**Proceedings of the ASME 2009 International Design Engineering Technical Conferences &
Computers and Information in Engineering Conference
IDETC/CIE 2009
August 30 - September 2, 2009, San Diego, California, USA**

DETC2009-86199

ELLIPSOIDAL APPROXIMATIONS OF INVARIANT SETS IN STABILIZATION PROBLEM FOR A WHEELED ROBOT FOLLOWING A CURVILINEAR PATH

Alexander V. Pesterev*
Javad GNSS, Moscow,
Institute of Control Sciences
Russian Academy of Sciences

Lev B. Rapoport
Javad GNSS, Moscow,
Institute of Control Sciences
Russian Academy of Sciences

ABSTRACT

A stabilization problem for a wheeled robot following a curvilinear target path is studied. In [1], a method for constructing invariant ellipsoids—quadratic approximations of the attraction domains for the target trajectory under a given control law—was developed. A basic result of that study is a theorem by means of which construction of the invariant ellipsoids reduces to solving a system of linear matrix inequalities (LMIs) and checking a scalar inequality. This paper is a sequel to work [1] and is devoted to practical implementation of the results obtained in that paper. It is discussed how to select the parameters in terms of which the theorem is formulated. An algorithm is developed that, for a given value of maximal deviation from the target trajectory, constructs an invariant ellipsoid of as large volume as possible.

1 INTRODUCTION

There exist many applications (e.g., in agriculture or road construction) where a vehicle is to be driven along a target path with high level of accuracy [2, 3]. Such tasks are performed by automatic vehicles (further referred to as *wheeled robots*, or simply *robots*) equipped with navigational and inertial tools and satellite antennas [1–7]. In the majority of studies on this subject, the well-known kinematic model of a vehicle moving without lateral slippage described by three nonlinear differential equations (e.g., [3–7]) is considered. To take into account dynamics of the steering gear mechanism, a simple actuator model is sometimes introduced. The control goal is to bring the robot to a given curvilinear path and to stabilize its motion along the target curve.

In view of essential nonlinearity of the considered problem, it seems impossible to synthesize a control law that would stabilize robot motion along an arbitrary curvilinear trajectory under arbitrary initial conditions. Therefore, in practice, it is desirable to have a criterion that would allow one to check in the course of motion whether the robot state belongs to the attraction domain of the target trajectory, or, in other words, whether the synthesized control can stabilize motion of the robot along the given trajectory from a given initial position. For such a criterion, it is suggested in [1] to construct invariant ellipsoids in the system state space. The desired ellipsoids are found by applying the approach proposed in [6] for the case where the target trajectory is a straight line or an arc of a circle. The approach is based on the absolute stability theory [8–10] and reduces construction of an invariant ellipsoid to solving an LMI system [11]. The problem studied in [1] differs from that discussed in [6] in that the target trajectory may be an arbitrary smooth curve and in the presence of phase constraints. This paper is a sequel to [1] and devoted to practical implementation of the results reported in [1].

2 PROBLEM STATEMENT

The wheeled robot considered in this work is a vehicle moving without lateral slippage with two rear driving wheels and one

*Contact author: Lab. 16, Institute of Control Sciences, Russian Academy of Sciences, Profsoyuznaya 65, Moscow, 117997 Russia. Phone: +7(910) 435-0921, Fax: +7(495) 926-5210, Email: A.Pesterev@javad.com

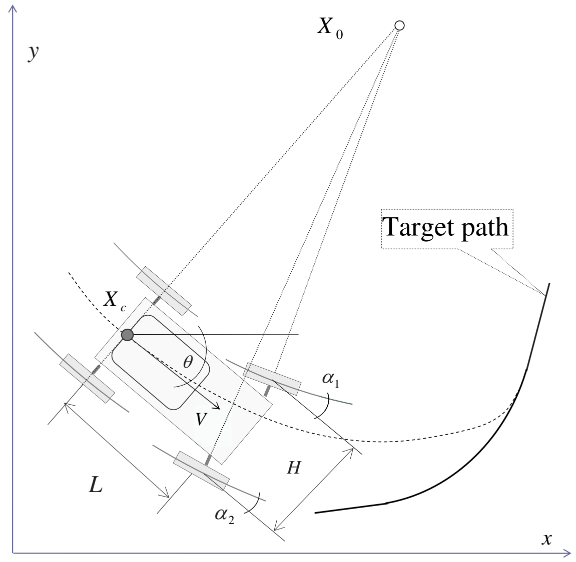

Figure 1. The kinematic scheme of the wheeled robot.

(or two) front wheel responsible for steering the platform. In the planar case, the robot position is described by two coordinates (x_c, y_c) of some point of the platform, the so-called *target point*, and one angle describing orientation of the platform with respect to an immovable reference system $x0y$.

For the target point, the point located in the middle of the rear axle is usually taken, and for the angle, the angle θ between the central line of the platform (which coincides with the direction of the velocity vector) and the x-axis. The kinematic equations of such a robot are well known (see, for example, [1, 6, 7]):

$$\begin{aligned} \dot{x}_c &= v\cos\theta, \\ \dot{y}_c &= v\sin\theta, \\ \dot{\theta} &= vu. \end{aligned} \tag{1}$$

Here, dot denotes differentiation with respect to time, $v \geq v_0 > 0$ is a scalar linear velocity of the target point, and u is the instant curvature of the trajectory described by the target point. The quantity u is uniquely related to the turning angle of the front wheel α by the equation

$$u = \tan\alpha / L, \tag{2}$$

where L is the distance between the front and rear axles. In the case of two front wheels (as depicted in Fig. 1), their turning angles α_1 and α_2 are different, but are not independent and are related to the trajectory curvature, such that there exists an "efficient mean" angle α [6, 7]. To take into account dynamics of the steering gear mechanism, a simple actuator model is introduced, which is described by the first-order differential equation (such dynamics is typical of step motors often used in practice)

$$\dot{\alpha} = V, \tag{3}$$

where V is the angular speed of rotation of the actuator shaft, which is considered to be a control, such that angle α (or curvature u) becomes a state space variable. The control resource is assumed to be bounded, $|V| \leq \overline{V}$, where $\overline{V}$ is the maximum angular speed of the actuator shaft.

