Product Design Selection With Preference and Attribute Variability for an Implicit Value Function

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An important aspect of engineering product design selection is the inevitable presence of variability in the selection process. There are mainly two types of variability: variability in the preferences of the decision maker (DM) and variability in attribute levels of the design alternatives. We address both kinds of variability in this paper. We first present a method for selection with preference variability alone. Our method is interactive and iterative and assumes only that the preferences of the DM reflect an implicit value function that is differentiable, quasi-concave and non-decreasing with respect to attributes. The DM states his/her preferences with a range (due to the variability) for marginal rate of substitution (MRS) between attributes at a series of trial designs. The method uses the range of MRS preferences to eliminate "dominated designs" and then to find a set of "potentially optimal designs." We present a payload design selection example to demonstrate and verify our method. Finally, we extend our method for selection with preference variability to the case where the attribute levels of design alternatives also have variability. We assume that the variability in attribute levels can be quantified with a range of attribute levels. [DOI: 10.1115/1.2216728]

Keywords: multi-attribute decision making, preference variability, attribute variability, partial information

1 Introduction

Often in engineering design selection there is no one design alternative that is better in terms of all attributes, and the preferred designs are dependent on the preferences of the decision maker (DM). In addition, there is always uncontrollable variability, which is mainly of two types, that has to be accounted for [1–3]. The first type, preference variability, is caused due to the DM's lack of information on end users' needs or due to the DM's own inherent inability to state the preferences with certainty [4,5]. The second type, attribute variability, is caused due to uncontrollable engineering design parameters like material properties, manufacturing errors, or due to different usage situations of the product [1]. Both types of variability are inevitable in engineering design, as acknowledged widely [6,7]. If the variability is not accounted for, the preferred designs found might be erroneous.

Multi-attribute decision making (MADM) is a popular technique that is used for engineering design selection [1,8]. MADM methods for product design selection assume that the DM places a value on design alternatives in accordance with an unexpressed implicit value function of the attributes [9]. It is a convention to use the term utility function when there is variability in the attribute levels of alternatives and the term value function otherwise [9]. However, for simplicity (and to avoid confusion), we only use the term value function in this paper.

Many of the existing MADM methods make assumptions about the DM's value function to simplify the selection problem. The most common assumption is that the value function is additive with respect to the attributes [10–14]. Significant research has been reported in the MADM literature for selection with variability in preferences alone. Some researchers propose to assume different probability distributions for preferences and then study the effect of these distributions on the preferred designs [15,16]. Another way (popularly known as selection with partial information) for accounting preference variability in selection is to ask the DM to provide some constraints on the preferences [17–24]. Typical constraints could be some ranges on the preferences, like relative importance of attribute a_1 is between 0.3 and 0.4. The ranges on preferences are then used in finding the "non-dominated" and "potentially optimal designs" [18,23]. When there is variability in attributes of alternatives alone, many researchers have proposed to use lottery techniques and expectations to find the preferred designs [1,9,25]. Some research has also been reported for selection with variability in both the preferences and the attribute levels of alternatives. Methods for selection are proposed when the probability distributions governing the variability in attributes and some constraints on the preferences are known [26–28]. Recently, some work has been reported to find non-dominated and potentially optimal designs when the variability in attributes and preferences is expressed in the form of ranges [29–32].

However, to the best of our knowledge, existing methods for preference and/or attribute variability in the literature are applicable only when the DM's value function is presumed known (e.g., additive, multiplicative). It is well known that presuming a form for the value function is restrictive and applicable only to special cases [33–35]. In this paper, we first present a method for selection with preference variability alone when the DM's value function is implicit and unknown. We then extend this method further for selection with preference and attribute variability.

Our method for selection with preference variability is iterative and assumes only that the DM's value function is differentiable, quasi-concave and non-decreasing with respect to the attributes. In this method, we assume that the DM gives a range for the marginal rate of substitution (MRS) (refer to Sec. 2.3) between the attributes at a series of trial designs (each a particular design under consideration). Using the range of MRS preferences and gra-

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dient properties of the value function, we eliminate some design alternatives as dominated designs with respect to the trial designs. However, it is possible that some of the non-eliminated designs are not potentially optimal. So, we propose an approach that is based on some approximations to identify the potentially optimal designs from the set of non-eliminated designs. An advantage of the proposed method is that it does not need the probability distributions governing the MRS preferences, which are usually difficult to obtain.

In our method for selection with preference and attribute variability, we assume that the variability in the attribute levels can be quantified with a known range for each attribute of a design alternative. Using the ranges of attributes and the ranges of MRS preferences, we then eliminate some design alternatives as dominated designs.

The organization of this paper is as follows. In Sec. 2 we provide some definitions for the main terminology used in the paper. In Sec. 3 we present our method for selection with preference variability alone. Next, in Sec. 4 we discuss our method for selection with preference and attribute variability. Section 5 describes the application of our selection method with preference variability and our selection method with preference and attribute variability in a payload design selection problem. Finally, we conclude with a summary in Sec. 6.

2 Definitions

In this section we give definitions and some pertinent properties for the important terms used throughout the paper.

2.1 Selection Problem. The set of "n" discrete design alternatives from which the most preferred is to be selected is $\{D_1, \ldots, D_j, \ldots, D_n\}$. Each alternative D_j is represented by the set of attributes $[a_{1j}, \ldots, a_{mj}]$ in the m-dimensional design attribute space. Let the value function, $V(D_j)$ be an implicit value function of attributes $[a_{1j}, \ldots, a_{mj}]$ that represents the DM's preferences. Here, we assume that there is a single DM. Note that, for the application of the method developed in this paper, it is not important how the design alternatives for selection are generated.

When there is no variability in attributes, a_{ij} would be exact (deterministic or fixed). However, when there is variability in the attributes, we assume that the ranges of attribute levels for each design alternative are known. We use the symbol A_{ij}^L to represent the lower bound, A_{ij}^U to represent the upper bound, and A_{ij} to represent the range $[A_{ij}^L, A_{ij}^U]$ of the i^{th} attribute of design D_j. We use the symbol a_{ij} to represent a variable attribute level that belongs to the range A_{ij}. (Note that a_{ij} could be fixed or variable depending on whether or not the i^{th} attribute level of design D_j is deterministic.)

2.2 Quasi-Concave Function. A function V defined on a non-empty convex domain is said to be quasi-concave [36,37] if

$$V[\theta X_1 + (1-\theta)X_2] \geqslant \min[V(X_1), V(X_2)] \tag{1}$$

for all X_1, X_2 that belong to the domain of V and $\theta \in [0,1]$.

2.3 Marginal Rate of Substitution (MRS). At trial design D_T, let Δa_j be the amount DM will compromise in attribute a_j in order to gain an amount Δa_i in attribute a_i while maintaining constant value (i.e., remain indifferent [9] with respect to D_T). The MRS, s_{ijT}, between attributes a_i and a_j at D_T is the ratio $-\Delta a_j / \Delta a_i$. Note that the location of the design alternative in the design attribute space can influence the DM's MRS [9]. If the MRS preferences are consistent, it can be shown that $s_{ijT} = w_{iT}/w_{jT}$, where w_{iT} and w_{jT} are the gradient coefficients of the value function V at D_T with respect to attributes a_i and a_j, (i.e., $\partial V/\partial a_i$ and $\partial V/\partial a_j$), respectively [13,35].

When there is no variability in preferences, both Δa_j and s_{ijT} would be exact (or deterministic). However, if there is variability in preferences, the DM would give a range for Δa_j (for a fixed Δa_i) thus leading to a range of MRS. We use the symbol S_{ijT}^L to represent the lower bound, the symbol S_{ijT}^U to represent the upper bound, and the symbol S_{ijT} to represent the range $[S_{ijT}^L, S_{ijT}^U]$ of MRS preferences between two attributes a_i and a_j, at trial design D_T. We use the symbol s_{ijT} to represent a variable MRS that belongs to the range S_{ijT}. (Note that s_{ijT} could be fixed or variable depending on whether or not MRS is deterministic.)

