M. V. MOUNTFORT
Mathematician,
Research and Development Laboratories,
Amalgamated Brick and Pipe Co. Ltd.,
Auckland, New Zealand

Moisture Transfer in Porous Elastic Solids

A theory is developed to cover the simultaneous moisture transfer and stress development in porous elastic solids. It is shown that this theory, small-strain, is analogous to that governing coupled heat transfer. As an example of the general solution method, the case of a porous elastic beam under simple load is examined.

Introduction

There are many practical situations which call for a small-strain theory of moisture movement and the consequent stress development in porous solids. Much work already exists, but little is immediately applicable to the commercial drying problems of such common materials as timber, clay, and so on. For example, Soil Mechanics deals with moisture transfer using a completely realistic moisture variable, but is so oriented that states of tensile developed or applied stress are virtually excluded, and this restriction makes the work inapplicable to situations where tensile stress development leading to fracture is a major consideration. Again, work has been done on small-strain theories covering general stress states, but quite often in terms of unrealistic moisture transfer variables, and these are severely restricted for this reason.

In what follows, we sketch out enough basic work to show that the moisture variable we must work with is prescribed just as rigidly as is temperature in thermal analysis, and go on to fit a small-strain theory covering general stress states around this. What finally emerges is almost completely analogous with the theory of coupled thermoelasticity. The example given, i.e., the analysis of the behavior of a moist porous elastic beam under simple applied load, illustrates a general method of solution for practical problems.

Moisture Potential

On a macroscopic scale, flow of liquid water is traditionally studied by the methods of hydrodynamics. These concepts cannot be applied directly to porous media for many reasons, most notably the following:

1 In a porous medium, knowledge of pore shape and size is sketchy, and a small volume of material will usually contain a range of different pores.

2 In small pores it is no longer obvious that water cannot support shearing forces, i.e., consideration of general stress states may be necessary.

3 Transfer within a porous medium may be in liquid form, vapor form or a nonspecific combination of both.

The word "moisture" is used to designate water in any phase, and it is obvious a more generalized variable than simple fluid pressure will be needed to describe moisture transfer adequately. The variable must be a unique, state variable that can be shown to govern the equilibrium of moisture regardless of phase; and where a situation is capable of description in either moisture or hydrodynamic terms the two variables must represent the same physical property. For instance, one might make a unidirectional porous body by cementing together a bundle of capillary tubes with all generators parallel. Hydrodynamics provides a solution to liquid flow in such a medium and any porous medium theory devised must agree with this.

The necessity for coping with multiphase transfer indicates a likely line of attack. For the vapor pressure p of water over a flat surface, we have the Clausius-Clapeyron equation.

$$dp/dT = \lambda/T(v_2 - v_1) \tag{1}$$

with T the absolute temperature, $v_2 - v_1$ the volume change associated with phase change, and λ the latent heat of vaporization. This prescribes the water vapor to liquid water equilibrium over flat water sheets, or more particularly over surfaces of liquid water which is itself stress-free. We wish to generalize this, and prove the following:

Theorem. The vapor pressure p of water vapor in equilibrium with water in a stressed state of which the hydrostatic component (first invariant) is σ is given by

$$\rho RT/M \cdot \log_e p/p_0 = -\sigma \tag{2}$$

where p_0 is the vapor pressure over stress-free liquid water at temperature T, ρ the density of liquid water at T, R the universal gas constant, and M the molecular weight of water.

To prove this, consider the following:

Take the stressed water as occupying a specific finite region, σ is taken as negative for a compressive state, positive for tensile.

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and the means by which the water is maintained in its stressed state is of no interest provided only that it is assumed no work is being done in so maintaining it. Let h be a height given by

$$\sigma = \rho g h \tag{3}$$

h = positive or negative.

With a tube of finite bore, connect the region of stressed water to a free water surface at a height h below it. Then since h is determined by (3), the two water regions are in equilibrium.

So enclose the whole system inside an impermeable wall that is isolated from its surroundings, but leaving means for vapor access from one water region to the other. Then the system as a whole must also maintain water vapor equilibrium, since, if not, water vapor would transfer from one region to the other causing changes in relative levels and a steady flow of water through the connecting tube. This flow could be made to do work, contrary to the Second Law of Thermodynamics.