Limitation on the turning angle of the front wheels results in the two-sided constraints on the state space variables

$$-\overline{u} \leq u \leq \overline{u}\, (-\alpha_{\max} \leq \alpha \leq \alpha_{\max}), \tag{4}$$

where $\overline{u} = \tan\alpha_{\max}/L$ is the maximal possible curvature of an actual trajectory.

The stabilization problem for a wheeled robot consists in synthesizing a control law V that brings the robot to a given target trajectory and stabilizes its motion along the curve. In the considered planar case, the target trajectory (path) is described by a pair of functions $(X(s), Y(s))$, where $X(s)$ and $Y(s)$ are x- and y-coordinates of a point on the curve and s is a parameter. Functions $X(s)$ and $Y(s)$ are assumed to be three times differentiable, and s is arc length ("natural parametrization").

A control law ensuring exponential convergence of the actual trajectory to the target path in a neighborhood where the phase constraints hold as strict inequalities under the condition of the unbounded control resource was derived in [7]. However, in the general case, this control law does not guarantee stabilizability of the system for arbitrary initial conditions, and we arrive at the necessity of determining whether a current vehicle position belongs to the attraction domain.

In [1], it is suggested to divide the path into segments where parameters of the curve do not vary too much and, for each segment, construct an invariant ellipsoid [12], which is an approximation of the attraction domain for the given segment. If the trajectory of the system comes into such an ellipsoid, it will remain in it until, at least, the system moves along the given segment. When turning from one trajectory segment to another, the ellipsoid is, generally, changed, and it is required to check again whether the current state belongs to the new ellipsoid. Thus, the constructed system of invariant ellipsoids may serve as a stabilizability criterion.

For the control law synthesized in [7], the construction of the invariant ellipsoids reduces to solving an LMI system and checking a scalar inequality [1]. Theorem 1 proved in [1] gives one a constructive way for finding the matrix of a desired ellipsoid for given values of certain parameters and checking whether the ellipsoid obtained belongs to the attraction domain of the given trajectory segment. However, it says nothing on how to select these parameters and what to do when they are determined ambiguously. The goal of this paper is to try to address these questions and to develop an efficient algorithm for constructing ellipsoids of as large volume as possible. Before we set the above plan in motion, we need to briefly present results of our earlier studies, referring the reader to [1,6,7] for detail.

3 EARLIER RESULTS

3.1 Change of Variables:

By means of change of variables suggested in [7], equations (1), (3) reduce to the form that admits feedback linearization, at least, in a small neighborhood of the trajectory. For the independent variable, the path ξ passed by the robot is taken, and the state space variables are deviation (distance) of the target point from the target path z_1, angle deviation $z_2 = \sin\psi$, and

$$z_3 = u\sqrt{1-z_2^2} - \frac{k(1-z_2^2)}{1+kz_1}, \tag{5}$$

where ψ is the angle between the velocity vector and the tangent line to the trajectory at the point closest to the target point and k is the curvature of the target path at this point.

By means of this change of variables, equations (1), (3) reduce to the equations in deviations [7]:

$$\begin{aligned} z_1' &= z_2, \\ z_2' &= z_3, \\ z_3' &= \varphi(z)\tfrac{V}{v} - f(z), \end{aligned} \tag{6}$$

where

$$\varphi(z) = \sqrt{1-z_2^2}\left(Lu^2(z) + \frac{1}{L}\right), \tag{7}$$

$$f(z) = \frac{z_2 z_3^2}{1-z_2^2} - \frac{kz_2 z_3}{1+kz_1} - \frac{k^2 z_2(1-z_2^2)}{(1+kz_1)^2} + \frac{k_s'(1-z_2^2)^{\frac{3}{2}}}{(1+kz_1)^3}, \tag{8}$$

and $u(z)$ is the instant curvature in terms of the new variables,

$$u(z) = \frac{z_3}{\sqrt{1-z_2^2}} + \frac{k\sqrt{1-z_2^2}}{1+kz_1}. \tag{9}$$

Here and in what follows, k_s' is the derivative of curvature with respect to the arc length at the trajectory point closest to the target point. In all other cases, the prime denotes differentiation with respect to the independent variable ξ. In addition, the system must satisfy the constraints on the state space variables

$$|u(z)| \le \bar{u}. \tag{10}$$

3.2 Control Law Synthesis:

The problem of synthesizing control that stabilizes robot's motion along the target trajectory is formulated in the state space of z-coordinates as that of finding control V satisfying the constraints $|V| \le \bar{V}$ for which solution $z(\xi) \equiv [z_1(\xi), z_2(\xi), z_3(\xi)]^{\mathrm{T}}$ of equations (6) tends to zero, with the constraints on the state space variables (10) being fulfilled.

