The use of fuzzy relations in the assessment of information resources producers' performance

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Abstract. The producers assessment problem has many important practical instances: it is an abstract model for intelligent systems evaluating e.g. the quality of computer software repositories, web resources, social networking services, and digital libraries. Each producer's performance is determined according not only to the overall quality of the items he/she outputted, but also to the number of such items (which may be different for each agent).

Recent theoretical results indicate that the use of aggregation operators in the process of ranking and evaluation producers may not necessarily lead to fair and plausible outcomes. Therefore, to overcome some weaknesses of the most often applied approach, in this preliminary study we encourage the use of a fuzzy preference relation-based setting and indicate why it may provide better control over the assessment process.

Keywords: Fuzzy relations, preference modeling, producers assessment problem, StackOverflow, bibliometrics, h-index.

1 Introduction

It is evident that the intensive development of information storage centers causes that their users are likely to suffer from the so-called information overload. As a consequence, there is an urgent need to develop methods for automated quality management of information units as well as their producers. Such a task is of interest in a field of research that deals with measurable aspects of information science, called informetrics.

Let $P = \{p_1, \dots, p_k\}$ be a finite set consisting of k producers. The i-th producer outputs n_i products. Additionally, each product is given some kind of quantitative rating, e.g. concerning its overall quality. Consequently, the state of p_i may be described by a sequence $\mathbf{x}^{(i)} = \left(x_1^{(i)}, \dots, x_{n_i}^{(i)}\right) \in \mathbb{I}^{1,2,\dots} = \bigcup_{n \geqslant 1} \mathbb{I}^n$ with elements in $\mathbb{I}$, e.g. $\mathbb{I} = [0, \infty)$. Most importantly, we should note that the numbers of products may vary from producer to producer. The main aim of

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the *Producers Assessment Problem* (PAP, cf. [4]) is to construct methods for quantitative (numerical) assessment of producers, their ranking, or automatic selection of the most interesting (with respect to some aspects) ones. These computational tools must necessarily meet only some moderate assumptions: they shall somehow take into account a producer's ability to output highly-valuated products, and his/her overall productivity.

Among the most widely-used assessment methods one may find the family of mathematical functions motivated by the introduction of the famous Hirsch h-index [12] or other so-called informetric indices of impact, cf. [1]. Even though they may be used in many important practical problems, it is worth noting that their usage and recognition is, quite unfortunately, often reduced only to the domain of bibliometrics, see [7,13] for some of a few notable exceptions to this rule. Such tools are called aggregation operators and in our setting they are just functions that map the space of vectors of arbitrary length into a single number. Notably, the aggregation theory has a quite long history and its foundations are well-established. For example, due to a strong connection between aggregation operators and monotone measures and integrals, Hirsch-like indices were already studied by Sugeno in [18]. For example, it is known that indices of the form $\mathsf{H}(\mathbf{x}) = \max\{i : \mathsf{w}(x_{(n-i+1)}) \geqslant i\}$, where $x_{(i)}$ is the ith smallest order statistic and $\mathsf{w} : \mathbb{I} \to \mathbb{I}$ is a non-decreasing function, are universal integrals [10,14].

However, it becomes more and more evident that aggregation operators may not provide a proper way to assess information resources producers in PAP. First of all, intuitively, such functions are used to describe particular *aspects* of given numeric vectors, like central tendency, dispersion, or shape of the empirical distribution of data. Although in some cases one easily sees what does an aggregation operator *measure*, e.g. the sample mean describes some central tendency of data or the sample variance reflects its dispersion, it is difficult to tell what in fact do we measure with the h-index.

Moreover, recent results presented in [9] and briefly summarized in Sec. 2 indicate that aggregation operators give us too small control over cases in which we state that a sequence in $\mathbb{I}^{1,2,\cdots}$ is "better" than some other ones. Such an induced order often does not suit our intuition or needs well, c.f. also [3].

Therefore, in Sec. 3 we propose a pairwise comparison-based approach for PAP. As in some cases a decision maker's preferences cannot be expressed precisely, we will study the properties of an exemplary fuzzy preference relation. Then, in Sec. 4 we discuss simple methods to extract useful knowledge from the relation graph, e.g. to obtain a ranking of producers. Importantly, together with the results we are able to obtain some numeric measures of their quality (understood as the degree of conformance of the resulting ranking to the input relation). The discourse is illustrated with a case study consisting of the most active users of StackOverflow. Finally, we conclude the paper in Sec. 5.