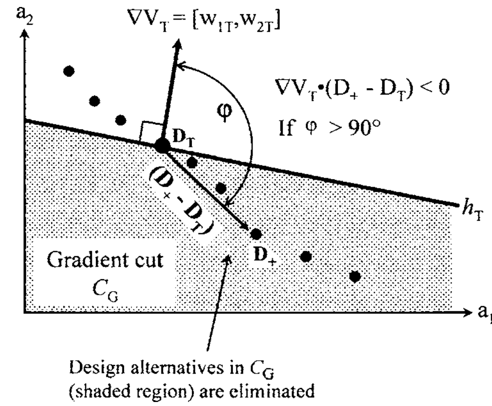

Fig. 1 Illustration of gradient cut

2.4 Gradient Cut. The gradient cut [33] is the half space C_G bounded by the normal to the gradient of a value function V at a point D_T, ∇V_T, with the gradient pointing in the outward direction from C_G; see Fig. 1. C_G does not include the boundary line h_T in Fig. 1.

If the value function, V, is differentiable and quasi-concave, then it can be shown that for all $D \in C_G$, $V(D) < V(D_T)$ [33,35,37]. That is, *any design alternative in* C_G has lower value than D_T, and can be eliminated. If the gradient of V at D_T is ∇V_T:$[w_{1T}, \ldots, w_{mT}]$, then a design D_+:$[a_{1+}, \ldots, a_{m+}]$ is in C_G of D_T:$[a_{1T}, \ldots, a_{mT}]$ if [37]

$$\sum_{i=1}^{m} w_{iT} \cdot (a_{i+} - a_{iT}) < 0 \tag{2}$$

(Here $[w_{1T}, \ldots, w_{mT}]$, $[a_{1+}, \ldots, a_{m+}]$, $[a_{1T}, \ldots, a_{mT}]$ are deterministic). Note that design alternatives that are not in C_G might have either higher value or lower value than D_T [37]. The gradient of the value function at D_T can be obtained from the deterministic MRS preferences by solving an optimization problem. The details of the formulation for that optimization problem are not presented here and the interested reader can refer to the literature [35].

2.5 Dominated Design. When there is no variability in the attribute levels, but there is variability in MRS preferences, a design D_+ is said to be dominated by another design D_T, if D_+ has lower value than D_T (i.e., $V(D_+) < V(D_T)$) for the whole range of MRS preferences, S_{ijT}, at D_T. If the attribute levels also have variability, then D_+ is said to be dominated by D_T if $V(D_+) < V(D_T)$ for the whole range of S_{ijT} and the whole range of attribute levels A_{i+} and A_{iT} (where $i=1$ to m).

2.6 Potentially Optimal Design. When there is no variability in the attribute levels, but there is variability in MRS preferences, a design D_+ is said to be potentially optimal if D_+ has the highest value among all design alternatives for some subset of S_{ijT}. If the attribute levels also have variability, then D_+ is potentially optimal if D_+ is the highest valued design alternative for some subsets of S_{ijT}, A_{i+} and A_{id} (where $d=1$ to n).

Note that, from the above definition, a design which is not potentially optimal cannot be most preferred for any realization of MRS that belongs to the range of preferences and/or for any re-

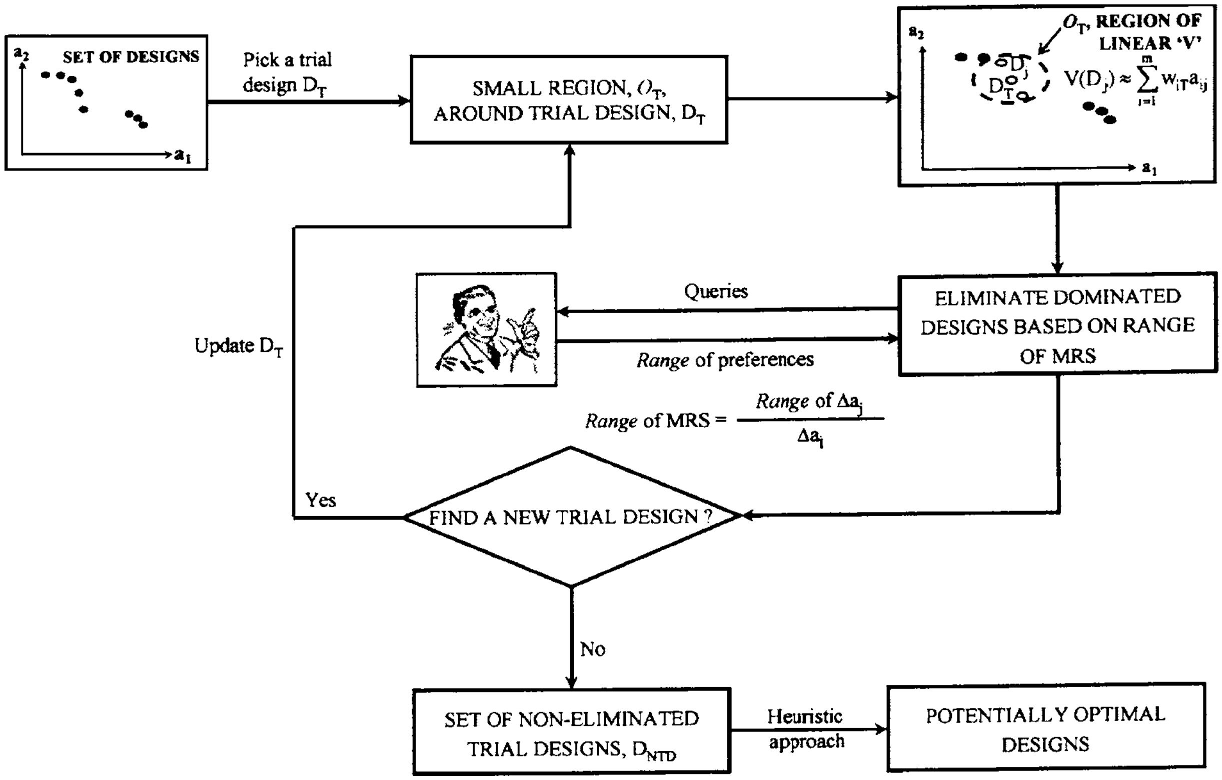

Fig. 2 Flow chart of our method for selection with preference variability

alization of attribute levels that belongs to the range of attributes. Eum et al. [30] gave similar definitions for dominated design and potentially optimal design when the value function is assumed to be additive. Here we have extended their definitions for the more general case of an implicit value function.

3 Selection with Preference Variability

Figure 2 shows the flow chart of our method for selection with preference variability. Since we assume the DM's value function to be non-decreasing with respect to attributes, for selection, it is enough to consider only those designs that are Pareto optimal from the original set of design alternatives [33].

In this method (see Fig. 2), we start by picking an initial trial design, D_T, from the set of design alternatives. We choose trial design D_T either as an alternative that would have maximum value if the value function were linear with equal importance to the attributes, or as a random pick. In a small region O_T around D_T we then approximate the value function to be linear with respect to the attributes. The gradient of V at D_T is $\nabla V_T = [w_{1T}, \ldots, w_{mT}]$. The general form for a linear approximation of $V(D_j)$ in O_T would be (considering only the differences between V for design alternatives near D_T):

$$V(D_j) = \sum_{i=1}^{m} w_{iT} \cdot a_{ij} \tag{3}$$

Note that our method does not do a "piecewise linear approximation" of the value function at a trial design. The linear approximation is only used to obtain the gradient of the value function at a trial design, which is then used to eliminate dominated designs with respect to the trial design.

To obtain the gradient coefficients, w_{iT}, we query the DM for the MRS preferences at the trial design. We ask the DM to provide the MRS between attributes a_i and a_{i+1} $(i=1, \ldots, 'm-1)$ and the MRS between attribute a_m and attribute a_1 (if $m>2$). As mentioned earlier, when there is variability in preferences, the DM gives a range for MRS preferences, S_{ijT}. For example, in the selection of a cordless power tool, the DM might say: "I would give up 40–50 operations per battery to reduce the weight by 0.1 pounds."

When the DM gives a range for MRS preferences, the gradient coefficients (which are a function of MRS preferences [9,35]), w_{iT}, will also have a range. Because of this, the gradient cut shown in Fig. 1 is not applicable for eliminating dominated designs. So, we use a modified version of gradient cut for eliminating dominated designs based on the range of MRS preferences (see Sec. 3.1 for details).

Next, we try to find a new trial design from the non-eliminated design alternatives. If a new trial design is found, we repeat the above steps (recall Fig. 2), referred to as an "iteration" from here on. Otherwise, we stop the process and collect the non-eliminated trial designs in a set, D_{NTD}. All the designs that are not in D_{NTD} are dominated by at least one design in the original set of design alternatives as per our definition in Sec. 2.5.