Without phase and control constraints, system (6) can be linearized by means of an appropriate feedback. Indeed, confining ourselves to the case where $|\psi| < \pi/2$ (which guarantees that $\varphi(z) \ne 0$), and taking V in the form[1]

$$V = \frac{v[f(z) - \sigma(z)]}{\varphi(z)}, \tag{11}$$

where $\sigma(z)$ is a linear function of the coordinates, $\sigma = c^{\mathrm{T}}z$, $c, z \in R^3, c = (c_1, c_2, c_3)^{\mathrm{T}}$, $z = (z_1, z_2, z_3)^{\mathrm{T}}$, we arrive at the linear closed-loop system

$$z' = Az, \tag{12}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c_1 & -c_2 & -c_3 \end{pmatrix}. \tag{13}$$

Clearly, by means of appropriate selection of vector c (which can be done in many ways), matrix A can always be made Hurwitz, resulting thus in a stable system. For example, selecting c as

$$c = (\lambda^3, 3\lambda^2, 3\lambda),\ \lambda > 0, \tag{14}$$

we obtain an exponentially stable linear system with the exponent $-\lambda$. In what follows, we assume that vector c is always selected in such a way that A is a Hurwitz matrix.

To meet constraint $|V| \le \bar{V}$, control is taken in the form [7]

$$V = s_{\bar{V}}\left(\frac{v[f(z) - \sigma(z)]}{\varphi(z)}\right), \tag{15}$$

[1]The case where $\varphi(z)$ may vanish was discussed in [7].

where $s_{\bar{V}}(V)$ is the saturation function,

$$s_{\bar{V}}(V) = \begin{cases} -\bar{V}, \text{ for } V \leq -\bar{V}, \\ V, \text{ for } |V| < \bar{V}, \\ \bar{V}, \text{ for } V \geq \bar{V}. \end{cases} \tag{16}$$

However, it is not guaranteed that this control law can stabilize robot's motion; i.e., it is important to know whether a current state belongs to the attraction domain of the given target trajectory.

3.3 Approximation of the State Space by a Cylinder:

The boundary manifold of the system state space, which is the set of points where constraint (10) turns to equality, is rather involved. The inscribing of an ellipsoid into a region bounded by this manifold is a too complicated task. Moreover, the shape of the manifold depends on variable curvature of the trajectory $k(s)$; i.e., we have a family of boundary manifolds and the corresponding state spaces parameterized by the arc length s. In [1], it is suggested to approximate the entire family of the state spaces associated with a given segment by a second-order surface, namely, a cylinder, and to inscribe ellipsoids into this region.

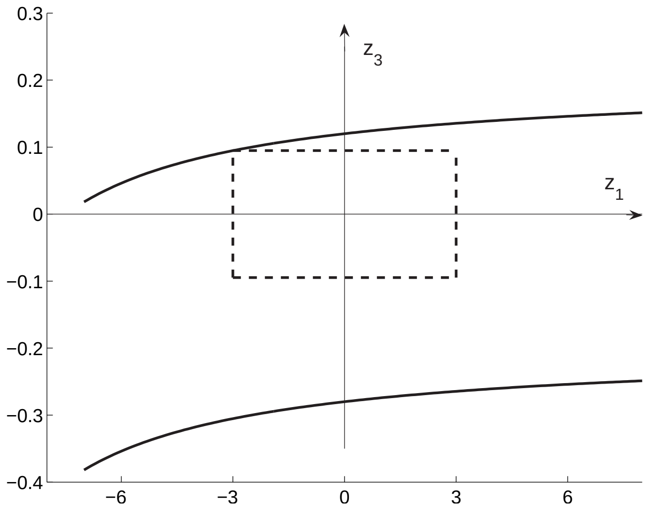

Figure 2. Sections of the state space (solid line) and the cylinder (dashed line) approximating the state space in the strip $|z_1| \leq 3$ by the plane $z_2 = 0$.

As shown in [1], the cylinder

$$|z_3| \leq \tilde{u}\sqrt{1 - z_2^2}, \; -\alpha_1 \leq z_1 \leq \alpha_1, \tag{17}$$

where

$$\tilde{u} = \bar{u} - \frac{\bar{k}}{1 - \bar{k}\alpha_1}, \; \bar{k} = \max_s |k(s)|$$

and

$$\alpha_1 < \bar{\alpha}_1 \equiv \frac{1}{\bar{k}} - \frac{1}{\bar{u}}, \tag{18}$$

belongs to all members of the family of the state spaces corresponding to the given segment in the strip $|z_1| \leq \alpha_1$.

The selection of value $0 < \alpha_1 < \bar{\alpha}_1$ is dictated by two circumstances. In the first turn, it is determined by a specific character of a particular applied problem, since the value of α_1 determines how far we allow the robot to move away from the target curve, with its state being kept within the ellipsoid. On the other hand, it should be taken into account that α_1 affects volume of the approximating cylinder and, thus, volume of the ellipsoid. Indeed, for the extreme values of α_1, $\alpha_1 = 0$ and $\alpha_1 = \bar{\alpha}_1$, volume of the cylinder vanishes, and it is not difficult to find the value of α_1 for which the cylinder volume is maximal.