The equation governing the equilibrium of the vertical vapor column is, with z measured upward

$$dp = -g\rho_{\text{vapor}}\cdot dz$$

that is,

$$dp = -Mg/RT\cdot p dz$$

which integrates to

$$RT/Mg \log_e p/p_0 = -h$$

or

$$\rho RT/M \log_e p/p_0 = -\sigma,$$

the required result.

This theorem will appear to give an occasional anomolous result: Notably regions of liquid water containing solutes will be alloted nonzero σ by virtue of their altered vapor pressures even when they appear stress-free from general considerations. Further, this nonzero σ will be significant in respect of moisture equilibrium. This apparently anomolous σ in a stress-free region should not be taken as a drawback, since a stress-free body is rarely achievable, and stress states essentially incremental.

We are now in a position to define a moisture equilibrium variable. We call it moisture potential and use the symbol p_E.

Definition. We define the moisture potential of a moist system as the first invariant of the stress tensor of liquid water which would be in equilibrium with the moist system under consideration.

The following properties hold as a consequence of the definition:

1 p_E is a scalar quantity measured in stress units, and as a tensor invariant is independent of coordinate systems.

2 Where the moist system under consideration is liquid water in the usual sense, p_E reduces to the normal water pressure, though with negative sign.

3 p_E is a state variable of moisture. It may also be used as a state variable of a moist porous medium. If some other variable, such as percentage moisture content, be used to categorize the condition of the medium, there will exist an equation of state between the two.

4 Equality of p_E for two media insures moisture equilibrium between them.

It is hardly practical to test the equilibrium of a moist medium against a stressed water system. However, equation (2) enables us to establish the values of p_E appropriate to air at various relative humidities, and so use constant humidity environments as standard moisture states [1].[1] It should be noted that although a medium in equilibrium with air, for which p_E is necessarily positive, must have a positive p_E itself, states of negative p_E are not excluded by the definition and will often exist.

Dependence of p_E on Stress State of Medium. Cases where moisture transfer is affected by applied medium stress are common, for instance, the squeezed bath sponge, and, since we are primarily concerned with moisture transfer and stress development, it is essential to deduce the form of dependence of p_E on the medium stress state.

Consider a small volume of medium saturated with liquid water, and subject it to some stress σ_{ij}. Then volume change will be proportional to the first invariant $\sigma_{kk}/3$, and will be opposed partly by medium skeleton and partly by water. From this consideration we postulate as a general assumption

$$(\sigma_{kk}/3)_{\text{water}} = b(\sigma_{kk}/3)_{\text{medium stress}} \text{ with } 0 \leq b \leq 1 \tag{4}$$

Obviously b will depend on such things as the individual bulk moduli of skeleton and water, and certainly on whether or not the medium is fully saturated, but it will be reasonable to treat b as a constant over small ranges of moisture condition. Then

$$p_E = p_c + b\sigma_{kk}/3 \tag{5}$$

where now σ_{ij} categories the stress due to applied loads and any stress developed due to moisture gradients, and p_c is the moisture potential of the medium/water as a function of all state variables other than stress [2].

p_c would include, for example, the apparently anomolous σ in the case of a stress-free solution, already mentioned. It must be emphasized that (4) is not the only possible relation, merely a simple satisfactory one. It does hold for some cases; for instance, for clays in a fully saturated condition under small applied loads $b = 1$; however (4) should not be thought of as necessarily a universal law.

Flow of Moisture and Stress Development. Assured that equality of moisture potential gives moisture equilibrium, it is a small step to assume that where differences of moisture potential exist, flow is proportional to these differences. It is therefore assumed that, within a moist medium,

$$\mathbf{q}_i = k\partial p_E/\partial x_i, \qquad k > 0 \tag{6}$$

where $\mathbf{q}_i$ is the flow vector, k a constant called the permeability, and it should be noticed that moisture moves toward regions of high moisture potential.

