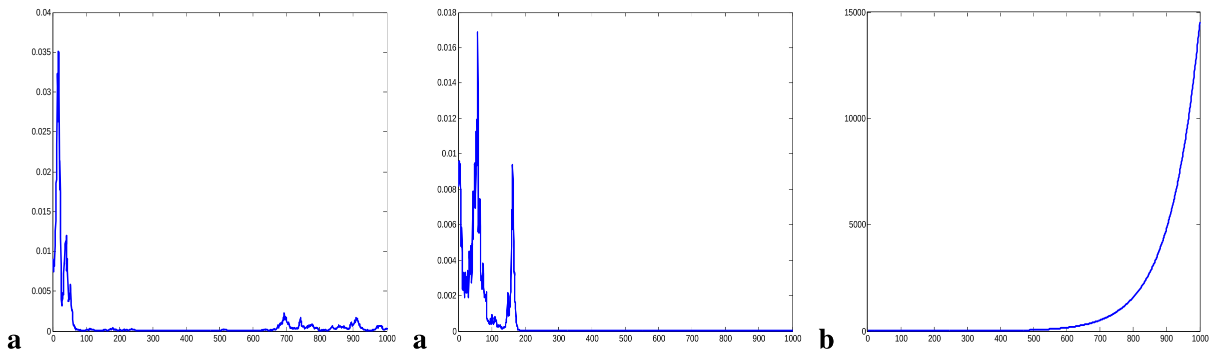

Figure 4: **a**) Timeseries $|x_n|$ generated by the stochastic two-dimensional system defined in Example 4.4. The values of parameters are $\epsilon = 0.01$, $\rho = 5$. **b**) The same as in **a** but with $\rho = 10$. **c**) The timeseries generated by the underlying deterministic system ($\sigma = 0$) is included for comparison.

Example 4.3. *Consider (4.3) with matrix B defined in (4.4) and*

$$A = \begin{pmatrix} 1+\epsilon & 0 \\ 0 & 0.5 \end{pmatrix}, \quad 0 < \epsilon \ll 1.$$

*Figure 3**a** shows a typical trajectory of the randomly perturbed system. The noise keeps the trajectory from diverging from the origin which takes place in the deterministic system (Figure 3**b**).*

Example 4.4. *In this example, we consider a nonnormal matrix with multiple eigenvalues*

$$A = \begin{pmatrix} 1+\epsilon & 0.1 \\ 0 & 1+\epsilon \end{pmatrix}, \quad 0 < \epsilon \ll 1.$$

*Figure 4 shows the results of the stabilization by noise for this case. The experiments with the noise intensity in plots **a** and **b** show that stronger (albeit small) noise results in a more robust stabilization.*

Acknowledgements. This work was supported in part by a grant from Simons Foundation (grant #208766 to PH) and by the NSF grants DMS 1109367 and DMS 1412066 to GM. The second author benefitted from participating in a SQuaRe group ‘Stochastic stabilisation of limit-cycle dynamics in ecology and neuroscience’ sponsored by the American Institute of Mathematics.

References

[1] V. S. Afraĭmovich, N. N. Verichev, and M. I. Rabinovich, *Stochastic synchronization of oscillations in dissipative systems*, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **29** (1986), no. 9, 1050–1060. MR 877439 (88g:58110)

[2] J. Appleby, G. Berkolaiko, and A. Rodkina, *On local stability for a nonlinear difference equation with a non-hyperbolic equilibrium and fading stochastic perturbations*, J. Difference Equ. Appl. **14** (2008), no. 9, 923–951. MR 2439782 (2009g:39007)

[3] John Appleby, Cónall Kelly, Xuerong Mao, and Alexandra Rodkina, *On the local dynamics of polynomial difference equations with fading stochastic perturbations*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **17** (2010), no. 3, 401–430. MR 2656407 (2011d:39028)

[4] John A. D. Appleby, Gregory Berkolaiko, and Alexandra Rodkina, *Non-exponential stability and decay rates in nonlinear stochastic difference equations with unbounded noise*, Stochastics **81** (2009), no. 2, 99–127. MR 2571683 (2010j:39028)

[5] John A. D. Appleby and Xuerong Mao, *Stochastic stabilisation of functional differential equations*, Systems Control Lett. **54** (2005), no. 11, 1069–1081. MR 2170288 (2006d:34156)

[6] John A. D. Appleby, Xuerong Mao, and Alexandra Rodkina, *On stochastic stabilization of difference equations*, Discrete Contin. Dyn. Syst. **15** (2006), no. 3, 843–857. MR 2220752 (2007b:39031)

[7] L. Arnold, *Stabilization by noise revisited*, Z. Angew. Math. Mech. **70** (1990), no. 7, 235–246. MR 1066866 (91j:93119)

[8] Nils Berglund and Barbara Gentz, *Noise-induced phenomena in slow-fast dynamical systems*, Probability and its Applications (New York), Springer-Verlag London, Ltd., London, 2006, A sample-paths approach. MR 2197663 (2007b:37115)

[9] Gregory Berkolaiko and Alexandra Rodkina, *Almost sure convergence of solutions to nonhomogeneous stochastic difference equation*, J. Difference Equ. Appl. **12** (2006), no. 6, 535–553. MR 2240374 (2007b:39002)

[10] P. Billingsley, *Probability and measure*, 3rd ed., Wiley, 1995.