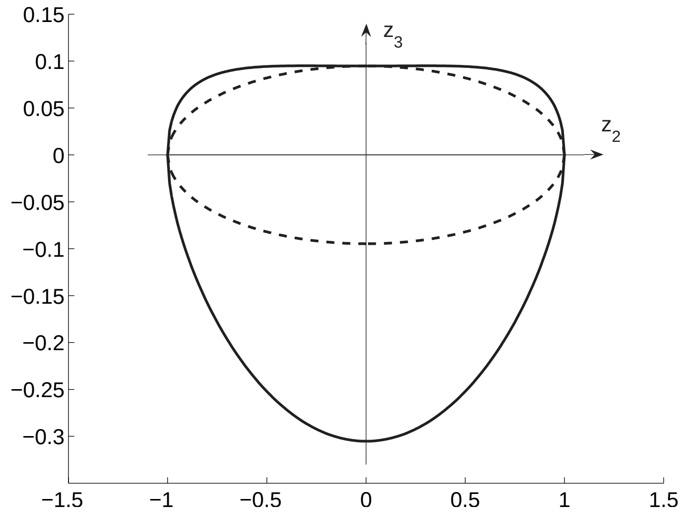

Figure 3. Sections of the state space (solid line) and the cylinder (dashed line) approximating the state space in the strip $|z_1| \leq 3$ by the plane $z_1 = -3$.

As an example, we consider approximation of the state space by the cylinder for the robot with $\bar{u} = 0.2 \text{ m}^{-1}$ (minimal curvature radius 5 m) following along a segment of a target trajectory with maximum curvature $\bar{k} = 0.08 \text{ m}^{-1}$. Let us set α_1 equal to 3 m.

Figures 2 and 3 show the sections of the boundary manifold and approximating cylinder corresponding to the point of the target trajectory with curvature $k = \underline{k}$ by the planes $z_2 = 0$ and $z_1 = -3$, respectively. The height of the cylinder for the given values of the parameters is equal to $\tilde{u} \approx 0.1\ \mathrm{m}^{-1}$.

It follows from the definition of the approximating cylinder that any ellipsoid inscribed in cylinder (17) automatically meets phase constraints (10). Thus, the problem of construction of an ellipsoid satisfying the constraints on the state space variables is replaced by a simpler problem of inscribing an ellipsoid into a region bounded by a second-order surface, which is equivalent to solving the LMI system

$$P \geq I_1,\ P \geq \Pi, \tag{19}$$

where $I_1 = \mathrm{diag}[1/\alpha_1^2, 0, 0]$ and $\Pi = \mathrm{diag}[0, 1, 1/\tilde{u}^2]$.

3.4 Reduction of the Problem to Solving LMI System:

Consider an arbitrary segment of the target trajectory. Let us rewrite system (6) closed by feedback (15), which takes into account the constraint on the control resource, as

$$\begin{aligned} z_1' &= z_2, \\ z_2' &= z_3, \\ z_3' &= -\Phi(z, \sigma), \end{aligned} \tag{20}$$

where

$$\Phi(z, \sigma) = \varphi(z) \frac{1}{v} s_{\overline{V}} \left(\frac{v[\sigma(z) - f(z)]}{\varphi(z)} \right) + f(z), \tag{21}$$

and consider the function

$$U(z) = \varphi(z) \frac{\overline{V}}{v} - |f(z)| \tag{22}$$

and its restriction $\overline{U}(s)$ to the target trajectory,

$$\overline{U}(s) \equiv U(0) = \left(Lk^2(s) + \frac{1}{L} \right) \frac{\overline{V}}{v} - |k_s'(s)|. \tag{23}$$

Definition 1. *[1] A segment of a trajectory $(X(s), Y(s))$, where $X(s)$ and $Y(s)$ are three times differentiable functions, is said to be* admissible for a given robot *(further, simply* admissible*) if, on this segment,*

$$\max_s |k(s)| < \overline{u}. \tag{24}$$

The basic result of paper [1]—reduction of the construction of invariant ellipsoids to solving an LMI system—is formulated as the following theorem.

Theorem 1. *[1] Consider an arbitrary segment of an admissible trajectory for which function (23) is positive for any s and an ellipsoid $\Omega = \{z : z^{\mathrm{T}} P z \leq 1\}$, $P \geq 0$. Let, for some $\alpha_1 > 0$ and $0 < \beta \leq 1$,*
(a) condition

$$\tilde{u} = \overline{u} - \frac{\overline{k}}{1 - k\alpha_1} > 0 \tag{25}$$

hold; (b) LMIs (19) and

$$PA + A^{\mathrm{T}} P < 0,\ PA_\beta + A_\beta^{\mathrm{T}} P < 0, \tag{26}$$

where

$$A_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\beta c_1 & -\beta c_2 & -\beta c_3 \end{pmatrix}, \tag{27}$$

be resolvable in P; and (c) inequality

$$\beta \leq \frac{U_0}{\sigma_0} \tag{28}$$

hold, where $U_0 = \min_{z \in \Omega} U(z)$ and $\sigma_0 = \max_{z \in \Omega} \sigma(z)$. Then, Ω is an invariant ellipsoid for system (6) closed by feedback (15).

Theorem 1 gives us a constructive way for finding a matrix P for given values of parameters α_1 and β and checking whether the ellipsoid generated by matrix P is invariant. However, it says nothing on how to select these parameters. Clearly, the desired matrix is to be sought iteratively, and the efficiency of the construction of the invariant ellipsoids greatly depends on the efficiency of strategy of selecting β, since each iteration includes solution of an LMI system. Moreover, the right-hand side of inequality (28) depends on the desired matrix P and, hence, on β. This means that the desired procedure of searching β should ensure monotone variation of the right-hand side of (28) in order to guarantee the fulfillment of this condition. Note also that, from practical point of view, it is important to obtain not simply an arbitrary invariant ellipsoid but rather an ellipsoid of as large volume as possible, which imposes additional requirements on the algorithm for searching β. In the remaining sections, we discuss how to choose parameter β and develop an algorithm for constructing invariant ellipsoids.