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2 Crisp dominance relation for sets of producers

Let us consider the following binary relation $\trianglelefteq \subseteq \mathbb{I}^{1,2,\cdots} \times \mathbb{I}^{1,2,\cdots}$, cf. [9]. For any $\mathbf{x} \in \mathbb{I}^n$ and $\mathbf{y} \in \mathbb{I}^m$ we write $\mathbf{x} \trianglelefteq \mathbf{y}$ (or, equivalently, $(\mathbf{x}, \mathbf{y}) \in \trianglelefteq$) if and only if $n \leqslant m$ and $x_{(n-i+1)} \leqslant y_{(m-i+1)}$ for $i = 1, \ldots, n$. In other words, we say that a producer X is (weakly) dominated by a producer Y, if X has yielded no more products than Y and each of the i-th most highly valuated product by X is of no better quality than the i-th most highly valuated item by Y. We assume that the order of entries of vectors from $\mathbb{I}^{1,2,\cdots}$ is irrelevant by considering order statistics. Of course, $\trianglelefteq$ is a preorder, i.e. it is reflexive and transitive.

What is most important, we have the following result, tightly linking the post-Hirsch "indices of scientific impact" (impact functions) – which take into account the producer's productivity and quality of its products, see e.g. [17,20] – with the above-introduced preorder.

Theorem 1. *Let* $\mathsf{F} : \mathbb{I}^{1,2,\cdots} \to \mathbb{I}$ *be an aggregation operator. Then* F *is symmetric (independent of the order of products in a sequence, i.e.* $(\forall \mathbf{x} \in \mathbb{I}^{1,2,\cdots})$ $\mathsf{F}(\mathbf{x}) = \mathsf{F}(x_{(n)}, \ldots, x_{(1)})$*), nondecreasing with respect to each variable (improvement of a product's quality does not result in a decrease in a producer's valuation, i.e.* $(\forall n)$ $(\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)$ $(\forall i)$ $x_i \leqslant y_i \Longrightarrow \mathsf{F}(\mathbf{x}) \leqslant \mathsf{F}(\mathbf{y})$*) and arity-monotonic (additional elements do not result in a decrease in a producer's valuation, i.e.* $(\forall \mathbf{x} \in \mathbb{I}^{1,2,\cdots})$ $(\forall y \in \mathbb{I})$ $\mathsf{F}(\mathbf{x}) \leqslant \mathsf{F}(\mathbf{x}, y)$*) if and only if for any* $\mathbf{x}, \mathbf{y} \in \mathbb{I}^{1,2,\cdots}$ *if* $\mathbf{x} \trianglelefteq \mathbf{y}$*, then* $\mathsf{F}(\mathbf{x}) \leqslant \mathsf{F}(\mathbf{y})$.

It should be noted that $\trianglelefteq$ represents the information on pairs of vectors which comparison may be performed in such a way that we obtain rationally plausible results. However, it is easily seen that $\trianglelefteq$ is not necessarily total (or complete), i.e. there exist $\mathbf{x}, \mathbf{y} \in \mathbb{I}^{1,2,\cdots}$ such that $\mathbf{x} \not\trianglelefteq \mathbf{y}$ and $\mathbf{y} \not\trianglelefteq \mathbf{x}$. Thus, the linear order $\trianglelefteq''_{\mathsf{F}}$ induced by any impact function F, $\trianglelefteq \subseteq \trianglelefteq''_{\mathsf{F}}$, possibly resolves the comparison problems in a way that is beyond our control. For example, it is known that a *fair* impact function must necessarily be trivial, cf. [9, Theorem 3]: if we would like to obtain $\neg(\mathbf{x} \trianglelefteq \mathbf{y}$ or $\mathbf{y} \trianglelefteq \mathbf{x}) \Longrightarrow \mathsf{F}(\mathbf{x}) = \mathsf{F}(\mathbf{y})$, we surely get $\mathsf{F}(\mathbf{x}) = c$ for some c and all $\mathbf{x}$. On the other hand, for a set of incomparable (with $\trianglelefteq$) vectors $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$, we may always construct an impact function such that $\mathsf{F}(\mathbf{x}^{\sigma(1)}) < \cdots < \mathsf{F}(\mathbf{x}^{\sigma(k)})$, given any permutation σ of the set $\{1, \ldots, k\}$, see [9, Theorem 4]. We see that the minimal requirements for F are too mild. This is partially because the "sure knowledge" represented in $\trianglelefteq$ does not include "almost sure knowledge" for example concerning the comparison results of e.g. $(11, 11)$ vs $(100, 10, 10, 1)$.