At a boundary between media

$$\mathbf{q}_i = \alpha(p_{E2} - p_{E1}), \qquad \alpha > 0 \tag{7}$$

Most cases of interest will concern transfer from a moist solid to air, and (5–7) then combine to give an amended boundary condition

$$\frac{1}{h}\frac{\partial}{\partial x}(p_c + b\cdot\sigma_{kk}/3) + (p_c + b\cdot\sigma_{kk}/3) = p_a, \qquad h = \alpha/k > 0 \tag{8}$$

To complete a description of flow, a continuity condition must be deduced. This condition expresses mathematically the fact that where moisture flows into any volume of medium there is a corresponding change in moisture condition. The condition is

$$\mathbf{q}_{i,i} = -\rho/(1 + \rho\theta)^2\cdot d\theta/dp_c\cdot\partial p_c/\partial t$$

where ρ is the specific gravity of the porous skeleton, θ the moisture content as a fraction by weight of the dry solid, and p_c has been assumed a unique function of θ. Using this condition in conjunction with (6), the equation governing flow within a medium becomes

$$(p_c + b\cdot\sigma_{kk}/3)_{,ii} = 1/\kappa'\partial p_c/\partial t \tag{9}$$

where

$$1/\kappa' = -\rho/k(1 + \rho\theta)^2\cdot d\theta/dp_c > 0$$

[1] Numbers in brackets designate References at end of paper.

κ' is taken as constant. Most problems will require a solution of (9) subject to boundary conditions of form (8).

It is immediately obvious that we will not be able to obtain a solution without knowledge of the medium stress state. For elastic bodies in the absence of body forces and with no other connection between moisture transfer and stress, it can be shown that $(\sigma_{kk}/3)_{,ii} = 0$, reference [3], and (9) then reduces to the usual heat transfer equation. When change in moisture state is accompanied by shrinkage or expansion the equations of stress development must be considered simultaneously with (8) and (9).

Consider now a shrinking elastic medium, and postulate an expansion coefficient $\alpha > 0$, so that free strain due to changé of moisture potential is given by $-\alpha(p_c - p_0)\delta_{ij}$; δ_{ij} is the Krönecker delta. Then the equations governing stress development are

Equilibrium Equations

$$\sigma_{ij,j} = 0$$

Stress-Strain Relations

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} + (3\lambda + 2\mu)\alpha(p_c - p_0)\delta_{ij} \qquad (10)$$

Strain-Displacement Relations

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

within the body, and boundary conditions on stress

$$\sigma_{ij}\eta_j = X_i \qquad (11)$$

Equations (8)–(11) make up a complete system determining stress and moisture conditions in porous elastic media. The remainder of this article is restricted to such media, that is, elastic-plastic or viscoelastic media are excluded from further consideration. There is no specific restriction on the range of the moisture potential variable other than may be imposed by variations in the quantities taken as constants. The complete analogy with coupled thermoelasticity [4] should be noted except that in the present case $b\cdot\sigma_{kk}/3$ is not negligible.

Reduction of Elastic Case. In the present case $\sigma_{kk}/3$ is no longer harmonic. However, a relation can be deduced from equations (10) between $\sigma_{kk}/3$ and p_c, and this relation used to eliminate $\sigma_{kk}/3$ from (9).

Using the stress-strain and strain-displacement relations in the equations of equilibrium,

$$\lambda\epsilon_{kk,i} + 2\mu\epsilon_{ji,j} + (3\lambda + 2\mu)\alpha p_{c,i} = 0$$

i.e.,

$$\lambda u_{k,ki} + \mu(u_{j,ij} + u_{i,jj}) + (3\lambda + 2\mu)\alpha p_{c,i} = 0$$

i.e.

$$(\lambda + \mu)u_{k,ki} + \mu u_{i,kk} + (3\lambda + 2\mu)\alpha p_{c,i} = 0$$

Differentiating with respect to x_i and adding

$$(\lambda + 2\mu)u_{k,kii} + (3\lambda + 2\mu)\alpha p_{c,ii} = 0$$

Also, from the stress-strain relations alone

$$\sigma_{kk} = (3\lambda + 2\mu)\epsilon_{kk} + 3(3\lambda + 2\mu)\alpha(p_c - p_0)$$