[11] E. Braverman and A. Rodkina, *On difference equations with asymptotically stable 2-cycles perturbed by a decaying noise*, Comput. Math. Appl. **64** (2012), no. 7, 2224–2232. MR 2966858

[12] Evelyn Buckwar and Cónall Kelly, *Towards a systematic linear stability analysis of numerical methods for systems of stochastic differential equations*, SIAM J. Numer. Anal. **48** (2010), no. 1, 298–321. MR 2608371 (2011b:60271)

[13] R. E. Lee DeVille, Eric Vanden-Eijnden, and Cyrill B. Muratov, *Two distinct mechanisms of coherence in randomly perturbed dynamical systems*, Phys. Rev. E (3) **72** (2005), no. 3, 031105, 10. MR 2179903 (2006f:37074)

[14] Brent Doiron, John Rinzel, and Alex Reyes, *Stochastic synchronization in finite size spiking networks*, Phys. Rev. E (3) **74** (2006), no. 3, 030903, 4. MR 2282117 (2007k:92017)

[15] Mark Freidlin, *On stochastic perturbations of dynamical systems with fast and slow components*, Stoch. Dyn. **1** (2001), no. 2, 261–281. MR 1840196 (2003a:60032)

[16] Denis S. Goldobin and Arkady Pikovsky, *Synchronization and desynchronization of self-sustained oscillators by common noise*, Phys. Rev. E (3) **71** (2005), no. 4, 045201, 4. MR 2139983 (2005m:82085)

[17] Desmond J. Higham, *Mean-square and asymptotic stability of the stochastic theta method*, SIAM J. Numer. Anal. **38** (2000), no. 3, 753–769 (electronic). MR 1781202

[18] Desmond J. Higham, Xuerong Mao, and Chenggui Yuan, *Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations*, SIAM J. Numer. Anal. **45** (2007), no. 2, 592–609 (electronic). MR 2300289 (2008c:60064)

[19] Paweł Hitczenko and Georgi S. Medvedev, *Bursting oscillations induced by small noise*, SIAM J. Appl. Math. **69** (2009), no. 5, 1359–1392. MR 2487064 (2010f:60169)

[20] ———, *The Poincaré map of randomly perturbed periodic motion*, J. Nonlinear Sci. **23** (2013), no. 5, 835–861. MR 3101836

[21] Roger A. Horn and Charles R. Johnson, *Matrix analysis*, second ed., Cambridge University Press, Cambridge, 2013. MR 2978290

[22] Rafail Khasminskii, *Stochastic stability of differential equations*, second ed., Stochastic Modelling and Applied Probability, vol. 66, Springer, Heidelberg, 2012, With contributions by G. N. Milstein and M. B. Nevelson. MR 2894052

[23] Hüseyin Koçak and Kenneth J. Palmer, *Lyapunov exponents and sensitivity dependence*, J. Dynam. Differential Equations **22** (2010), no. 3, 381–398. MR 2719912 (2012f:37075)

[24] Carlo Laing and Gabriel J. Lord (eds.), *Stochastic methods in neuroscience*, Oxford University Press, Oxford, 2010. MR 2640514 (2010m:60006)

[25] André Longtin, *Neural coherence and stochastic resonance*, Stochastic methods in neuroscience, Oxford Univ. Press, Oxford, 2010, pp. 94–123. MR 2642697

[26] Xuerong Mao, *Stochastic stabilization and destabilization*, Systems Control Lett. **23** (1994), no. 4, 279–290. MR 1298174 (95h:93089)

[27] Maurizio Porfiri and Roberta Pigliacampo, *Master-slave global stochastic synchronization of chaotic oscillators*, SIAM J. Appl. Dyn. Syst. **7** (2008), no. 3, 825–842. MR 2443024 (2009h:93117)

[28] Yoshihiro Saito and Taketomo Mitsui, *Stability analysis of numerical schemes for stochastic differential equations*, SIAM J. Numer. Anal. **33** (1996), no. 6, 2254–2267. MR 1427462 (98c:65138)

[29] ———, *Mean-square stability of numerical schemes for stochastic differential systems*, Vietnam J. Math. **30** (2002), no. suppl., 551–560. MR 1964242 (2003m:65014)