Thus, we would like to turn our attention to the extension of the "crisp", $\trianglelefteq$-based, approach to fuzzy preferences. With these means we hope to handle uncertainty and pairwise comparisons in a more subtle way than by the "black and white" crisp setting.

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3 From crisp to fuzzy preference relations

3.1 Fuzzy relations

First we shall recall some notions from fuzzy preference modeling theory, see e.g. [6]. The following definition gives a generalization of a crisp binary relation to a relation in the fuzzy sense. We assume that we are given a set of alternatives A, whose elements are to be compared with one another.

Definition 1. *A fuzzy relation on the set A is a pair (R, μ), where μ is the membership function of R, $\mu : A \times A \to [0, 1]$, measuring the degree to which R holds.*

For brevity we further on write "a relation R" instead of (R, μ) as it should be clear from the context what its membership function is.

With such a tool we may model the concept of partial dominance. It allows us to say that one producer's output is only slightly or indisputably more advantageous than another producer's output by saying that the dominance relation holds between their outputs with a certain degree $\in [0, 1]$ (the membership function in case of a crisp relation is a binary-valued function in $\{0, 1\}$ which means that the relation either holds or does not hold at all).

We say that a relation R is (fuzzy) reflexive if $\mu(a, a) = 1$ for all $a \in A$. We say that a relation R is (fuzzy) total if $\mu(a, b) + \mu(b, a) \geqslant 1$. Note that these definitions naturally extend their crisp counterparts when we consider a crisp relation as a function into $\{0, 1\}$. Additionally, if $\mu(a, b) + \mu(b, a) = 1$, then we say that R is additive reciprocal (or probabilistic).

We are primarily interested in fuzzy preference relations. Thus, we shall recall the notion of (fuzzy) transitivity. There are several definitions of this concept unified with the use of t-norms.

Definition 2. *A t-norm is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ that for any $x, y, z \in [0, 1]$ satisfies the following conditions: (a) $T(1, x) = x$ for all $x \in [0, 1]$, (b) T is symmetric, i.e., $T(x, y) = T(y, x)$, (c) T is non-decreasing, i.e., $T(x, y) \leqslant T(z, y)$ whenever $x \leqslant z$, (d) T is associative, i.e. $T(x, T(y, z)) = T(T(x, y), z)$.*

An example of a t-norm is $T_L(x, y) = \max\{x + y - 1, 0\}$, which is called the Łukasiewicz t-norm. As it is the smallest 1-Lipshitz t-norm, we will adopt it in our considerations for proving transitivity of an appropriate relation.

We are ready to define the composition of fuzzy relations R_1, R_2.

Definition 3. *The t-composition of fuzzy relations (R_1, μ_1) and (R_2, μ_2) w.r.t. a t-norm T is a relation R_3 with the membership function μ_3 given by*

$$\mu_3(a, b) = \sup_{c \in A} T(\mu_1(a, c), \mu_2(c, b)).$$

Again, this definition naturally extends the composition of binary-valued relations. In a crisp situation, a relation R is transitive iff $R^2 = R \circ R \subseteq R$. This motivates the following definition in case of a fuzzy relation: a fuzzy relation (R, μ) is (fuzzy) T-transitive if

$$\mu(a, b) \geqslant \sup_{c \in A} T(\mu(a, c), \mu(c, b)),$$

T is a given t-norm. If $T = T_L$, then we call this property the max-Δ-transitivity, see [16].

Clearly, the transitivity property of fuzzy relations depends on the choice of the corresponding t-norm, see e.g. [5,6,11] for discussion. The given definition will serve us in the construction of a fuzzy preference relation that extends the dominance relation $\trianglelefteq$ in the next subsection.