However, it is possible that the elements of the set D_{NTD} are not all potentially optimal (i.e., they might be dominated by some designs belonging to D_{NTD}, see Sec. 3.1 for a detailed explanation). So, we present a heuristic approach to test whether or not the elements of D_{NTD} are potentially optimal. This heuristic approach is based on a linear approximation and the gradient properties of the quasi-concave value function (see Sec. 3.2 for details).

In the next two sections, we discuss the individual components of our method for selection with preference variability. In Sec. 3.1, we present our approach for eliminating dominated designs based on a range of MRS preferences. In Sec. 3.2, we present the heuristic approach to find potentially optimal designs from the set of non-eliminated trial designs D_{NTD}. Our approach for finding a new trial design is not presented here due to space restrictions and the interested reader can refer to [35].

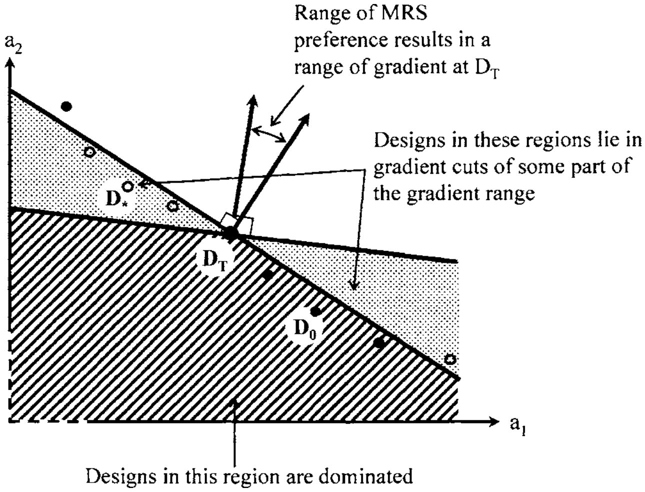

Fig. 3 Illustration of approach for eliminating dominated designs based on range of MRS preferences

3.1 Eliminating Dominated Designs Based on a Range of MRS Preferences. As mentioned earlier, when the DM gives a range of MRS at a trial design D_T, the corresponding gradient coefficients at D_T, w_{iT}, also have a range. Due to this, some designs might lie only in the gradient cuts of some part of the range and not in the gradient cuts of some other part of the range, e.g., D_* in Fig. 3. I.e., D_* is guaranteed to have a lower value than D_T only for some part of the range of MRS preferences. So, we adopt a conservative approach and eliminate, as dominated designs, those designs that lie in all possible gradient cuts for the entire range of w_{iT} (e.g., D_0 in Fig. 3).

Note that, the ranges of gradient coefficients at D_T, w_{iT}, in fact result in a hyper-cone of gradients with D_T as the apex. In two attribute space, the gradients corresponding to the extremes of MRS range (i.e., S_{ijT}^U and S_{ijT}^L) define this hyper-cone. Hence, in a two attribute space, the hyper-cone can be viewed as a range of gradient and a design is said to be dominated by D_T if and only if Eq. (2) is satisfied for both extremes of the range of the gradient (recall Fig. 3). But, when the number of attributes is more than two, there is no easy (general) way to define the hyper-cone using the gradients corresponding to the extremes of MRS range and a simple check of Eq. (2) is not enough to determine whether or not a design is dominated. (For simplicity, we continue to use the term range of gradient in the rest of the paper for referring to the hyper-cone of gradients.)

Below, we present a formulation that uses the range of MRS preferences, S_{ijT}, directly (i.e., without mapping them to a range of gradient coefficients) for checking whether or not a design $D_+:[a_{1+},\ldots,a_{m+}]$ is dominated by $D_T:[a_{1T},\ldots,a_{mT}]$. This linear programming (LP) problem is simple to solve by any LP solver (e.g., "linprog" from the MATLAB® optimization toolbox). In this formulation, w_{iT} are the variables and $[a_{1+},\ldots,a_{m+}]$, $[a_{1T},\ldots,a_{mT}]$ are deterministic.

$$\text{Maximize } Z^* = \sum_{i=1}^{m} w_{iT}\cdot(a_{i+}-a_{iT}) \tag{4a}$$

$$\text{subject to:} \sum_{i=1}^{m} w_{iT}=1; \quad w_{iT}\geq 0 \tag{4b}$$

$$S_{ijT}^L \leq \frac{w_{iT}}{w_{jT}} \leq S_{ijT}^U; \text{"}m-1\text{" such constraints} \tag{4c}$$

The objective function Z^* in the above formulation, Eq. (4a), is used to check whether or not D_+ is dominated by D_T (recall Fig. 3). If there exists a vector $\nabla V_T:[w_{1T},\ldots,w_{mT}]$ from the possible range of gradient at D_T for which D_+ does not lie in the corresponding gradient cut, then the value of Z^* in Eq. (4a) will be non-negative (recall Eq. (2)) otherwise Z^* will be negative. So, if the maximum value of Z^* is negative then we can conclude that D_+ lies in the gradient cuts of all the gradients for the given range of MRS preferences at D_T. Hence D_+ is dominated by D_T.

Equation (4b) is a normalization constraint on the gradient coefficients, w_{iT}. We impose the constraint that the gradient coefficients, w_{iT}, are non-negative because we assume that the value function is non-decreasing with respect to the attributes.

Equation (4c) imposes the constraint that the MRS preferences $s_{ijT}=w_{iT}/w_{jT}$ should belong to the range of MRS $S_{ijT}:[S_{ijT}^L,S_{ijT}^U]$ given by the DM at D_T. Note that the condition $s_{ijT}=w_{iT}/w_{jT}$ holds when the MRS preferences are assumed to be exact and consistent, i.e., Eq. (5) is satisfied [13].

$$s_{ijT}=\frac{w_{iT}}{w_{jT}} \quad \text{and} \quad s_{ijT}\cdot s_{jkT}=\frac{w_{iT}}{w_{kT}} \tag{5}$$

Since $m-1$ MRS values are independent when they are consistent, we use only $m-1$ constraints for the bounds on $s_{ijT}=w_{iT}/w_{jT}$ (recall Eq. (4c)). However, if one feels that the exactness and consistency assumption is not appropriate, then Eq. (4) can be easily modified by adding two more constraints as given by Eq. (6).

$$\sum_{i,j}\left[s_{ijT}-\frac{w_{iT}}{w_{jT}}\right]^2 \leq \varepsilon, \quad \text{where } \varepsilon \text{ is arbitrarily small} \tag{6a}$$

$$\sum_{i,j,k}\left[s_{ijT}\cdot s_{jkT}-\frac{w_{iT}}{w_{kT}}\right]^2 \leq \varepsilon \tag{6b}$$

Equation (6a) would be used to check how close the s_{i5j}^T's are to the w_{iT}'s and Eq. (6b) would be used to check that s_{ijT} are consistent. However, note that adding the constraints in Eq. (6) (which are non-linear and non-convex) to the formulation in Eq. (4) would increase the computational burden in eliminating the dominated designs. Also we obtain only m MRS preferences (instead of the possible $m\cdot(m-1)/2$) in our method to reduce the question burden on the DM. Note, however, that Eqs. (4) and (6) can be easily modified if the DM provides more than m MRS preferences.

Note that Eq. (4) should be applied to each design D_+ (that belongs to the original set of design alternatives) to check whether or not that design is dominated by D_T. Based on the definition of Sec. 2.5, for a design D_+, if Z^* in Eq. (4) is negative then it is guaranteed that D_+ is dominated by the trial design D_T. However, it is possible that D_+ might be dominated by D_T even if Z^* is positive. This is because of the property of the quasi-concave value function. Recall that design alternatives not in the gradient cut C_G, i.e., above the hyper-plane, h_T, (recall Fig. 1) might have either higher or lower value than D_T, i.e., gradient cut does not eliminate all designs that have lower value than D_T. Added to that, for eliminating dominated designs when the MRS preferences have a range, we use a worst case (i.e., conservative) approach and eliminate only those designs that are in all possible gradient cuts (recall Fig. 3).

Because Eq. (4) cannot guarantee that all dominated designs with respect to a trial design are eliminated, it is possible that some designs in the set of non-eliminated designs D_{NTD} are dominated. We present, in the next section, a heuristic to identify dominated designs from D_{NTD}, and hence find the set of potentially optimal designs.

