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Experimental Solution of Flexed Plates by the Method of Caustics

A new experimental technique based on the optical method of caustics for the determination of bending moments in laterally loaded plates is developed. The method involves the direct by the experiment calculation of the partial slopes of the flexed plate or the deflection of the plate in a power series form. Two particular cases were considered to show the potentialities of the method: the simply supported triangular plate bent by moments uniformly distributed along its boundary and the axisymmetric plates. In the latter case the method leads to the direct determination of the curvatures of the plate.

Introduction

Well-known experimental stress analysis methods, such as photoelasticity, interferometry, holography, moiré and strain gauges, have extensively been used for the evaluation of deformations in laterally loaded plates. Although plate bending problems have been solved by these methods, all these present serious limitations and disadvantages which make the application of the methods either cumbersome, or tedious and necessitating elaborate experimental work. Indeed, the novel methods of interferometry and holography [1],[1] giving directly by the experiment the full-field picture of deflections of a flexed plate present the disadvantage that two differentiations are needed in order to obtain the strain and moment distribution of the plate, while methods based on photoelasticity require special model preparation because of the cancellation of the optical effects created by the symmetric and opposite in sign distribution of stresses through the thickness of the plate [2]. The well-known technique of strain gauges presents the limitation of measuring surface strains only and that it is a point by point procedure.

On the other hand the methods based on moiré phenomenon have extensively been used [3] for the experimental solution of laterally loaded plates. These methods were devoted to measure directly by the moiré phenomenon the deflections of the plate [4] by the so-called shadow moiré method or the partial slopes of the flexed plate [5, 6] by the Ligtenberg method or the partial slope contour method developed by Theocaris [7]. Furthermore, the well-known property of the moiré phenomenon for the determination of the derivatives of a family of curves by transversely shifting the family of moiré fringes and creating the moiré of moiré of the two families of curves was applied for the determination of the partial curvatures of the plate from the corresponding slopes [8, 9]. This method of graphical differentiation was introduced in order to avoid numerical differentiation for the determination of partial curvatures of the plate needed for the evaluation of its partial moments, twists and strains, which is a deviating process. However, graphical differentiation is only effective for a dense initial moiré pattern all over the field, a condition which is rarely satisfied with elastically flexed plates. For a critical evaluation of all these moiré methods, their possibilities and disadvantages the reader is referred to reference [3].

Recently, an optical method has been developed by the first author [10] for the experimental determination of the stress-intensity factor and the order of singularity in singular elastic stress fields. This method, called the *method of caustics,* was applied to a large number of such singular stress fields (cracks and other discontinuities, contact and multiwedge problems) in a series of recent publications by the first author and his collaborators [11–13].

The same method has found further applications to nonsingular elastic stress fields in places where stress concentrations appear and proved to be equally effective as an optical stress rosette yielding directly the orientation of the principal stresses and their difference at the point of application of the rosette [14]. The same principle may be used for the study of elastically flexed plates. Thus, in the present paper, this optical method of caustics was used to formulate a new experimental technique for the determination of bending moments in such flexed plates. The method leads to the direct by the experiment calculation of the deflection of the plate in a power series form. Two particular cases, the simply supported equilateral triangular plate bent by moments uniformly distributed along its boundary and the simply supported circular plate loaded by a concentrated load at its center were considered to show the potentialities of the method.

[1] Numbers in brackets designate References at end of paper.

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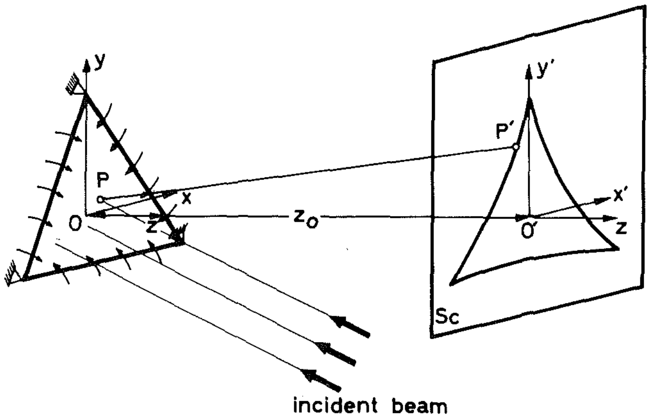

Fig. 1 Geometry of the flexed plate illuminated by a parallel light beam and relative position of the reference screen where the caustic is formed.

It was concluded that the method developed, constitutes a powerful and accurate tool for flexed-plate problems.