3.2 An exemplary class of fuzzy dominance relation

Let us consider the space $\mathcal{S}$ of infinite nonincreasing sequences with elements in $\mathbb{I}$. Let $\widetilde{\cdot} : \mathbb{I}^{1,2,\cdots} \to \mathcal{S}$ be an operator such that for $\mathbf{x} \in \mathbb{I}^n$ we have $\widetilde{\mathbf{x}} = (x_{(n)}, x_{(n-1)}, \ldots, x_{(1)}, 0, 0, \ldots)$. It is a quite natural way to embed $\mathbb{I}^{1,2,\cdots}$ into the space of vectors of infinite lengths $\mathbb{I}^\infty$, cf. [10].

Let us propose one (as this is a preliminary study) of the possible approaches to the construction of fuzzy preorders that are in some way concordant with $\trianglelefteq$.

Definition 4. *Let $\mathbf{x}, \mathbf{y} \in \mathcal{S}$, and $\mathbf{w} = (w_1, w_2, \ldots)$, $w_i > 0$ for all i. The **fuzzy producers dominance relation** is a fuzzy preference relation $\blacktriangleleft$ with the membership function given by:*

$$\mu(\mathbf{x}, \mathbf{y}) = \begin{cases} \frac{\pi_{yx}}{\pi_{xy} + \pi_{yx}} & \text{if } \pi_{xy} + \pi_{yx} > 0, \\ 0.5 & \text{otherwise,} \end{cases}$$

where $\pi_{xy} = \sum_i w_i \cdot \max\{x_i - y_i, 0\}$.

Note that the $\mathbf{w}$ vector has a nice interpretation here: it may be used to put bigger weights to the producer's productivity or to make products of high quality more significant, cf. [10] for discussion and more formal treatment in a monotone measure setting. Fig. 1 shows the interpretation of π_{xy} and π_{yx} for $\mathbf{x} = (10, 9, 8, 4, 2, 1, 1)$, $\mathbf{y} = (7, 7, 6, 5, 4, 4, 3, 2, 1, 1)$, and $\mathbf{w} = (1, 1, \ldots)$. Here we have $\pi_{xy} = 7$, $\pi_{yx} = 13$, and $\mu(\mathbf{x}, \mathbf{y}) = 0.65$.

Given such a definition of the preference relation we would like to study its properties. First of all, it is easily seen that this relation is additive reciprocal. We view it as a generalization of the crisp approach and we interpret values of its membership function close to 0.5 as indifference between objects. Whenever we have that $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{x} \trianglelefteq \mathbf{y}$ then $\mu(\mathbf{x}, \mathbf{y}) = 1$ as in the crisp case. More generally, if $\mathbf{x} \trianglelefteq \mathbf{y}$, then $\mathbf{I}(\mu(\mathbf{x}, \mathbf{y}) \geqslant 0.5) = 1$ (0.5 α-cut of $\blacktriangleleft$ is a superset of $\trianglelefteq$). In the Proposition to follow we will show that this relation is also transitive when Łukasiewicz t-norm is considered. Based on these properties, the relation is a fuzzy preference relation in the sense of [19].

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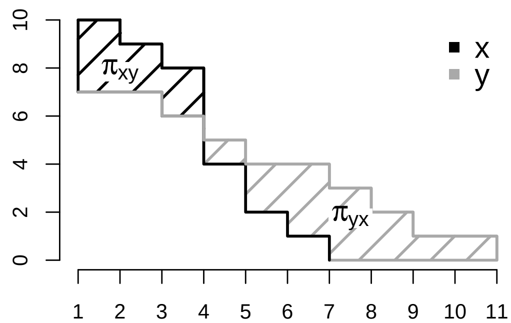

Fig. 1. Illustration of Def. 4; $\mathbf{x} = (10, 9, 8, 4, 2, 1, 1)$, $\mathbf{y} = (7, 7, 6, 5, 4, 4, 3, 2, 1, 1)$.

Proposition 1. *The fuzzy producers dominance relation* $\blacktriangleleft$ *is max-*Δ*-transitive, i.e. for all* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{S}$ *we have*

$$\mu(\mathbf{x}, \mathbf{z}) \geqslant \max\left\{\mu(\mathbf{x}, \mathbf{y}) + \mu(\mathbf{y}, \mathbf{z}) - 1, 0\right\}. \tag{1}$$

Proof. First of all, if any two elements of the set $\{x, y, z\}$ are equal, then the proposition obviously holds. Thus, from now on we assume that all the considered vectors are distinct.