3.2 Heuristic Approach to Find Potentially Optimal Designs. Figure 4 illustrates our heuristic approach in a two attribute space. Let D_{T1} and D_{T2} be two non-eliminated trial designs (i.e., they belong to D_{NTD}). Let $S_{ijT1}:[S_{ijT1}^L,S_{ijT1}^U]$ and $S_{ijT2}:[S_{ijT2}^L,S_{ijT2}^U]$ be the range of MRS preferences between at-

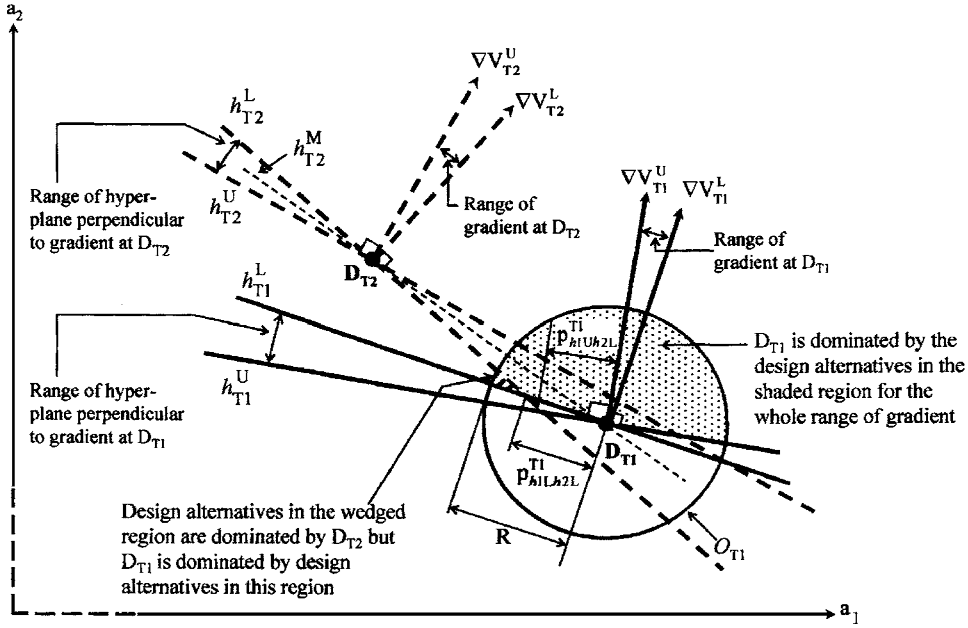

Fig. 4 Illustration of heuristic approach for eliminating dominated designs

tributes a_i and a_j at D_{T1} and D_{T2}, respectively. Recall that for a two attribute space the range of gradient corresponding to the range of the MRS preferences at the trial designs can be represented using the extremes of gradient. Let ∇V^L_{T1} and ∇V^U_{T1} be the extremes of the range of gradient at D_{T1} and ∇V^L_{T2} and ∇V^U_{T2} be the extremes of the range of gradient at D_{T2}.

Lines h^L_{T1} and h^U_{T1} pass through D_{T1} and are perpendicular to the extremes of the range of gradient, ∇V^L_{T1} and ∇V^U_{T1}, respectively. Lines h^L_{T2} and h^U_{T2} pass through D_{T2} and are perpendicular to the extremes of the range of gradient, ∇V^L_{T2} and ∇V^U_{T2}, respectively. O_{T1} is the region around D_{T1} in which we approximate the value function to be linear (recall Fig. 2), i.e., at every point inside O_{T1}, the range of MRS preferences is the same as the range of MRS preferences given by the DM at D_{T1} [35,37]. Note that neither of the two trial designs D_{T1}or D_{T2} is dominated by the other (recall Fig. 3).

As shown in Fig. 4, all points in the shaded region of O_{T1} have a higher value than D_{T1} for the entire range of gradient at D_{T1} (i.e., those points dominate D_{T1}). For the case shown in Fig. 4, all the lines that lie between the extremes h^L_{T2} and h^U_{T2} at D_{T2} pass through the shaded region of O_{T1}. Hence D_{T2} dominates some points (recall Fig. 3) in the shaded region of O_{T1} that have higher value than D_{T1}. Hence D_{T2} dominates D_{T1}.

Note that in Fig. 4, D_{T1} lies in the gradient cuts of the gradients perpendicular to the lines in the range h^M_{T2} and h^U_{T2}. So it is enough to check that D_{T2} dominates D_{T1} for the other part of the range, i.e., h^L_{T2} and h^M_{T2}. Let h_{T1} be a line that lies between h^L_{T1} and h^U_{T1} at D_{T1}. Let h_{T2} be a line that lies between h^L_{T2} and h^M_{T2} at D_{T2}. Also, let p^{T1}_{h1h2} be the perpendicular distance from $D_{T1}$to the intersection of h_{T1} and h_{T2}. If the maximum p^{T1}_{h1h2} is less than R (i.e., radius of O_{T1}, typical value of R is 0.1) then all lines between h^L_{T2} and h^M_{T2} pass through the shaded region of O_{T1} and hence D_{T2} dominates D_{T1}. Equation (7) states the condition mathematically.

$$\left\{ \underset{h_{T1}\in[h^L_{T1},h^U_{T1}],h_{T2}\in[h^L_{T2},h^M_{T2}]}{\text{maximum}} \quad p^{T1}_{h1h2} \right\} \leq R \tag{7}$$

It can be seen that for the case shown in Fig. 4, the maximum distance from D_{T1} to the intersection of h_{T1} and h_{T2} (i.e., maximum p^{T1}_{h1h2}) corresponds to p^{T1}_{h1Lh2L} (i.e., intersection of the lines h^L_{T1} and h^L_{T2}). Also p^{T1}_{h1Lh2L} is less than R, the radius of O_{T1}. So we can say that D_{T2} dominates D_{T1} for the case shown in Fig. 4.

The case shown in Fig. 4 is simple in that any line h_{T1} that lies between h^L_{T1} and h^U_{T1} at D_{T1} is not parallel to any line h_{T2} that lies between h^L_{T2} and h^U_{T2} at D_{T2}. But, this might not hold for some cases in the given range of preferences $S_{ijT1}:[S^L_{ijT1},S^U_{ijT1}]$ and $S_{ijT2}:[S^L_{ijT2},S^U_{ijT2}]$ at D_{T1} and D_{T2}, respectively, resulting in the maximum p^{T1}_{h1h2} to be infinity.

However, for the case where h_{T1} is parallel to h_{T2}, it is implied that ∇V_{T1} is equal to ∇V_{T2}, where ∇V_{T1} and ∇V_{T2} are the gradients perpendicular to h_{T1} and h_{T2} at D_{T1} and D_{T2}, respectively. When ∇V_{T1} is equal to ∇V_{T2}, we can find the value of the designs directly by using Eq. (3) based on a linear approximation of value function. In such a case, Eq. (8) can be used to check that $D_{T2}:[a_{1T2},\ldots,a_{mT2}]$ dominates $D_{T1}:[a_{1T1},\ldots,a_{mT1}]$ (here $[a_{1T2},\ldots,a_{mT2}]$,$[a_{1T1},\ldots,a_{mT1}]$ are deterministic).

$$\left\{ \underset{\nabla V_{T2}=\nabla_{T1}:[w_{1T1},\ldots,w_{mT1}]}{\text{maximum}} \sum_{i=1}^{m} w_{iT1}\cdot(a_{iT1}-a_{iT2}) \right\} \leq 0 \tag{8}$$

In our heuristic approach, to mathematically check that a trial design D_{T2} dominates another trial design D_{T1}, we need to conduct two tests. The first test, Eq. (7), is for the case in which any line h_{T1} at D_{T1} is not parallel to any line h_{T2} at D_{T2}. The second test, Eq. (8), is for the case in which some of the lines at D_{T1}, i.e., h_{T1}'s, are parallel to some of the lines at D_{T2}, i.e., h_{T2}'s. However, visualizing the range of gradients at D_{T1} and D_{T2} as shown in Fig. 4 is possible in two attribute space but unrealistic when the number of attributes is more than two. So, we developed mathematical formulations, that find the variables (left hand side of the inequality in Eq. (7) and (8)) needed for the two tests directly without mapping them to a range of gradient coefficients. These formulations were originally published in [38] and are reproduced in Appendix A (Eq. (7)) and Appendix B (Eq. (8)) for completeness.