4 PROPERTIES OF THE LMI SYSTEM

First of all, we note that parameters α_1 and β can be selected independently from one another. The value of α_1 is selected based on specific features of a particular applied problem under the condition that inequality (25) is satisfied. It affects volume of the constructed ellipsoid and, thus, the fulfillment of the scalar inequality (28). On the other hand, solvability of the LMIs does not depend on α_1; i.e., if system (19), (26) has a solution for some α_1, it has a solution for any other admissible value of α_1. Therefore, in what follows, we assume that α_1 is admissible and fixed and exclude it from the subsequent consideration. For the same reason, we will not discuss LMIs (19) assuming that they are always satisfied. We will use notation $P(\beta)$ for the solution of the LMIs to emphasize its dependence on β.

The algorithm to be described below relies on the following important properties of LMIs (26):

1. There exist at least one value of β for which system (26) has a nontrivial solution. Indeed, for $\beta = 1$, both inequalities in (26) coincide and have a solution since $A_1 \equiv A$ is a Hurwitz matrix.
2. There exists a range of small values $0 < \beta < \beta_0$ where the LMI system (26) has no solutions. Indeed, in order that LMI (26) have a solution, matrices A and A_β must be Hurwitz. It is not difficult to show that there exists a small β for which matrix A_β is not Hurwitz.
3. If system (26) has a solution for $\beta = \beta_1 < 1$, then it has a solution for any β satisfying the condition $\beta_1 < \beta \leq 1$, and the set of solutions corresponding to this β is not narrower than that corresponding to $\beta = \beta_1$. Indeed, let $P(\beta_1)$ be a solution to (26) for $\beta = \beta_1$. Then [8], $P(\beta_1)$ satisfies (26) for any $\beta > \beta_1$.

It follows from the second property that, in searching the best β, we should confine ourselves to the range $(\beta_0, 1)$, and the third property implies that, among all values of β for which LMIs (26) have solutions, the greatest value for which the scalar inequality (28) holds is the best one.

It can easily be seen from the form of matrices (13) and (27) that β_0 may depend only on vector c, i.e., on the localization of poles of the closed-loop system. Thus, for any β, a particular trajectory segment does not affect the solvability of the LMI system (26). Hence, one and the same β_0 can be used for any trajectories (at least, for a fixed control law). Since it is not currently clear how to find an exact value of β_0, we use its upper estimate $\tilde{\beta}_0 \geq \beta_0$, which guarantees that this LMI system is solvable for any values of β satisfying the condition $\tilde{\beta}_0 \leq \beta \leq 1$. Such an estimate can be obtained, for example, experimentally before constructing ellipsoids under assumption that the poles of the closed-loop system are fixed, or, in the case of the multiple pole, for some range of λ variation. To this end, it is sufficient to solve LMIs (26) for several trial values of β, for example, for $\beta = 0.15,\ 0.2,\ 0.25, \ldots$, and take $\tilde{\beta}_0$ equal to the first value for which the LMIs have solutions. In our numerical experiments, when λ varied from 0.3 to 1, LMIs (26) always had solutions for $\beta \geq 0.25$.

5 CHECKING INEQUALITY (28)

Calculation of the right-hand side of (28) requires finding extrema of functions $U(z)$ and $\sigma(z)$ on the constructed ellipsoid. It is shown in [6] that the maximum σ_0 of linear function $\sigma(z)$ on the ellipsoid is given by the formula

$$\sigma_0 = \sqrt{c^T P^{-1} c}. \tag{29}$$

It seems likely that the minimum of function $U(z)$ can be found only numerically. To avoid this, instead of U_0, we will use its lower estimate. Consider the number

$$\tilde{U}_0 = \sqrt{1-\alpha_2^2}\left(\frac{\bar{V}}{vL} - \frac{\bar{k}'}{(1-k\alpha_1)^3} - \frac{\alpha_2 \bar{k}\bar{u}}{1-k\alpha_1}\right) - \alpha_2 \tilde{u}^2, \tag{30}$$

where $\bar{k}' = \max_s |k_s'(s)|$ on the considered segment and $\alpha_2 = \max|z_2|$ on the ellipsoid. Using (22) and (8), it is not difficult to show that $\tilde{U}_0 \leq U_0$. The number α_2 is found in the same way as σ_0. Indeed, representing z_2 as $z_2 = e_2^T z$, where $e_2^T = (0,1,0)$, we obtain

$$\alpha_2 = \sqrt{e_2^T P^{-1} e_2}; \tag{31}$$

i.e., α_2 is the second diagonal element of the inverse matrix of the constructed ellipsoid. Let us introduce the notation (here and in what follows, arguments of the estimates show values of β for which the ellipsoid was constructed)

$$\tilde{\beta}(\beta) = \frac{\tilde{U}_0(\beta)}{\sigma_0(\beta)} \tag{32}$$

for a lower estimate of the right-hand side of (28) and check condition

$$\beta \leq \tilde{\beta}(\beta) \tag{33}$$

instead of (28).

Efficient finding of an optimal value of β is possible if the right-hand side of inequality (33) is a monotone function of β. It was noted earlier (property 3) that the set of solutions of the LMIs (26) may only extend as β grows. Unfortunately, this does not imply monotonicity of estimate $\tilde{\beta}(\beta)$. The latter property can be ensured if we require that the ellipsoid corresponding to the smaller value of β be nested into the ellipsoid corresponding to the larger value of β. To have this property in the course of searching the desired optimal value of β, we modify the LMI system solved on each step of the algorithm. Namely, let $P(\beta_1)$ be a solution of the LMI system (19), (26) for some $\beta = \beta_1$. When constructing an ellipsoid for a new value $\beta < \beta_1$, we replace the LMIs (19) by

$$P \geq P(\beta_1), \tag{34}$$

i.e., we require that the new ellipsoid be nested in that corresponding to β_1 (which implies also that it belongs to the approximating cylinder).