For the sake of simplicity, let us introduce the following notation. Let $a := \pi_{yx}$, $b := \pi_{zy}$, $c := \pi_{xz}$, $a' := \pi_{xy}$, $b' := \pi_{yz}$, $c' := \pi_{zx}$. For instance, we have: $c' = \pi_{zx} = \sum_i w_i \cdot \max\{z_i - x_i, 0\}$, $a = \pi_{yx} = \sum_i w_i \cdot \max\{y_i - x_i, 0\}$, $b = \pi_{zy} = \sum_i w_i \cdot \max\{z_i - y_i, 0\}$. As for any $p, q, r \in \mathbb{R}$ the triangle inequality $\max\{p - r, 0\} \leqslant \max\{p - q, 0\} + \max\{q - r, 0\}$ holds, thus we have $0 \leqslant c' \leqslant a + b$. In an analogous way we derive 5 more inequalities and arrive at a set of 6 constraints:

$$0 \leqslant a \leqslant b' + c' \tag{2}$$
$$0 \leqslant a' \leqslant b + c \tag{3}$$
$$0 \leqslant b \leqslant a' + c' \tag{4}$$
$$0 \leqslant b' \leqslant a + c \tag{5}$$
$$0 \leqslant c \leqslant a' + b' \tag{6}$$
$$0 \leqslant c' \leqslant a + b \tag{7}$$

In terms of the introduced notation, ineq. (1) becomes equivalent to (note we can omit the max operator):

$$\frac{c'}{c + c'} + 1 \geqslant \frac{a}{a + a'} + \frac{b}{b + b'}. \tag{8}$$

After some transformations this may be rewritten as:

$$ab'c' + a'bc' + a'b'c + 2a'b'c' - abc \geqslant 0. \tag{9}$$

The left-hand side of the above inequality may be viewed as a function $f : \mathbb{R}^6 \to \mathbb{R}$, $f(a, a', b, b', c, c') = ab'c' + a'bc' + a'b'c + 2a'b'c' - abc$. Now, to prove that (1) holds, it suffices to show that at all minima of f under constraints (2)–(7) are non-negative. However, the domain of the function given by these constraints is unbounded. To restrict our considerations only to bounded domains we note that we may assume additionally that

$$a + a' + b + b' + c + c' = 1. \tag{10}$$

This is because if $f(\mathbf{h}) < 0$ for some $\mathbf{h} = (a, a', b, b', c, c')$ then we also have $f(\lambda\mathbf{h}) = \lambda^3 f(\mathbf{h}) < 0$ for any $\lambda > 0$ (that is to say the function f is negative in direction $\mathbf{h}$). Scaling with an appropriate factor λ we see that we may assume that (10) holds. Now the subset of $\mathbb{R}^6$ defined by constraints (2)–(7) and (10) is bounded and closed hence a compact set. If f attains non-negative values on this set we will conclude that f is non-negative on its whole domain.

We proceed to show that the function on the above defined set is non-negative. First of all, note that if any of a, b, c equals to 0, then (9), and in consequence (8), obviously holds. Moreover, if $a' = 0$, then (9) becomes

$$ab'c' - abc \geqslant 0,$$

and such an inequality holds, as $c' \geqslant b$ and $b' \geqslant c$ from (4) and (6). For $b' = 0$ and $c' = 0$ we may obtain similar conclusions.

To prove ineq. (9) we will apply the well-known Karush-Kuhn-Tucker (KKT) theorem, see [15]. Basing on the previous considerations, with no loss in generality we may assume that all a, a', b, b', c, c' are positive. In such a situation the constraints of the form $a \geqslant 0, a' \geqslant 0$, etc. are inactive, therefore their corresponding Lagrange multipliers are equal to 0. From now on we should focus only on the constraints given by the second inequalities in (2)–(7) and the constraint (10).