If there are more than two non-eliminated trial designs in the set D_{NTD}, we apply the heuristic approach between the trial designs pairwise. Also note that the maximum of p^{T1}_{h1h2} (i.e., perpendicular distance from D_{T1} to the intersection of h_{T1} and h_{T2}) might not be the same as the maximum of p^{T2}_{h1h2} (i.e., perpendicular distance from D_{T2} to the intersection of h_{T1} and h_{T2}). If it so happens that the maximum values of both p^{T1}_{h1h2} and p^{T2}_{h1h2} are less

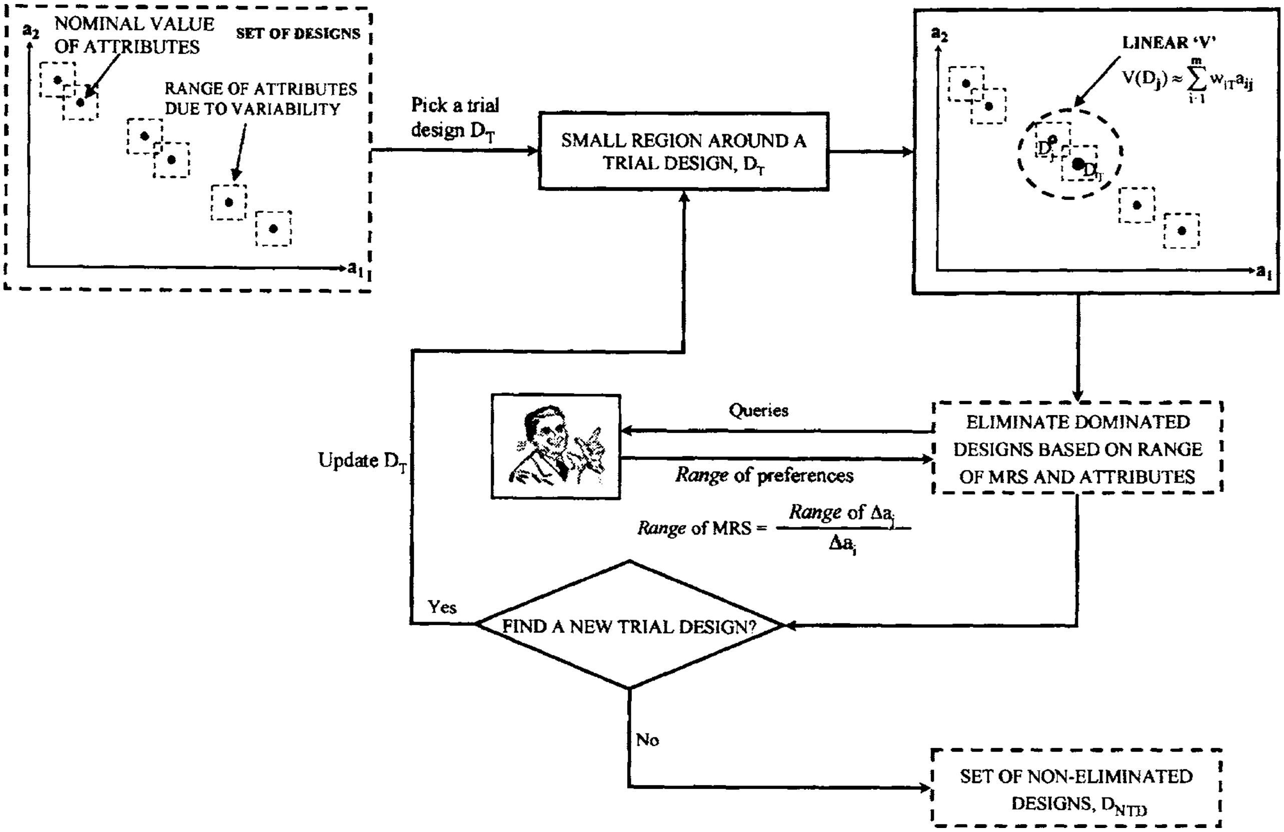

Fig. 5 Flow chart of our method for selection with preference and attribute variability

than the given radius R of O_{Ti}, it means that R is too large for the linear approximation to be valid. The designs that are not eliminated after the application of heuristic approach will be denoted as the potentially optimal designs. Note, however, it is possible that some dominated designs are not eliminated even after applying our heuristic approach.

In the next section we extend our selection method with preference variability to the case when there is variability in the attribute levels also.

4 Selection With Preference and Attribute Variability

Figure 5 shows the flow chart of our method for selection with preference and attribute variability. The individual components of the method shown in Fig. 5 are similar to that of the method shown in Fig. 2 except for the dashed boxes.

In our selection method with preference and attribute variability, we assume that the ranges of the attributes (shown by dotted rectangles in Fig. 5) quantifying the variability in the attribute levels of the design alternatives are known. The black dot in the middle of the dashed rectangle we call the nominal attribute levels of the design alternatives. By nominal attribute levels we mean the attribute levels that would occur when there is no variability in the attributes.

With the range of MRS preferences (obtained by querying the DM at the trial design) and the range of the attribute levels of design alternatives, we use a modified version of gradient cut for eliminating some of the dominated designs with respect to the trial design (see Sec. 4.1 for details). Next, we try to find a new trial design from the non-eliminated alternatives. If a new trial design is found, we repeat the above steps (see Fig. 5). Otherwise, we stop the process and collect the non-eliminated trial designs in a set, D_{NTD}. Ideally the set D_{NTD} should consist only of the potentially optimal designs. But due to the properties of quasi-concave function (explained in Sec. 3.1), it is possible that some dominated designs belong to D_{NTD}. In Sec. 4.1, we present our approach for eliminating dominated designs based on the range of MRS preferences and the range of attribute levels of design alternatives.

4.1 Eliminating Dominated Designs Based on the Range of MRS Preferences and Attribute Levels. Figure 6 illustrates, in two attribute spaces, our approach for eliminating dominated designs based on the range of MRS preferences and the range of

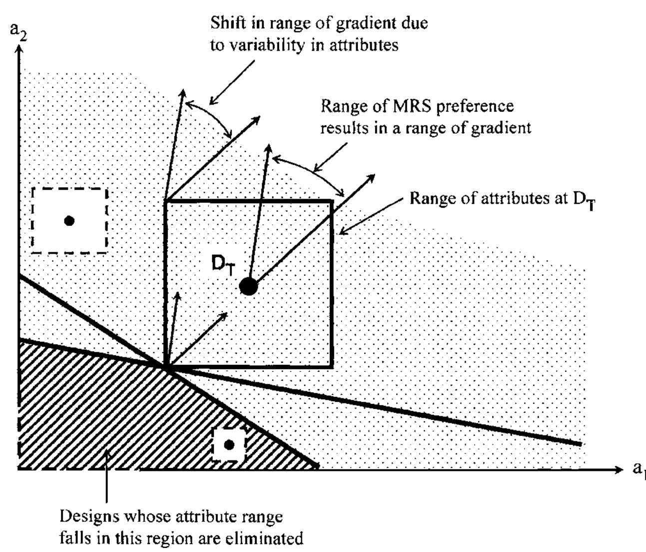

Fig. 6 Illustration of approach for eliminating dominated designs based on the range of MRS preferences and the range of attribute levels of design alternatives

attribute levels. Let D_T be the current trial design with the solid rectangle as the range of attributes and the black dot in the middle as the nominal attribute levels. Because the DM gives a range of MRS (due to variability) at D_T, the corresponding gradient coefficients at D_T, w_{iT}, also have a range as shown in Fig. 6. Note that it is assumed that the DM gives the range of MRS preferences at D_T keeping in mind the range of attribute levels of D_T. In other words, the given range of MRS preferences should be applicable for any attribute levels belonging to the range of attributes at D_T.

Because of the variability in the MRS preferences and the attribute levels, a number of gradient cuts are possible at D_T, the union of which is shown by the dotted region in Fig. 6. The shaded area in Fig. 6 is the intersection of all the gradient cuts possible at D_T. We eliminate those designs whose range of attribute levels lies completely inside the shaded area of Fig. 6 as the dominated designs. However, visualizing the range of gradient corresponding to the range of MRS preferences as shown in Fig. 6 is trivial in two dimensions but is difficult for higher dimensions. So, we present a mathematical formulation in Eq. (9) for checking whether or not a design D_+ is dominated by a trial design D_T. In this formulation, w_{iT}, a_{i+} and a_{iT} are the variables.

$$\text{Maximize} \quad Z^* = \sum_{i=1}^{m} w_{iT} \cdot (a_{i+} - a_{iT}) \tag{9a}$$

$$\text{subject to:} \quad \sum_{i=1}^{m} w_{iT} = 1; w_{iT} \geqslant 0 \tag{9b}$$

$$S_{ijT}^L \leqslant \frac{w_{iT}}{w_{jT}} \leqslant S_{ijT}^U; \quad m-1 \text{ such constraints} \tag{9c}$$

$$A_{i+}^L \leqslant a_{i+} \leqslant A_{i+}^U; \quad m \text{ such constraints} \tag{9d}$$

$$A_{iT}^L \leqslant a_{iT} \leqslant A_{iT}^U; \quad m \text{ such constraints} \tag{9e}$$

The formulation in Eq. (9) is similar to the formulation in Eq. (4) except that two new set of constraints are added to account for the variability in attribute levels. Equation (9*d*) is to check that the variable attribute levels of D_+, a_{i+}, belong to the range of attribute levels at D_+. Equation (9*e*) imposes a similar constraint on the variable attribute levels of D_T, a_{iT}.