The Method of Caustics Used to Determine Slopes in Flexed Plates

Let us consider a thin plate and a system of Cartesian coordinates Oxy related to the plate, Fig. 1. The deflection curve of the plate after the application of a system of lateral loads can be represented by an equation of the form

$$z = f(x, y) \tag{1}$$

where the deflection z is assumed to be small, relatively to the thickness t of the plate. If we consider a parallel light beam normally incident on the polished surface of the plate, the reflected rays are deviated by different amounts according to the slopes at each point of the plate. The deviation vector $\mathbf{w}$ of the reflected ray from a generic point $P(x, y)$ of the deformed surface of the plate on a reference screen Sc placed parallelly to the plane Oxy and at some distance z_0 from it (Fig. 1), is given according to Snell's reflection law by

$$\mathbf{w} = w_x \mathbf{i} + w_y \mathbf{j} \tag{2}$$

with

$$w_x = (z - z_0) \tan 2\alpha; \qquad w_y = (z - z_0) \tan 2\beta$$

$$\tan \alpha = \frac{\partial f(x, y)}{\partial x}; \qquad \tan \beta = \frac{\partial f(x, y)}{\partial y} \tag{3}$$

where $\mathbf{i}$ and $\mathbf{j}$ are Cartesian unit vectors referred to the parallel projection $O'x'y'$ of the frame Oxy on the screen. It is self-evident that in order to avoid the interference of the incident and the reflected light beams the plate is slightly inclined to the direction of incidence. This angle of incidence is enough small, so that its influence on the reflected light rays from the plate may be considered as negligible.

Relations (2) and (3) for our case, where the deflections z of the plate are negligible relative to the distance z_0 and the slopes $\partial f(x, y)/\partial x, y$ are small, can be written in the form

$$\mathbf{w} = \left(-2z_0 \frac{\partial f(x, y)}{\partial x}\right) \mathbf{i} + \left(-2z_0 \frac{\partial f(x, y)}{\partial y}\right) \mathbf{j} \tag{4}$$

Referring vector $\mathbf{w}$ on the origin of the system $O'x'y'$ (Fig. 1) we obtain that the image of any point $P(x, y)$ of the plate on the screen is given by

$$\mathbf{W} = W_x \mathbf{i} + W_y \mathbf{j} \tag{5}$$

with

$$W_x = x - 2z_0 \frac{\partial f(x, y)}{\partial x}; \qquad W_y = y - 2z_0 \frac{\partial f(x, y)}{\partial y}$$

For the more general case of a point light source lying on the z-axis and placed at distance z_i from the plate we obtain for the vector $\mathbf{W}$ [10]:

$$\mathbf{W} = \left[\lambda_m x - 2z_0 \frac{\partial f(x, y)}{\partial x}\right] \mathbf{i} + \left[\lambda_m y - 2z_0 \frac{\partial f(x, y)}{\partial y}\right] \mathbf{j} \tag{6}$$

where λ_m is the magnification factor of the optical arrangement defined by

$$\lambda_m = \frac{z_0 + z_i}{z_i} \tag{7}$$

Relation (6) for the case of a parallel light beam ($z_i \to \infty$, $\lambda_m = 1$) degenerates to relation (5).

We can observe that via relations (5) or (6) to each point $P(x, y)$ of the plate corresponds a point P' (W_x, W_y) on the screen. In order that points P' on the screen Sc belong to a particular curve, the Jacobian determinant of the transformation defined by relations (5) or (6) must be zero. Thus the zeroing of the Jacobian expressed by

$$J = \frac{\partial(W_x, W_y)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial W_x}{\partial x} & \dfrac{\partial W_x}{\partial y} \\ \dfrac{\partial W_y}{\partial x} & \dfrac{\partial W_y}{\partial y} \end{vmatrix} = 0 \tag{8}$$

defines a curve on the plate called the *initial curve*, while the system of equations (5) or (6) and (8) defines on the screen its corresponding *caustic*.

For the case when the slopes of the flexed plate are not negligible, as it was assumed in the foregoing, relations (2) and (3) can be used for the determination of the caustic and its initial curve, instead of the simplified relations (5) or (6). Thus the method presented in the paper can be equally well applied to surfaces with small or large slopes. However, the case of plates with small slopes is examined in the present paper due to its major significance in practical applications.

Relation (8) defines to each position of the reference screen placed at some distance z_0 from the plate an initial curve on the plate, which is connected to the caustic on the reference screen through relations (5). Thus, by placing the reference screen at different distances z_0 from the plate different caustics can be obtained from the reflected light rays from the corresponding initial curve on the plate. This correspondence between the initial curve on the plate and the caustic on the reference screen can be established experimentally, by tracing for example a marked grid on the plate and observing the images of points of the grid on the reference screen. Thus, to each point $P'(W_x, W_y)$ on the screen the corresponding point $P(x, y)$ on the plate can be determined. If this correspondence between pairs of points on the specimen and the screen is established, relations (5) enable the direct calculation of the partial derivatives $\partial f(x, y)/\partial x, y$ at the point $P(x, y)$ on the plate. By placing the reference screen at various distances from the plate the partial derivatives at all points of the plate can be determined. By a single differentiation of the obtained partial slopes of plate its bending moments M_x, M_y, M_{xy} can be determined from the following relations [15]:

$$M_x = -D\left(\frac{\partial^2 f}{\partial x^2} + \nu \frac{\partial^2 f}{\partial y^2}\right), \quad M_y = -D\left(\frac{\partial^2 f}{\partial y^2} + \nu \frac{\partial^2 f}{\partial x^2}\right),$$

$$M_{xy} = -D(1 - \nu) \frac{\partial^2 f}{\partial x \partial y} \tag{9}$$

where D is the flexural rigidity of the plate given by

$$D = \frac{Et^3}{12(1 - \nu^2)} \tag{10}$$

with E the elastic modulus, ν Poisson's ratio, and t the thickness of the plate.