Eliminating ϵ_{kk} ($\equiv u_{k,k}$) from these last two equations

$$\sigma_{kk,ii} = (3\lambda + 2\mu)4\mu\alpha/(\lambda + 2\mu)\cdot p_{c,ii} = 2E\alpha/(1 - \nu)\cdot p_{c,ii}$$

If this is now used, (9) becomes

$$[1 + 2E\alpha b/3(1 - \nu)]\cdot p_{c,ii} = \frac{1}{\kappa'}\cdot\partial p_c/\partial t$$

or

$$p_{c,ii} = 1/\kappa\partial p_c/\partial t \qquad (12)$$

where $\kappa = \kappa'[1 + 2E\alpha b/3(1 - \nu)]$ will still be taken as a constant.

Note that this reduction does not uncouple the problem since the boundary conditions contain both p_c and $\sigma_{kk/3}$. However *this reduction will help toward a general method of solution* [5].

Practical problems tend to fall into two main categories (*a*) applied load under equilibrium boundary moisture conditions, and (*b*) moisture transfer under no load.

The linearity of the equations will permit more complex cases to be compounded from these two. As the method of solution is not substantially different, we confine ourselves to an example of the first case.

Moist Porous Body Under Load. Suppose a body and its surroundings be in equilibrium at a moisture potential p_a and uniform throughout. Let the body now be loaded by surface forces X_i which produce a stress system σ_{ij}^*. If subsequent to loading moisture moves within the body or to the surroundings *an additional set of stresses will develop as moisture gradients* appear. Call this system σ_{ij}', ϵ_{ij}', u_i' and the total system σ_{ij}, ϵ_{ij}, u_i so that $\sigma_{ij} = \sigma_{ij}^* + \sigma_{ij}'$, etc. Then the starred system satisfies

$$\sigma_{ij,j}^* = 0$$

$$\sigma_{ij}^* = \lambda\epsilon_{kk}^*\delta_{ij} + 2\mu\epsilon_{ij}^*$$

throughout and

$$\sigma_{ij}^*\eta_j = X_i$$

at the boundaries, and hence the dashed system must satisfy

$$\sigma_{ij,j}' = 0 \qquad (13a)$$

$$\sigma_{ij}' = \lambda\epsilon_{kk}'\delta_{ij} + 2\mu\epsilon_{ij}' + (3\lambda + 2\mu)\alpha p_c\delta_{ij} \qquad (13b)$$

throughout and

$$\sigma_{ij}'\eta_j = 0 \qquad (13c)$$

at the boundaries, where p_c is now reckoned relative to p_a as zero.

Further,

$$(p_c + b\cdot\sigma_{kk}^*/3 + b\cdot\sigma_{kk}'/3)_{,ii} = \frac{1}{\kappa'}\cdot\partial p_c/\partial t$$

becomes

$$(p_c + b\cdot\sigma_{kk}'/3)_{,ii} = \frac{1}{\kappa'}\cdot\partial p_c/\partial t$$

since $\sigma_{kk}^*/3$ is harmonic; that is

$$p_{c,ii} = \frac{1}{\kappa}\cdot\partial p_c/\partial t$$

The boundary conditions on moisture are

$$\frac{1}{h}\,\partial/\partial x(p_c + b\cdot\sigma_{kk}'/3) + p_c + b\cdot\sigma_{kk}'/3 = p_a$$

$$-\frac{b}{h}\,\partial/\partial x(\sigma_{kk}^*/3) - b\cdot\sigma_{kk}^*/3 \qquad (14)$$

We see that *the loading has had the effect of destroying the* equilibrium at the boundaries, and moisture will indeed transfer between medium and surroundings. The initial conditions for the dashed system are

$$p_c = 0, \qquad \sigma_{ij}' = 0 \qquad (15)$$

As an illustrative example we take the simple case of a porous elastic rectangular beam bent by couples, and with its sides and ends impermeable to moisture.