If there exists a vector $\nabla V_T{:}[w_{iT}, \ldots, w_{mT}]$ in range of gradient at D_T, and vectors $[a_{1+}, \ldots, a_{m+}]$ and $[a_{1T}, \ldots, a_{mT}]$ in the ranges of attribute levels at D_+ and D_T respectively, for which D_+ does not lie in the corresponding gradient cut, then the value of Z^* in Eq. (9*a*) will be non-negative (recall Eq. (2)) otherwise Z^* will be negative. So, if the maximum value of Z^* is negative, then we can conclude that D_+ lies in the gradient cuts of all the gradients at D_T. Hence D_+ is dominated by D_T.

The formulation in Eq. (9) has a non-linear objective function with linear constraints and is simple to solve by any existing commercial software (e.g., the MATLAB® optimization toolbox). Note that in Eq. (9), we assumed the MRS preferences s_{ijT} are exact and consistent (recall Eq. (5)). However, if one feels such an assumption is not appropriate, then additional constraints similar to Eq. (6) can be added to Eq. (9). Also Eq. (9) can be easily modified with additional constraints if more than m MRS preferences are given by the DM.

In the next section, we demonstrate the application of our selection method with preference variability and selection method with preference and attribute variability in the selection of a payload design.

5 Payload Design Selection Example

This example is for a two-attribute problem and involves the selection of a payload design for an undersea autonomous vehicle. Typically, the payload must be effective in several different uses, called "scenarios." Effectiveness in a scenario is measured by a probability of success P_S of the payload in that scenario. The design goal is to simultaneously maximize individual P_S's for all scenarios. The payload design is constrained by upper limits on the weight and radiated noise of the payload. For our example, we maximized P_{S1} and P_{S2} for two different scenarios using a multi-objective genetic algorithm (refer to [39] for details). Figure 7 shows the resulting ten Pareto optimum design alternatives from which we select, with the P_{Si}'s being the attributes. Note that in Fig. 7, some of the design alternatives (namely D_1 and D_2; D_3 and D_4; and D_7 and D_8) are located close to each other in the attribute space and are difficult to distinguish.

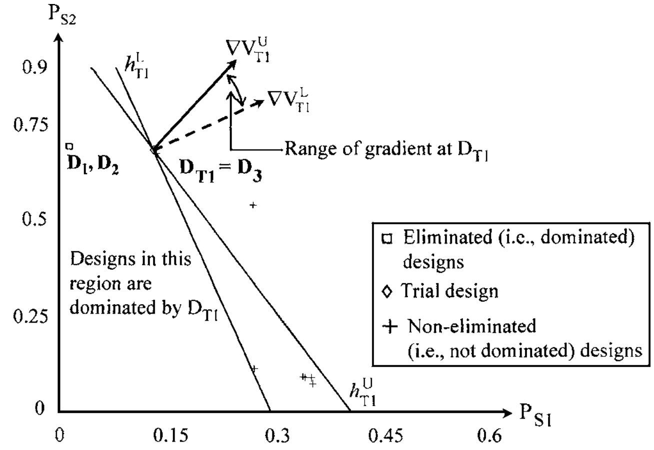

Fig. 7 Dominated designs when β lies between 11 and 18 at D_{T1}

In Sec. 5.1, we show the application of our selection method with preference variability to payload design selection. Next, in Sec. 5.2 we discuss the application of our selection method with preference and attribute variability to the same example.

5.1 Payload Design Selection With Preference Variability. Since it is difficult for a human DM to verify that the potentially optimal designs found by our method with preference variability are indeed accurate (i.e., the designs are indeed most preferred for some subset of original range of preferences), we use a simulated DM in this example. We constructed the DM's implicit value function to be of the form

$$V = -[(1 - P_{S1})^{\beta} + (1 - P_{S2})^2]. \tag{10}$$

In Eq. (10), parameter β creates variability in the MRS preferences between the attributes. We assign a range to β (note in Eq. (10), V is quasi-concave, differentiable and non-decreasing for any β greater than or equal to one). As β varies in its specified range, the MRS preference between attributes also varies. As the range of β increases, the variability in the MRS preferences also increases, and vice versa. We emphasize that the variability construct of Eq. (10) is not a presumed value function. Rather, it simulates a human DM who is supposedly being queried by our selection method, providing a range of MRS preferences. The only reason we use this variability construct is to verify that the potentially optimal designs obtained by our method are indeed accurate. Note that Eq. (10) is a simple concave function and we use it in this example for demonstration. However, we did test our method with more complex value functions in other examples [40] (these examples had 2–6 attributes and up to 50 design alternatives).

We applied our method to three cases with different ranges for β in each case. We discuss in detail the case where variability in β is large (thus resulting in large variability in MRS preferences) in Sec. 5.1.1. Next we briefly discuss the results for the other two

Table 1 Z^* values for payload designs alternatives at D_{T1}

Design alternative No.	Attributes of design alternative $[P_{S1}, P_{S2}]$	Z^* values at D_{T1}, objective function in Eq. (4)	Z^* values at D_{T2}, objective function in Eq. (4)	Z^* values at D_{T3}, objective function in Eq. (4)
1	[0.016, 0.695]	−0.0812		
2	[0.016, 0.693]	−0.0814		
3	[0.134, 0.684]	0	0.1215	−0.0001
4	[0.139, 0.675]	0.0018	0.1127	0
5	[0.274, 0.541]	0.0847	0	0.0792
6	[0.275, 0.114]	0.0024	−0.2866	
7	[0.343, 0.093]	0.0532	−0.2786	
8	[0.346, 0.091]	0.0549	−0.2792	
9	[0.355, 0.090]	0.062	−0.2768	
10	[0.357, 0.075]	0.0612	−0.2858	

cases in Sec. 5.1.2. Finally, we present our verification results in Sec. 5.1.3.

5.1.1 Large Variability in MRS Preferences. For this case, we fix the range of β to be "11–18." The range of MRS preferences at a trial design corresponding to a range of β can be found from Eq. (10) by solving a simple optimization problem (we do not show the optimization problem here due to space restrictions).

Following the iterative method discussed in Sec. 3 (recall Fig. 2), we start by picking a trial design from the set of ten design alternatives. We randomly pick design alternative 3, D_3, as the trial design for the first iteration, and set it as D_{T1}: [0.134, 0.684]. Since this is a two attribute problem, we ask the DM to provide the range of only one MRS preference, i.e., MRS preference between P_{S1} (attribute 1) and P_{S2} (attribute 2). Our simulated DM, Eq. (10), responds by saying: "I would give up between 0.246 and 0.413 in P_{S2} to gain 0.1 in P_{S1}," i.e., the range of MRS preferences at D_{T1} is, S_{12T1}: [2.46, 4.13].

We then use Eq. (4) with the given MRS range to eliminate some dominated designs. Table 1 (column 3) shows the Z^* values (objective function in Eq. (4)) at D_{T1} for the payload design alternatives. We can see that Z^* is negative for D_1, D_2 (hence dominated by D_{T1}) and non-negative for the rest of the design alternatives. Z^* of D_3 is zero because it is the trial design for this iteration.

As there are only two attributes in this example, the upper bound, S^U_{12T1}, and the lower bound, S^L_{12T1}, of MRS preferences correspond to the extremes, ∇V^U_{T1} and ∇V^L_{T1}, of the range of gradient at D_{T1}. So we can visualize the attribute space with the range of gradients as shown in Fig. 7. From Fig. 7, we can see that only D_1 and D_2 lie in all the possible gradient cuts that belong to the range of gradient at D_{T1}. Hence, only D_1 and D_2 are dominated by D_{T1} and can be eliminated.