If, additionally, we have an invariant ellipsoid corresponding to some $\beta_2 < \beta$, we supplement the LMI system with the LMI

$$P \leq P(\beta_2), \tag{35}$$

The fulfillment of both LMIs for $\beta_2 < \beta < \beta_1$ means that the corresponding ellipsoids are nested into one another, which results in the following chains of inequalities: $\sigma_0(\beta_2) \leq \sigma_0(\beta) \leq \sigma_0(\beta_1)$, $\tilde{U}_0(\beta_2) \geq \tilde{U}_0(\beta) \geq \tilde{U}_0(\beta_1)$ and $\tilde{\beta}(\beta_2) \geq \tilde{\beta}(\beta) \geq \tilde{\beta}(\beta_1)$. Thus, the right-hand side of (33) decreases monotonically as β grows, which allows us to efficiently find the desired optimal value of β. In our numerical experiments, two–three iterations were sufficient to get an optimal value such that further improvement of β did not result in a noticeable increase of the ellipsoid.

It may happen that, for given values of all other parameters, inequality (33) does not hold for $\beta = \tilde{\beta}_0$ (and, hence, by virtue of monotonicity of $\tilde{\beta}(\beta)$, for all greater values). Further decrease of β is impossible, since, by virtue of the above property 2, the LMIs (26) will have no solutions. In this case, it is possible to ensure the fulfillment of inequality (33) for the same value of $\beta = \tilde{\beta}_0$ by supplementing LMIs (26) with the LMIs

$$P \geq cc^T \tilde{\beta}_0^2 / \tilde{U}_0^2(\tilde{\beta}_0),\ P \geq P(\tilde{\beta}_0), \tag{36}$$

where $\tilde{U}_0(\tilde{\beta}_0)$ and $P(\tilde{\beta}_0)$ are the lower estimate of U_0 and matrix P, respectively, obtained for $\beta = \tilde{\beta}_0$. Let $\tilde{P}$ denote a solution of the LMI system obtained, and let us show that condition (33) holds. Indeed, multiplying both sides of the first inequality in (36) by $c^T \tilde{P}^{-1}$ from the left and by $\tilde{P}^{-1}c$ from the right, we obtain

$$c^T \tilde{P}^{-1} c \geq (c^T \tilde{P}^{-1} c)^2 \tilde{\beta}_0^2 / \tilde{U}_0^2,$$

or, taking into account (29), $\tilde{\beta}_0^2 \leq \tilde{U}_0^2 / \sigma_0^2(\tilde{P})$, where $\sigma_0(\tilde{P})$ is the maximum of linear function $\sigma(z)$ on the new ellipsoid. By virtue of the second inequality in (36), the new ellipsoid is nested into that corresponding to matrix $P(\tilde{\beta}_0)$. Hence, estimate $\tilde{U}_0(\tilde{P})$ on this ellipsoid is greater than $\tilde{U}_0$. Then, it follows that $\tilde{\beta}_0 \leq \tilde{U}_0(\tilde{P}) / \sigma_0(\tilde{P})$; i.e., condition (33) holds. Note that the fulfillment of the first LMI in (36) is equivalent to inscribing the ellipsoid into the strip between the planes $\sigma(z) = \tilde{U}_0/\tilde{\beta}_0$ and $\sigma(z) = -\tilde{U}_0/\tilde{\beta}_0$; i.e., inequality (33) holds owing to the increasing of its right-hand side, which, in turn, is achieved by contracting the ellipsoid in the "direction" of σ.

6 ALGORITHM

Based on the above, we arrive at the following algorithm for finding the best invariant ellipsoid.

Step 1. Solve the LMI system (19), (26) for the maximum possible β, $\beta = 1$, and check whether the scalar inequality (33) holds. If it holds, the ellipsoid constructed is the desired invariant ellipsoid for the given α_1.

Step 2. Otherwise, solve the LMI system (26), (34) (with $\beta_1 = 1$) for the least possible $\beta = \tilde{\beta}_0$ and check condition (33). If it holds, find a greater value of β in the range $\tilde{\beta}_0 < \beta < 1$ for which (33) holds (Step 3). Otherwise, go to Step 4.

Step 3 is repeated while the difference of two successive values of β is greater than a given threshold. On each iteration, we have a current interval (β_2, β_1) ($\beta_2 < \beta_1$) that contains the desired optimal value. The iteration consists in selecting an intermediate value $\beta_m \in (\beta_2, \beta_1)$ (for example, by bisecting the interval); solving the LMI system (26), (34), and (35); and checking condition (33). Depending on whether this condition holds or does not hold, for the next current interval, either $(\beta_m, \tilde{\beta}(\beta_m))$ or $(\tilde{\beta}(\beta_m), \beta_m)$, where $\tilde{\beta}(\beta_m) \in (\beta_2, \beta_1)$, is taken.

Step 4. Invariant ellipsoid is found by solving the LMI system (26), (36) for $\beta = \tilde{\beta}_0$.

It should be noted that the algorithm finds an invariant ellipsoid for not more than three iterations (on Steps 1, or 2, or 4). Several (in practice, two or three) subsequent iterations are spent for finding an ellipsoid of maximum size for the given α_1. By virtue of monotonicity of estimate $\tilde{\beta}(\beta)$, the iteration process converges to the optimum value of β.