Let us rewrite (2)–(7) in terms of KKT constraint functions; (2) becomes:

$$g_1(a, a', b, b', c, c') = b' + c' - a \geqslant 0,$$

and five other constraints are rewritten analogously as $g_2, \ldots, g_6$. We also have an additional constraint of the form

$$g_7(a, a', b, b', c, c') = a + a' + b + b' + c + c' - 1 = 0.$$

By the KKT theorem, if (a, a', b, b', c, c') is a local minimum of f, then there exist constants $\lambda_i \geqslant 0$, $i = 1, 2, \ldots, 6$, for which:

$$\nabla f^T(a, a', b, b', c, c') = \begin{pmatrix} b'c' - bc \\ bc' + b'c + 2b'c' \\ a'c' - ac \\ ac' + a'c + 2a'c' \\ a'b' - ab \\ ab' + a'b + 2a'b' \end{pmatrix} = \begin{pmatrix} -\lambda_1 + \lambda_4 + \lambda_6 + \lambda_7 \\ -\lambda_2 + \lambda_3 + \lambda_5 + \lambda_7 \\ -\lambda_3 + \lambda_2 + \lambda_6 + \lambda_7 \\ -\lambda_4 + \lambda_1 + \lambda_5 + \lambda_7 \\ -\lambda_5 + \lambda_2 + \lambda_4 + \lambda_7 \\ -\lambda_6 + \lambda_1 + \lambda_3 + \lambda_7 \end{pmatrix} \tag{11}$$

with

$$\lambda_i\, g_i(a, a', b, b', c, c') = 0, \quad i = 1, 2, \ldots, 7. \tag{12}$$

Let us note that from (11) regardless of which constraints are active it holds:

$$\begin{cases} \frac{\partial f}{\partial a} + \frac{\partial f}{\partial b'} = \frac{\partial f}{\partial a'} + \frac{\partial f}{\partial b} \\ \frac{\partial f}{\partial a} + \frac{\partial f}{\partial c'} = \frac{\partial f}{\partial a'} + \frac{\partial f}{\partial c} \\ \frac{\partial f}{\partial b} + \frac{\partial f}{\partial c'} = \frac{\partial f}{\partial b'} + \frac{\partial f}{\partial c}, \end{cases}$$

which yields that

$$(c + c')(a + a' - b - b') = (b + b')(c + c' - a - a') = (a + a')(b + b' - c - c') = 0.$$

Therefore, in combination with (10) it follows that at a minimum we necessarily have:

$$a + a' = b + b' = c + c' = \frac{1}{3}, \tag{13}$$

since we assumed that all the variables are positive. Substituting $a := \frac{1}{3} - a'$, $b := \frac{1}{3} - b'$ and $c := \frac{1}{3} - c'$ in (9) we obtain

$$\left(\frac{1}{3} - a'\right) b'c' + a'\left(\frac{1}{3} - b'\right) c' + a'b'\left(\frac{1}{3} - c'\right) + 2a'b'c'$$

$$- \left(\frac{1}{3} - a'\right)\left(\frac{1}{3} - b'\right)\left(\frac{1}{3} - c'\right) \geqslant 0$$

or

$$a' + b' + c' \geqslant \frac{1}{3}. \tag{14}$$

On the other hand, from (2), (4), (6) and (10) we have that

$$1 = a + a' + b' + c + c' \leqslant a' + b' + c' + 2(a' + b' + c') = 3(a' + b' + c'),$$

which shows that inequality (14) holds. We conclude that at a minimum subject to constraints (2)–(7) and (10) it holds that $f(\cdot) \geqslant 0$. Hence, ineq. (8) holds, and the relation of concern is max-$\varDelta$-transitive, QED.

3.3 Aggregation of preferences

Clearly, information on the comparison results for a given set of n producers $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^n\}$ may be stored in a $[0,1]$-valued $n \times n$ matrix M, where $m_{ij} := \mu(\mathbf{x}^i, \mathbf{x}^j)$ is the value of the membership function of the fuzzy preference relation $\mathbf{x}^i \blacktriangleleft \mathbf{x}^j$. Such a matrix, which we refer to as the preference matrix, has to be processed so that some valuable knowledge may be extracted from it.

This may be achieved e.g. with the net flow method [2,6]. It provides a way of aggregating a preference profile from the preference matrix. This method assigns scores according to the formula

$$S_{\text{net}}(\mathbf{x}^i) = \sum_{\mathbf{x}^j \in \mathcal{X}} \mu(\mathbf{x}^j, \mathbf{x}^i) - \mu(\mathbf{x}^i, \mathbf{x}^j), \tag{15}$$

which in our case reduces to $S_{\text{net}}(\mathbf{x}^i) = \sum_{\mathbf{x}^j \in \mathcal{X}} 2\mu(\mathbf{x}^j, \mathbf{x}^i) - 1$, because $\blacktriangleleft$ is additive reciprocal. This is quite analogous to the classical approach in which the impact functions are used. However, the assigned scores depend on the "environment" $\mathcal{X}$ in which the object $\mathbf{x}^i$ is considered, i.e. the preference matrix of the objects being compared. By ordering the objects with respect to the scores, we obtain the final ranking of a *given* set of producers $\mathcal{X}$.