Since more than one design is not eliminated, we find a new trial design using our approach discussed in [35]. Our approach finds D_5 as the new trial design. So we set D_5 as D_{T2}: [0.274, 0.541] and start the second iteration. Our simulated DM, Eq. (10), gives the range of MRS preferences at D_{T2} as, S_{12T2}: [0.09, 0.49]. We then use Eq. (4) to eliminate dominated designs based on the given range of MRS, S_{12T2}. Table 1 (column 4) shows the Z^* values at D_{T2} for the payload design alternatives. We can see that Z^* is negative for D_6, D_7, D_8, D_9, and D_{10} (hence dominated by D_{T2}) and positive forD_3 and D_4. Z^* of D_5 is zero because it is the trial design for this iteration. Z^* is empty for D_1 and D_2 because they are already eliminated by D_{T1}.

Since more than one design is not eliminated (recall D_3, D_4, and D_5 are not eliminated), we find a new trial design. Perforce, D_4 is the new trial design because it is the only non-eliminated design which has not been a trial design. So we set D_4 as D_{T3}: [0.139, 0.675] and start the third iteration. Our simulated DM, Eq. (10), gives the range of MRS preferences at D_{T3} as, S_{12T3}: [2.18, 3.79]. We then use Eq. (4) to eliminate dominated designs based on the given range of MRS, S_{12T3}. Table 1 (column 5) shows the Z^* values at D_{T3} for the payload design alternatives. We can see that Z^* is negative for D_3 (hence dominated by D_{T3}) and positive for D_5.

D_4 and D_5 are the only non-eliminated designs at this stage. Since both of them have already been trial designs we stop the iterative process and collect the two designs in the set D_{NTD} (recall Fig. 2). We then apply our heuristic approach to see if any of the two trial designs can be eliminated (recall Sec. 3.2). We fix the radius of the region, $O_{Ti}(i=2,3)$, around $D_{Ti}(i=2,3)$ where the linear approximation of value function is estimated to be valid as: $R=0.10$ (the R value is chosen arbitrarily). Using the formulation in Appendix A, we then find the maximum p^{T2}_{h2h3} for D_{T2} as 0.13 and the maximum p^{T3}_{h2h3} for D_{T3} as 0.14. Since the maximum values of both p^{T2}_{h2h3} and p^{T3}_{h2h3} are greater than R, neither design dominates the other. So we conclude that D_4 andD_5 are potentially optimal for the case when β lies between 11 and 18.

In the next section, we briefly discuss our results for the selection of payload design when the variability in β (hence the variability in MRS preferences) is moderate.

5.1.2 Moderate Variability in MRS Preferences. We applied the selection method with preference variability for different β ranges. In the case where $\beta \in [11, 14.4]$, the method found D_5 as the singleton potential optimal (hence the most preferred) design. In the case where $\beta \in [14.6, 18]$, the method found D_4 as the most preferred design. Our results show that when the variability in MRS preferences is large we obtain a couple of potentially optimal designs and as the variability in MRS preferences decreases we find a single most preferred design.

In the next section, we verify that the potentially optimal designs obtained for the above-discussed cases (with different ranges for β in each case) are indeed accurate.

5.1.3 Verification of Results. To verify the results obtained by our method we use the variability construct shown in Eq. (10). Substituting different values for β in Eq. (10), we can obtain the values of the design alternatives for that β (see Fig. 8). Note that the maximum of Eq. (10) (which is zero), for each β, is obtained when both P_{S1} and P_{S2} are equal to one.

From Fig. 8 we can see that when β lies between 11 and 14.5, D_5 has the highest value. When β is equal to 14.5, both D_5 and D_4 have the highest value. When β lies between 14.5 and 18, D_4 alone has the highest value. Even though we showed in Fig. 8, the values of the design alternatives for only some discrete β in the range 11–18, it can be verified that D_5 has highest value for $\beta \in [11, 14.5)$ and D_4 has highest value for $\beta \in (14.5, 18]$.

Recall that using our method we obtained D_4 and D_5 as the potentially optimal designs when $\beta \in [11, 18]$. From Fig. 8 this is expected because D_4 has the highest value for some part of the β

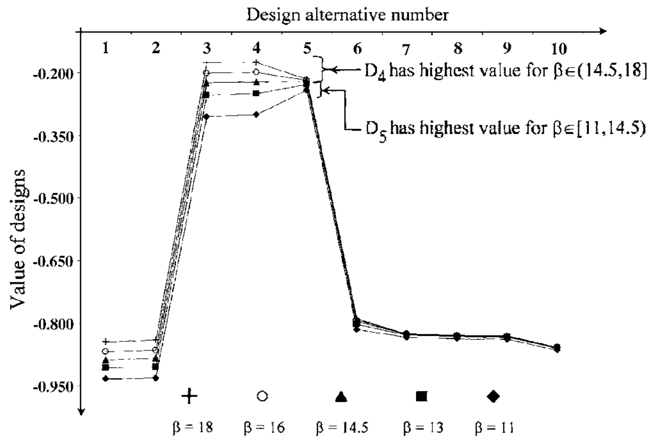

Fig. 8 Value of payload design alternatives for different β's

range and D_5 has the highest value for some other part of the β range. When $\beta \in [11, 14.4]$, our method obtained a single most preferred design D_5 and when $\beta \in [14.6, 18]$, our method again obtained a single most preferred design D_4, as expected from Fig. 8. This verifies the results of our method.

In the next section we present the application of our selection method with preference and attribute variability to the payload design selection example.

5.2 Payload Design Selection With Preference and Attribute Variability. Once again we use the simulated DM given by Eq. (10) for verifying the results obtained by our method. However, for this example, in Eq. (10), in addition to the parameter β, the attributes P_{S1} and P_{S2} have variability quantified by a range. We applied our method to two cases with different ranges for β, P_{S1} and P_{S2} in each case.

For the first case, we fix the range of β to be 11–18. Also, we fix the range of attribute levels P_{S1} and P_{S2} to be ±5% around the nominal attribute levels, i.e., if the nominal attribute level of a design alternative, say D_1, for the attribute, say P_{S1}, is 0.016, then the variability in the attribute P_{S1} for D_1 is quantified by the range [0.015, 0.017]. Note that, the nominal attribute level should not be confused with the probabilistic mean. The range of MRS preferences at a trial design for the given ranges of β, P_{S1} and P_{S2} can be found from Eq. (10) by solving a simple optimization problem (we do not show the optimization problem here due to space restrictions).

Following the iterative method shown in Fig. 5, we started with an initial trial design D_3 and our method, in three iterations, found that the set of non-eliminated trial designs D_{NTD} consists of designs D_3, D_4, and D_5. The details of the dominated designs eliminated at each iteration are not shown due to space restrictions. Also we verified that D_3, D_4, and D_5 are indeed the potential optimal designs using Eq. (10) for the given ranges of β, P_{S1}, and P_{S2}.

For the second case, we once again fix the range of β to be 11–18. However, we changed the range of attribute levels P_{S1} and P_{S2} to be ±15% around the nominal attribute levels. Starting with an initial trial design of D_3, our method found designs D_1, D_2, D_3, D_4, and D_5 to be the elements of D_{NTD}. However, using Eq. (10), we found that only D_3, D_4, and D_5 are the potentially optimal design for this case also. This verifies our earlier statement that at present our method for selection with preference and attribute variability can include some designs that are actually dominated in the set of non-eliminated trial designs D_{NTD}. However, on the bright side, the set D_{NTD} found by our method always contains the actual potentially optimal designs.

6 Summary

In this paper, we presented a method for product design selection with variability in preferences for an implicit value function and later extended it to account for variability in attribute levels of design alternatives. The only assumption we made in our method was that the DM's implicit value function is differentiable, quasi-concave and non-decreasing with respect to the attributes. This assumption is more general and less restrictive than other popular assumptions as reported in the literature (e.g., additive value function) [30–32].

Our method for selection with preference variability is iterative and requires that the DM gives a range for MRS preference between attributes at a series of trial designs. We presented an approach to eliminate dominated designs using the range of MRS preferences directly. The mathematical formulation of this approach under certain condition becomes a linear programming problem and can be solved quickly to obtain the set of non-eliminated trial designs. We also presented a heuristic to identify potentially optimal designs from the set of non-eliminated trial designs. Finally, we demonstrated the applicability of our method with the help of a payload design selection example. We showed and verified that when the variability in MRS preferences is large, we obtain a couple of potentially optimal designs and as the variability in MRS preferences decreases, we find a single most preferred design.

In addition to the range of MRS preferences, if the DM can provide the probability distributions (within the given range) of the MRS preferences, our method for selection with preference variability can be extended for finding the preferred designs. Since the designs not in the set of non-eliminated trial designs D_{NTD} are dominated irrespective of the probability distributions for the given ranges of MRS preferences, D_{NTD} can be used as the set of designs from which the selection has to be made. Then a Monte Carlo simulation of our deterministic selection method [35] can be used to find the preferred designs from the set D_{NTD}.