Beam Bent by Couples. With z the longitudinal axis and x vertically upward, the only nonzero stress component is $\sigma_z^* =$

$-Ex/R$. If the bar has depth $2l$ and the maximum tensile stress is M then the distribution due to applied load is

$$\sigma_z^* = -Mx/l \tag{16}$$

Then

$$\sigma_{kk}^*/3 = -Mx/3l$$

and the boundary conditions become

Upper Boundary

$$\frac{1}{h}\,\partial/\partial x(p_c + b\cdot\sigma_{kk}'/3) + p_c + b\cdot\sigma_{kk}'/3 = bM(1 + 1/L)/3$$

Lower Boundary

$$-\frac{1}{h}\,\partial/\partial x\,(p_c + b\cdot\sigma_{kk}'/3) + p_c + b\cdot\sigma_{kk}'/3 = -bM(1 + 1/L)/3 \tag{17}$$

$$L = lh$$

these to be solved in conjunction with (12), (13), and the initial conditions (15)

We proceed to a solution in the following manner. The solution of

$$p_{c,ii} = \frac{1}{\kappa}\,\partial p_c/\partial t \qquad p_c = 0 \text{ initially} \tag{18}$$

and

$$\frac{1}{h}\,\partial p_c/\partial x + p_c = \varphi_l(\lambda) \quad \text{at} \quad x = l$$

$$-\frac{1}{h}\,\partial p_c/\partial x + p_c = \varphi_{-l}(\lambda) \quad \text{at} \quad x = -l$$

is

$$p_c = \frac{1}{2}\,[\varphi_l(\lambda) + \varphi_{-l}(\lambda)] + x[\varphi_l(\lambda) - \varphi_{-l}(\lambda)]/2(L+1)l$$

$$- [\varphi_l(\lambda) + \varphi_{-l}(\lambda)]\cdot\sum_{n=1}^{\infty}\frac{L\cos x\alpha_n/l}{[L^2 + L + \alpha_n^2]\cos\alpha_n}\cdot e^{-\kappa\alpha_n^2 t/l^2}$$

$$- [\varphi_l(\lambda) - \varphi_{-l}(\lambda)]\cdot\sum_{n=1}^{\infty}\frac{L\sin x\beta_n/l}{[L^2 + L + \beta_n^2]\sin\beta_n}\cdot e^{-\kappa\beta_n^2 t/l^2}$$

where the

$$\alpha_n, \beta_n$$

are the positive roots of

$$\alpha\tan\alpha = L, \qquad \beta\cot\beta = -L$$

and we note that

$$\sum_{n=1}^{\infty} 2L\cos x\alpha_n/l/[L^2 + L + \alpha_n^2]\,\cos\alpha_n = 1$$

$$\sum_{n=1}^{\infty} 2(L+1)\sin x\beta_n/l/[L^2 + L + \beta_n^2]\,\sin\beta = x/l \tag{19}$$

for $-l < x < l$.

Applying the Duhamel theorem, another solution of (18) is

$$p_c = \int_0^t [\varphi_l(\lambda) + \varphi_{-l}(\lambda)]\cdot\sum_{n=1}^{\infty}\frac{L\cos x\alpha_n/l}{[L^2 + L + \alpha_n^2]\cos\alpha_n}\cdot\frac{\kappa\alpha_n^2}{l^2}\cdot e^{-\kappa\alpha_n^2(t-\lambda)/l^2}\,d\lambda + \int_0^t [\varphi_l(\lambda) - \varphi_{-l}(\lambda)]\cdot\sum_{n=1}^{\infty}\frac{L\sin x\beta_n/l}{[L^2 + L + \beta_n^2]\sin\beta_n}\cdot\frac{\kappa\beta_n^2}{l^2}\cdot e^{-\kappa\beta_n^2(t-\lambda)/l^2}\,d\lambda \tag{20}$$

and this solution satisfies

$$\frac{1}{h}\,\partial p_i/\partial x + p_c = \varphi_l(t) \quad \text{at} \quad x = l$$

$$-\frac{1}{h}\,\partial p_c/\partial x + p_c = \varphi_{-l}(t) \quad \text{at} \quad x = -l \tag{21}$$