3.4 Quality of rankings

In this subsection we propose some quality measures for rankings of elements in $\mathcal{X}$. Such measures shall be based on the preference matrix derived from a fuzzy preference relation.

Let $r : \mathcal{X} \to \{1, 2, \ldots, n\}$ be a ranking function. Some objects can be ranked equal, i.e. we may have $r(\mathbf{x}^i) = r(\mathbf{x}^j)$ for some $i \neq j$. We would like to suggest an evaluation (quality) measure Q for ranking r which describes the level of concordance of this ranking with the preference relation $\blacktriangleleft$. We require that the measure has at least the following properties:

1. $Q(r, \blacktriangleleft) \in [0, 1]$, where we assume that 0 and 1 are the lowest and the highest possible quality value, respectively;
2. $Q(r, \blacktriangleleft) = 1$ if $(\forall i, j)$ $\mu(\mathbf{x}^i, \mathbf{x}^j) = 1$ implies $r(\mathbf{x}^i) > r(\mathbf{x}^j)$ (strict preference) and $(\forall i, j)$ $\mu(\mathbf{x}^i, \mathbf{x}^j) = 0.5$ results in $r(\mathbf{x}^i) = r(\mathbf{x}^j)$ (indifference);
3. Accordingly, $Q(r, \blacktriangleleft) = 0$ if $(\forall i, j)$ $\mu(\mathbf{x}^i, \mathbf{x}^j) = 0$ implies $r(\mathbf{x}^i) > r(\mathbf{x}^j)$ and $(\forall i, j)$ $\mu(\mathbf{x}^i, \mathbf{x}^j) = 0$ or $\mu(\mathbf{x}^i, \mathbf{x}^j) = 1$ gives $r(\mathbf{x}^i) = r(\mathbf{x}^j)$.

The following function can constitute an exemplary quality measure:

$$Q(r, \blacktriangleleft) = \frac{\displaystyle\sum_{\substack{i,j:\\ r(\mathbf{x}^i) > r(\mathbf{x}^j)}} \mu(\mathbf{x}^i, \mathbf{x}^j) + \sum_{\substack{i<j:\\ r(\mathbf{x}^i) = r(\mathbf{x}^j)}} 1 - 2\left|\mu(\mathbf{x}^i, \mathbf{x}^j) - \tfrac{1}{2}\right|}{\binom{n}{2}}.$$

4 A case study

In this section we applied the introduced method to the data on the activity of users at the StackOverflow website[4]. StackOverflow allows users to ask or answer questions on various computer programming-related issues. Answers are graded by the community according to their quality and relevance (they may be voted up or down). Thanks to good answers the users have their "reputation" increased. In May 2014, the website has over 3 million users that posted over 7.2 million questions and provided around 13 million answers to them (each question may have several associated answers).

[4] See http://stackoverflow.com. The data of the users' activity is freely available for download at http://data.stackexchange.com/. For the purposes of our study the data were downloaded on April 30, 2014.

In our study we treat answers provided by the users as products and the number of votes as the quality measures of consecutive units. We decided to pick 100 users with the highest number of answers. Notably, these users provided ca. 634,000 answers which is roughly 5% overall. In general, 1% of the users with the greatest number of answers provided answers to the 62.5% of questions.

Since the answers may also be down-voted, some of them received negative score. In such cases we set their quality to 0. This is only a minor correction as the fraction of such answers in the considered group is relatively small (95% of users have at most 1.6% negatively evaluated answers and 7% is the highest fraction in the considered group). After this operation, the quality of each product is contained in the interval $[0, \infty)$.

We suggest several methods for evaluation of producers' output: reputation index compiled by the StackOverflow website to evaluate its users (i_R), mean quality ($\bar{\mathbf{x}}$), maximum of the quality ($\mathbf{x}_{(n)}$), sum of quality of answers ($\Sigma(\mathbf{x})$), number of answers (n), Egghe's g-index (i_G), Hirsch's h-index (i_H), and Woeginger's w-index (i_W), see [1,12,20]. For our base preference relation (denoted NF) we set $(\forall i)\ w_i = 1$ in Def. 4.