Our method for selection with preference and attribute variability requires that the range of attribute levels of design alternatives be known in addition to the range of MRS preferences. We presented a mathematical formulation for eliminating dominated designs using the ranges of attributes and MRS preferences. Finally, we demonstrated the applicability of the method with the help of payload design selection problem.

At present in our method for selection with preference and attribute variability, the non-eliminated designs might not be potentially optimal. We are currently working on developing a heuristic similar to the one discussed in Sec. 3.2 for identifying potential optimal designs from the set D_{NTD} when there is variability in preferences and attribute levels.

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Nomenclature

a_i = i^{th} attribute

a_{ij} = i^{th} attribute (variable or fixed) of design D_j

A_{ij}^L = lower bound of attribute a_i for design D_j when there is attribute variability

A_{ij}^U = upper bound of attribute a_i for design D_j when there is attribute variability

A_{ij} = range of attribute a_i for design D_j when there is attribute variability

A_{iT} = range of attribute a_i for design D_T when there is attribute variability
C_G = gradient cut at a trial design
D_+ = arbitrary design (other than trial design) that belongs to the original set of designs
D_i = i^{th} design alternative
D_{NTD} = set of non-eliminated trial designs
D_T = trial design
D_{Ti} = i^{th} trial design
h_T = hyper-plane perpendicular to ∇V_T (fixed or variable) at a trial design D_T
h_{Tk} = hyper-plane perpendicular to ∇V_{Tk} (fixed or variable) at the k^{th} trial design D_{Tk}
h^L_{Tk} = hyper-plane perpendicular to ∇V^L_{Tk} at the k^{th} trial design D_{Tk} in Fig. 4
h^U_{Tk} = hyper-plane perpendicular to ∇V^L_{Tk} at the k^{th} trial design D_{Tk} in Fig. 4
h^M_{T2} = hyper-plane in between $[h^L_{T2}, h^U_{T2}]$ such that D_{T1} does not lie in the gradient cut corresponding to the gradients perpendicular to $[h^L_{T2}, h^M_{T2}]$ in Fig. 4
m = number of attributes
n = number of designs
O_T = region around D_T where linear approximation of value function is valid
O_{Ti} = region around D_{Ti} where linear approximation of value function is valid
p^{Tj}_{hjhk} = perpendicular distance from the j^{th} trial design D_{Tj} to intersection of h_{Tj} and h_{Tk}
p^{Tk}_{hjhk} = perpendicular distance from the k^{th} trial design D_{Tk} to intersection of h_{Tj} and h_{Tk}
p^{T1}_{h1Lh2L} = perpendicular distance from D_{T1} to intersection of h^L_{T1} and h^L_{T2} in Fig. 4
p^{T1}_{h1Uh2L} = perpendicular distance from D_{T1} to intersection of h^U_{T1} and h^L_{T2} in Fig. 4
P_S = probability of success of a payload in a given scenario
P_{Si} = probability of success of a payload in the i^{th} scenario
R = radius of O_{Ti}
s_{ijT} = MRS between attributes a_i and a_j (variable or fixed) at a trial design D_T
s_{ijTk} = MRS between attributes a_i and a_j (variable or fixed) at the k^{th} trial design D_{Tk}
S^L_{ijT} = lower bound of MRS between attributes a_i and a_j at a trial design D_T
S^U_{ijT} = upper bound of MRS between attributes a_i and a_j at a trial design D_T
S^L_{ijTk} = lower bound of MRS between attributes a_i and a_j at the k^{th} trial design D_{Tk}
S^U_{ijTk} = upper bound of MRS between attributes a_i and a_j at the k^{th} trial design D_{Tk}
S_{ijT} = range of MRS between attributes a_i and a_j at a trial design D_T
S_{ijTk} = range of MRS between attributes a_i and a_j at k^{th} trial design D_{Tk}
V = value function
w_{iT} = coefficient of ∇V_T with respect to attribute a_i at a trial design D_T
w_{iTk} = coefficient of ∇V_{Tk} with respect to attribute a_i at the k^{th} trial design D_{Tk}
Z^* = objective function value in the formulations to identify dominated design
β = parameter to induce variability in the value function of simulated DM
∇V_T = variable gradient of V at a trial design D_T corresponding to variable MRS s_{ijT}
∇V_{Tk} = variable gradient of V at the k^{th} trial design D_{Tk} corresponding to variable MRS s_{ijTk}
∇V^L_{Tk} = extreme of gradient range at the k^{th} trial design D_{Tk} in Fig. 4
∇V^U_{Tk} = other extreme of gradient range at the k^{th} trial design D_{Tk} in Fig. 4

Appendix A

The maximum p^{T1}_{h1h2} that is required to conduct the test of Eq. (7) can be calculated using Eq. (A1).

$$\text{Maximize } p^{T1}_{h1h2} \tag{A1a}$$

$$\text{subject to: } \frac{\left[\sum_{i=1}^{m} w_{iT1}\cdot w_{iT2}\right]^2}{\left(\sum_{i=1}^{m} w^2_{iT1}\right)\left(\sum_{i=1}^{m} w^2_{iT1}\right)} < 1 \tag{A1b}$$

$$w_{iT1}\in W_{T1} \text{ and } w_{iT2}\in W_{T2};\ i=1 \text{ to } m \tag{A1c}$$

$$\sum_{i=1}^{m} w_{iT2}\cdot(a_{iT1}-a_{iT2}) \geqslant 0 \tag{A1d}$$

Eq. (A1b) is a constraint for checking that the angle between ∇V_{T1} and ∇V_{T2} is greater than zero (i.e., h_{T1}'s that are not parallel to any one of the h_{T2}'s, recall Sec. 3.2). Note that the angle between the vectors ∇V_{T1} and ∇V_{T2} is zero only when the cosine of the angle (given by the square root of the left hand side of Eq. (A1b)) is one. Equation (A1c) is a short notation for the normalization constraints on w_{iT1} and w_{iT2}, and the constraints that $s_{ijT1}=w_{iT1}/w_{jT1}$ and $s_{ijT2}=w_{iT2}/w_{jT2}$, should belong to the range of MRS preferences given by the DM at D_{T1} and D_{T2}, respectively (recall Eq. (4b) and Eq. (4c). Equation (A1d) is a constraint for checking that∇V_{T2} belongs to the range of the gradients that do not eliminate D_{T1} by gradient cut (recall Fig. 4).

Appendix B

The formulation required to conduct the test of Eq. (8) is given by Eq. (B1). As mentioned earlier (recall Sec. 3.2), when ∇V_{T1} is equal to ∇V_{T2} (i.e., h_{T1}'s that are parallel to some of the h_{T2}'s), we can find the value of the designs directly by using Eq. (3) based on a linear approximation of the value function. So, if the maximum of the difference between the value of D_{T1} and D_{T2} (i.e., objective function of Eq. (B1)) is negative we can conclude that D_{T2} dominates D_{T1} for the case where some of the h_{T1}'s are parallel to some of the h_{T2}'s. Equation (B1b) is a constraint for checking that ∇V_{T1} is equal to ∇V_{T2}. Equation (B1c) is similar to Equation (A1c).

$$\text{Maximize } \sum_{i=1}^{m} w_{iT1}\cdot(a_{iT1}-a_{iT2}) \tag{B1a}$$

$$\text{subject to: } \frac{\left[\sum_{i=1}^{m} w_{iT1}\cdot w_{iT2}\right]^2}{\left(\sum_{i=1}^{m} w^2_{iT1}\right)\left(\sum_{i=1}^{m} w^2_{iT1}\right)} = 1 \tag{B1b}$$

$$w_{iT1}\in W_{T1} \quad \text{and} \quad w_{iT2}\in W_{T2} \tag{B1c}$$

Note that, it is possible that Eq. (A1) or Eq. (B1) is infeasible. If Eq. (A1) is infeasible, the test of Eq. (8) alone is enough to conclude that D_{T2} dominates D_{T1}. Similarly, the test of Eq. (7) alone is enough to conclude that D_{T2} dominates D_{T1} if Eq. (B1) is infeasible. Note that Eq. (B1) becomes infeasible only when no

h_{T1} is parallel to any h_{T2} and Eq. (A1) becomes infeasible only when any h_{T1} is parallel to some h_{T2} (hence Eq. (A1) and (B1) cannot be infeasible simultaneously).

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