7 NUMERICAL EXAMPLE

For the sake of illustration, the proposed algorithm was applied to constructing invariant ellipsoids in the problem of stabilizing motion of a wheeled robot along a curvilinear trajectory approximating data of GNSS measurements. The same robot with the same control law as in [1] ($\bar{V} = 0.2584$ rad/s, $L = 2.45$ m, $\bar{u} = 0.2$ m^{-1}, and $\lambda = 0.3$)[2] moving with speed $v = 1.5$ m/s was used. The target trajectory was approximated by a cubic B-spline curve [13]. For the trajectory segment in this example, one elementary B-spline with the maximum curvature $\bar{k} = 0.105$ m^{-1} was used. The derivative of curvature for uniform B-splines is a piecewise constant function of arc length, which, in our case, was equal to $k'_s = 0.016$m^{-2}. The admissible deviation from the trajectory was taken equal to $\alpha_1 = 0.5$ m.

[2]The values of parameters $\bar{V}$, L, and $\bar{u}$ correspond to a wheeled robot created in the Javad GNSS company [14] on the basis of the "Niva–Chevrolet" car.

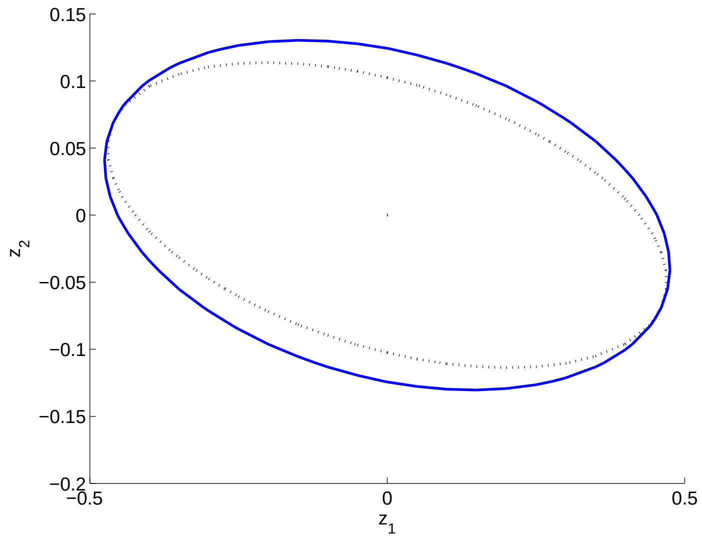

Figure 4. Sections of the invariant ellipsoids corresponding to $\beta = 0.25$ (dashed line) and $\beta = 0.78$ (solid line) by the plane $z_3 = 0$.

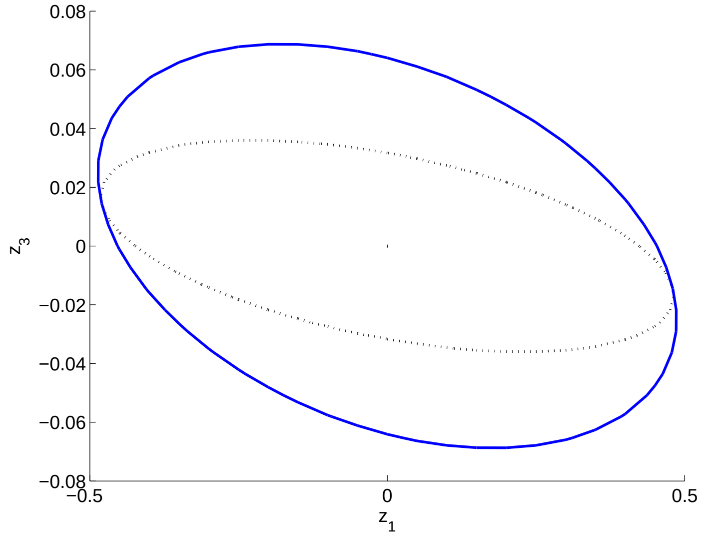

Figure 5. Sections of the invariant ellipsoids corresponding to $\beta = 0.25$ (dashed line) and $\beta = 0.78$ (solid line) by the plane $z_2 = 0$.

Figures 4–6 show sections of two constructed ellipsoids by the planes $z_3 = 0$, $z_2 = 0$, and $z_1 = 0$, respectively, and illustrate, in particular, importance of the problem of finding an optimal β. In accordance with the algorithm described in the previous section, on Step 1, an ellipsoid corresponding to $\beta = 1$ inscribed into the approximating cylinder (not shown in the figures) was constructed. On this ellipsoid, $\tilde{\beta}(1) = 0.62$; i.e., condition (33) does not hold, and, hence, the ellipsoid is not invariant. The ellipsoid

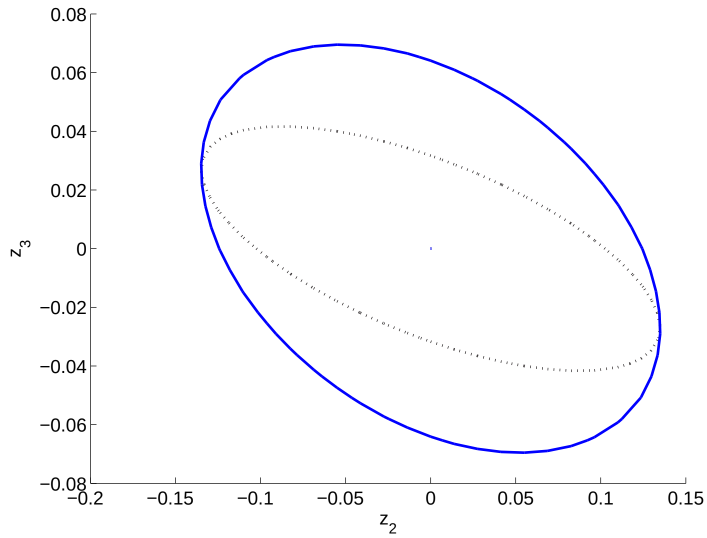

Figure 6. Sections of the invariant ellipsoids corresponding to $\beta = 0.25$ (dashed line) and $\beta = 0.78$ (solid line) by the plane $z_1 = 0$.