The correlations between pairs of rankings generated by the methods of interest are given in Table 1. The measures of rankings' quality are given in Table 2.

Table 1. Kendall's τ correlation coefficients for pairs of rankings obtained by different methods.

	i_R	$\bar{\mathbf{x}}$	$\mathbf{x}_{(n)}$	$\Sigma(\mathbf{x})$	n	i_G	i_H	i_W	NF
i_R	1	0.546	0.543	0.882	0.469	0.696	0.728	0.714	0.882
$\bar{\mathbf{x}}$	0.546	1	0.546	0.602	0.06	0.67	0.667	0.665	0.593
$\mathbf{x}_{(n)}$	0.543	0.546	1	0.562	0.22	0.708	0.606	0.624	0.564
$\Sigma(\mathbf{x})$	0.882	0.602	0.562	1	0.457	0.703	0.72	0.707	0.978
n	0.469	0.06	0.22	0.457	1	0.262	0.27	0.266	0.467
i_G	0.696	0.67	0.708	0.703	0.262	1	0.872	0.892	0.7
i_H	0.728	0.667	0.606	0.72	0.27	0.872	1	0.967	0.714
i_W	0.714	0.665	0.624	0.707	0.266	0.892	0.967	1	0.702
NF	0.882	0.593	0.564	0.978	0.467	0.7	0.714	0.702	1

Table 2. Quality measures of rankings.

i_R	$\bar{\mathbf{x}}$	$\mathbf{x}_{(n)}$	$\Sigma(\mathbf{x})$	n	i_G	i_H	i_W	NF
0.895	0.748	0.749	0.88	0.726	0.8	0.831	0.819	0.874

We see that the StackOverflow's reputation index provides a ranking of the highest quality[5]. Note that the reputation index uses more data than we have

[5] For the details on how reputation is compiled see http://stackoverflow.com/-help/whats-reputation; last access date: May 7, 2014.

employed in our illustration. Note that the highest possible quality ranking has not been listed in Table 2 – we were able to find better rankings according to our evaluation measure by stochastic optimization.

Interestingly, we have that the sum of total scores ($\Sigma(\mathbf{x})$) and net flow method gained the second and the third highest quality and virtually equal results. From Table 1 we see that these methods are highly correlated as indicated by Kendall's τ statistic. Since we employed uniform weights ($w_i = 1$) this is not surprising. In a pairwise comparison we have $\Sigma(\mathbf{x}) < \Sigma(\mathbf{y}) \iff \mu(\mathbf{x}, \mathbf{y}) > 0.5$. However, the net flow method evaluates an output in the "whole environment" as indicated by Eq. (15). In particular, since the rankings generated by the two methods are not concordant, we conclude that the net flow scoring method does not preserve the axiom of independence of irrelevant alternatives. However, we claim that the "environment" in which an object is considered is important during its evaluation process.

The other ranking methods, including bibliometric indices performed worse under our evaluation measure and preference relation. The agreement between different methods vary from 0.06 (number of answers n and average quality of an answer $\bar{\mathbf{x}}$, the latter in fact not being an arity-monotonic aggregation operator) to 0.978 for the already discussed case.

5 Conclusions and future work

In this paper we approached the *Producers Assessment Problem* by fuzzy pairwise comparisons. This method allows us to handle uncertainty in a more subtle way than by the crisp dominance relation approach. Our preliminary results indicate that the derived method can be successfully applied to the problem of producers evaluation.

Note the difference between this approach and the currently most popular one. In the latter case, one postulates an aggregation operator and then discusses how does it rank any possible pair of producers. In our case, we start from a fixed producers set and construct a ranking based on a easy-to-understand fuzzy relation. Here we have access to information on the degree of consistency between the ranking and the pairwise comparison results.

Further research in this area should focus on refinements of the preference relation which can be obtained by e.g. statistical or machine learning methods (utilizing experts' knowledge, for example) rather than given by an explicit formula [8]. Another direction is the construction of sensible quality measures for ranking evaluation specific to this task. These may also serve as cost functions for solving optimization problems of finding a ranking of the highest quality.

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