With $\varphi_l(t)$, $\varphi_{-l}(t)$ as yet unknown, the stress distribution can be evaluated corresponding to this distribution of p_c. We have the solution in the notation of Boley and Weiner [6],

$$\sigma_z' = -[-E\alpha p_c + P_T/A + M_{Tx}y/I_x + M_{Ty}x/I_y]$$

Since we use principal centroidal axes, and since p_c is a function of x only, $M_{Tx} = 0$. The minus sign arises since strain is shrinkage with increasing p_c not expansion as in the thermal case. Also, if we take the width of the beam as $2a$,

$$A = 4al,\ I_y = 4al^3/3, \qquad P_T = 2aE\alpha\int_{-l}^{l} p_c dx,$$

$$M_{Ty} = 2aE\alpha\int_{-l}^{l} p_c x dx$$

and we get

$$\sigma_z' = -E\alpha\left[-p_c + \frac{1}{2l}\int_{-l}^{l} p_c dx + \frac{3x}{2l^3}\int_{-l}^{l} p_c x dx\right]$$

giving, since σ_z' is the only nonzero stress component

$$\sigma_{kk}'/3 = -\frac{E\alpha}{3}\left[-p_c + \frac{1}{2l}\int_{-l}^{l} p_c dx + \frac{3x}{2l^3}\int_{-l}^{l} p_c x dx\right] \tag{22}$$

We now want to use (20) and (22) in the boundary conditions (17) to get a pair of equations from which to find $\varphi_l(t)$ and $\varphi_{-l}(t)$.

Combining (20) and (22), and noting that

$$\frac{1}{2l}\int_{-l}^{l}\cos x\alpha_n/l/\cos\alpha_n dx = L/\alpha_n^2,$$

$$\frac{3x}{2l^3}\int_{-l}^{l} x\cos x\alpha_n/l/\cos\alpha_n dx = 0,$$

$$\frac{1}{2l}\int_{-l}^{l}\sin x\beta_n/l/\sin\beta_n dx = 0,$$

$$\frac{3x}{2l^3}\int_{-l}^{l} x\sin x\beta_n/l/\sin\beta_n dx = 3xh(L+1)/\beta_n^2 L$$

we get

$$b\cdot\sigma_{kk}'/3 = -\frac{E\alpha b}{3}\left[-p_c + \int_0^t [\varphi_l(\lambda) + \varphi_{-l}(\lambda)]\cdot\sum_{n=1}^{\infty}\frac{L^2}{\alpha_n^2[L^2 + L + \alpha_n^2]}\cdot\frac{\kappa\alpha_n^2}{l^2}\cdot e^{-\kappa\alpha_n^2(t-\lambda)/l^2}\,d\lambda + 3xh\int_0^t [\varphi_l(\lambda) - \varphi_{-l}(\lambda)]\cdot\sum_{n=1}^{\infty}\frac{(L+1)}{\beta_n^2[L^2 + L + \beta_n^2]}\cdot\frac{\kappa\beta_n^2}{l^2}\cdot e^{-\kappa\beta_n^2(t-\lambda)/l^2}\,d\lambda\right]$$

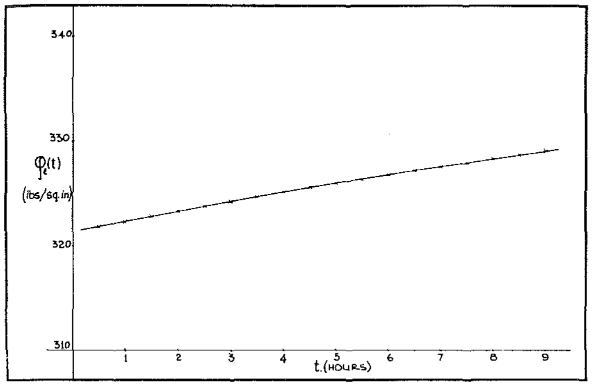

Fig. 1 Initial variation of $\varphi_l(t)$ with time

Using this in the desired conditions (17) and using the conditions (21), the equations in $\varphi_l(t)$, $\varphi_{-l}(t)$ are