constructed on Step 2 (for $\beta = 0.25$) satisfies all assumptions of Theorem 1 and is thus an invariant ellipsoid. In Figs. 4–6, this ellipsoid is depicted by the dashed line.

The "optimal" invariant ellipsoid obtained on Step 3 of the algorithm for $\beta = 0.784$ is depicted in Figs. 4–6 by the solid line. It took only two iterations to find this value, with the right-hand side of inequality (33) being equal to $\tilde{\beta}(0.784) = 0.786$. As can be seen, the resulting invariant ellipsoid is considerably bigger than the first invariant ellipsoid constructed on Step 2.

8 CONCLUSIONS

This work is a sequel to the study on stabilizing motion of a wheeled robot along a given curvilinear trajectory reported in [1]. Theorem 1 proved in [1] gives us a method for constructing an ellipsoid depending on two parameters α_1 and β and checking whether it is invariant. In this paper, an algorithm for constructing the invariant ellipsoids is presented. It is shown how to select admissible values of the above parameters. The value of parameter α_1 (maximum allowed deviation of the robot from the target trajectory) is assigned by the user based on specific features of the applied problem being solved. For a given value of α_1, an admissible value of β is found for not more than three iterations of the algorithm. The problem of finding an optimal value of β for which volume of the invariant ellipsoid is maximal is also solved.

The algorithm discussed was tested on a real wheeled robot created on the basis of the "Niva–Chevrolet" car in the Javad GNSS company [14]. The current state of the robot was measured by a receiver working in the carrier phase differential mode, which ensures centimeter accuracy. The information on whether the current- position of the robot belongs to the current invariant ellipsoid was displayed by means of a color (green/red) indicator to let the operator in the cabin know whether the synthesized control law is capable of stabilizing robot's motion along the given target trajectory.

ACKNOWLEDGMENT

This work was supported by Energetics, Mechanical Engineering, and Control Processes Branch of Russian Academy of Sciences (Program 15) and by the RF Presidential Grant from the Program for the Support of Leading Scientific Schools (project no. NSh-1676.2008.1).

REFERENCES

[1] Pesterev, A.V. and Rapoport, L.B., 2009, "Construction of Invariant Ellipsoids in the Stabilization Problem for a Wheeled Robot Following a Curvilinear Path," *Automation and Remote Control*, Vol. **70** (2), pp. 219–232.

[2] Cordesses, L., Cariou, C., and Berducat, M., 2000, "Combine Harvester Control Using Real Time Kinematic GPS," *Precision Agriculture*, Vol. **2** (2), pp. 147–161.

[3] Thuilot, B., Cariou, C., Martinet, P., and Berducat M., 2002, "Automatic Guidance of a Farm Tractor Relying on a Single CP-DGPS," *Autonomous Robots*, Vol. **13** (1), pp. 53–71.

[4] Samson, C., 1995, "Control of Chained Systems Application to Path Following and Time-Varying Point-Stabilization of Mobile Robots," *IEEE Transactions on Automatic Control*, Vol. **40** (1), pp. 64–77.

[5] Kolmanovsky, I. and McClamroch, N.H., 1995, "Developments in Nonholonomic Control Problems," *IEEE Control Systems Magazine*, 1995, Vol. **15** (6), pp. 20–36.

[6] Rapoport, L.B., 2006, "Estimation of Attraction Domain in a Wheeled Robot Control Problem," *Automation and Remote Control)*, Vol. **67** (9), pp. 1416–1435.

[7] Pesterev, A.V., Rapoport, L.B., and Gilimyanov, R.F., 2008, "Control of a Wheeled Robot Following a Curvilinear Path," *Sixth EUROMECH Nonlinear Dynamics Conference*, St. Petersburg, Russia (June 30–July 4, 2008).

[8] Pyatnitskii, E.S., 1970, "Absolute Stability of Nonstationary Nonlinear Systems," *Avtomatika i Telemekhanika (Automation and Remote Control)*, Vol. **31** (1), pp. 5–15.

[9] Gelig, A.Kh., Leonov, G.A., and Yakubovich, V.A., 1978, *Stability of Nonlinear Systems with a Nonunique Equilibrium State*, Nauka, Moscow (in Russian).

[10] Formal'skii, A.M., 1974, *Controllability and Stability of Systems with Limited Resources*, Nauka, Moscow (in Russian).

[11] Boyd, S., Ghaoui, L.E., Feron, E., and Balakrishnan, V., 1994, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia.

[12] Polyak, B.T. and Shcherbakov, P.S., 2002, *Robast Stability and Control*, Nauka, Moscow (in Russian).

[13] Pesterev, A.V., Rapoport, L.B., and Gilimyanov, R.F., 2007, "Global Energy Fairing of B-Spline Curves in Path Planning Problems," *Proceedings of the ASME 2007 IDETC*, Las Vegas, September 2007, paper no. 35306, CD ROM.

[14] www.javad.com