$$(1 + E\alpha b/3)\varphi_l(t) - \frac{E\alpha b}{3}\int_0^t [\varphi_l(\lambda) + \varphi_{-l}(t)] \cdot \sum_{n=1}^{\infty} \frac{L^2}{\alpha_n^2[L^2 + L + \alpha_n^2]} \cdot \frac{\kappa\alpha_n^2}{l^2} \cdot e^{-\kappa\alpha_n^2(t-\lambda)/l^2}\, d\lambda - \frac{E\alpha b}{3}\int_0^t [\varphi_l(\lambda) - \varphi_{-l}(\lambda)] \cdot \sum_{n=1}^{\infty} \frac{3(L+1)^2}{\beta_n^2[L^2 + L + \beta_n^2]} \cdot \frac{\kappa\beta_n^2}{l^2} \cdot e^{-\kappa\beta_n^2(t-\lambda)/l^2}\, d\lambda = bM\left(1 + \frac{1}{L}\right)/3 \quad (23)$$

and

$$(1 + E\alpha b/3)\varphi_{-l}(t) - \frac{E\alpha b}{3}\int_0^t [\varphi_l(\lambda) + \varphi_{-l}(\lambda)] \cdot \sum_{n=1}^{\infty} \frac{L^2}{\alpha_n^2[L^2 + L + \alpha_n^2]} \cdot \frac{\kappa\alpha_n^2}{l^2} \cdot e^{-\kappa\alpha_n^2(t-\lambda)/l^2}\, d\lambda + \frac{E\alpha b}{3}\int_0^t [\varphi_l(\lambda) - \varphi_{-l}(\lambda)] \cdot \sum_{n=1}^{\infty} \frac{3(L+1)^2}{\beta_n^2[L^2 + L + \beta_n^2]} \cdot \frac{\kappa\beta_n^2}{l^2} \cdot e^{-\kappa\beta_n^2(t-\lambda)/l^2}\, d\lambda = -bM(1 + 1/L)/3 \quad (24)$$

Once equations (23) and (24) are solved for $\varphi_l(t)$, $\varphi_{-l}(t)$ these values are used in (20) to give the desired p_c. The only non zero stress σ_z' of the dashed system is then found as usual. In this particular case, a simplification of the pair of simultaneous integral equations is possible. An obvious solution for all t is

$$\varphi_l(t) = -\varphi_{-l}(t)$$

and the remaining equation becomes

$$(1 + E\alpha b/3)\varphi_l(t) - \frac{2E\alpha b}{3}\int_0^t \varphi_l(\lambda) \cdot \sum_{n=1}^{\infty} \frac{3(L+1)^2}{\beta_n^2[L^2 + L + \beta_n^2]} \cdot \frac{\kappa\beta_n^2}{l^2}\, e^{-\kappa\beta_n^2(t-\lambda)/l^2}\, d\lambda = bM(1 + 1/L)/3 \quad (25)$$

A particular case has been evaluated numerically, using the values $\kappa = 0.30$ in.^{2}/hr, $h = 0.05$ in.$^{-1}$, $b = 0.25$, $E\alpha = 1.5$, and $M = 1000$ lb/sq in., for a beam 12 in. deep. Figs. 1 and 2 give an indication of the initial behaviour of $\varphi_l(t)$, σ_z' and p_c.

For the final distributions, it will be simpler to proceed as follows. The condition for equilibrium is

$$\partial/\partial x(p_c + b\cdot\sigma_{kk}/3) = 0$$

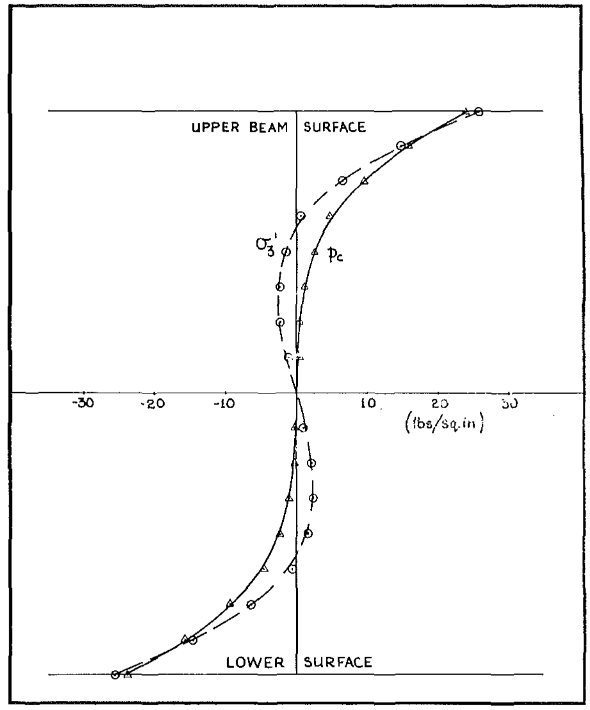

Fig. 2 Distribution of p_c and σ_z' across beam at t = 8 hr

everywhere, giving

$$p_c + b\cdot\sigma_{kk}'/3 = -b\cdot\sigma_{kk}^*/3 + \text{constant}$$

Since for this particular case p_c is an odd function of x, then the term $\frac{1}{2l}\int_{-l}^{l} p_c dx$ in equation (22) is zero and $\sigma_{kk}'/3$ is zero at $x = 0$. Also $\sigma_{kk}^*/3$ is zero at $x = 0$ and the constant of integration vanishes, that is,

$$p_c + b\cdot\sigma_{kk}'/3 = -b\cdot\sigma_{kk}^*/3 \quad (26)$$

This determines the final state of the composite variable $p_c + b\cdot\sigma_{kk}'/3$. To separate out the components we recast the stress-strain relations of equations (13) to obtain

$$\sigma_{ij,j}' = 0$$
$$\sigma_{ij}' = \lambda'\epsilon_{kk}'\delta_{ij} + 2\mu\epsilon_{ij}' + (3\lambda' + 2\mu)\alpha(p_c + b\cdot\sigma_{kk}'/3)\delta_{ij} \quad (27)$$
$$\sigma_{ij}'\eta_j = 0$$

where now

$$\lambda' = \lambda - b\alpha(3\lambda + 2\mu)^2/3[b\alpha(3\lambda + 2\mu) + 1]$$

and determine σ_z' as before. Since equations (27) are strictly analogous to the usual equations of thermoelasticity and since the distribution of $p_c + b\cdot\sigma_{kk}'/3$ is linear with respect to rectangular coordinates for this particular case, σ_{ij}' is everywhere zero and the final distributions are

$$p_c = Mx/3l, \ \sigma_z' = 0$$

Conclusion

The worked example has been presented to illustrate one case of the effects of stress-moisture coupling. For coupled transfer and these effects to exist, the key requirement is that the moisture transfer variable be stress-dependent. If coupling of this sort is ignored in the analysis of loaded porous media, the transient behavior that occupies a finite time as the medium comes to moisture equilibrium can be completely missed.

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References

1 G.B.D.S.I.R. Road Research Technical Paper No. 24, 1952; the suction of moisture held in soil and other porous materials.

2 We should compare (5) with the similar relation of Soil Mechanics. Croney, D., and Coleman, J. D., *Pore Pressure and Suction in Soils*, Conference of British National Society of Inter. Soc. of Soil Mech. and Foundation Engineers, Mar. 1960. They arrive at the relation $u = S + \alpha p$ which is identical in meaning with (5).

3 Love, A. E. H., *Mathematical Theory of Elasticity*, 4th ed., pp. 134 and 135.

4 Boley, B. A., and Weiner, J. H., *Theory of Thermal Stresses*, pp. 38–44.

5 Mountfort, M. V., "A Transformation in Coupled Thermoelasticity Problems," Journal of Applied Mechanics, Vol. 33, No. 1, Trans. ASME, Vol. 88, Series E, Mar. 1966, pp. 196–198.

6 Boley, B. A., and Weiner, J. H., *Theory of Thermal Stresses*, pp. 309 and 310.