Besides the aforementioned described technique for the direct determination of partial slopes in flexed plates, relations (5) enable the development of another useful technique. Let us consider that the deflection $z = f(x, y)$ of the plate can be written in a power series expansion of the form

$$z = f(x, y) = \sum\sum A_{mn} x^m y^n \tag{11}$$

Substituting this form of deflection into equations (5) we obtain

$$W_x = x - 2z_0 m \sum\sum A_{mn} x^{m-1} y^n,$$
$$W_y = y - 2z_0 n \sum\sum A_{mn} x^m y^{n-1} \tag{12}$$

Relations (12) applied to $m \times n$ points $P'(W_x, W_y)$ of the caustic, whose corresponding points $P(x, y)$ of the initial curve can be determined on the plate, as it was previously indicated, give a system of $m \times n$ linear algebraic equations from which the coefficients A_{mn} can be determined. It is self-evident that by this technique coefficients A_{mn} can be determined either by putting the reference screen at one position from the plate, if we take $m \times n$ points on the caustic at that place of the screen, or by putting the screen at $m \times n$ points and taking one point at each position. This last can be easily materialized if instead of putting the screen at $m \times n$ points, we place the reference screen in a horizontal position and making measurements on the two curves on the screen formed by intersection of the screen plane with the caustic surface. By either technique the coefficients A_{mn} of relation (12) can be easily determined, so that relations (9) enable the calculation of bending moments of plate. The number of $m \times n$ points is selected according to the desired accuracy.

Two particular cases were considered to show the importance of the previously outlined procedure to the experimental solution of flexed plates and to provide an estimation of the accuracy of the method: The simply supported equilateral triangular plate bent by moments uniformly distributed along its boundary and the general case of an axisymmetric plate. In the former case the theoretical and the experimental caustics of the plate are obtained and a comparison between them is made, while in the latter case the aforementioned described technique leads to a direct experimental determination of the curvature of the plate. In this latter case the simply supported circular plate loaded by a concentrated load at its center was examined in detail.

The Triangular Plate

We consider now the caustics formed by illuminating a simply supported equilateral triangular plate bent by moments uniformly distributed along its boundary. The deflection surface of the plate according to the thin plate theory is expressed by the following equation [15] in polar coordinates (r, ϑ) associated to a Cartesian coordinate system with its origin the center of gravity of the triangle and the x-axis coinciding with the height of the triangle.

$$z = f(r, \vartheta) = \frac{M_n}{4aD}\left[r^3 \sin^3\theta - 3r^3 \sin\theta \cos^2\theta - ar^2 + \frac{4}{27}a^3\right] \tag{13}$$

where M_n is the bending moment, a is the length of the height of the triangle and D is the rigidity of the plate given by relation (10).

If a parallel light beam is normally incident on the plate, then we obtain from relations (5) the following equations of the caustic formed on the reference screen:

$$(W_x/a) = \left[(r/a) + \frac{(r/a)}{3(r/a) - 1}\right]\cos\vartheta - \frac{3(r/a)^2}{2[3(r/a) - 1]}\cos 2\vartheta$$
$$(W_y/a) = \left[(r/a) + \frac{(r/a)}{3(r/a) - 1}\right]\sin\vartheta + \frac{3(r/a)^2}{2[3(r/a) - 1]}\sin 2\vartheta \tag{14}$$

By applying relation (8) we obtain for the radius r of the initial curve of the caustic the following equation:

$$(r/a) = \frac{1}{3}\left(\frac{z_0 M_n}{4D} + 1\right) \tag{15}$$

Relation (15) indicates that the initial curve on the plate is a circle, whose center coincides with the center of gravity of the triangle. The geometrical construction of the caustic on the reference screen for $(z_0 M_n / D) = 16.68$ is shown in Fig. 2. As it can readily be concluded from equations (14) the caustic is formed, if from every point $P(r, \vartheta)$ of the initial curve a vector **PR** of magnitude $(r/a)/[3(r/a) - 1]$ and in the direction of the radius **OP** is drawn. From the end of this vector another vector **RQ** of magnitude $1.5(r/a)^2/[3(r/a) - 1]$ is drawn, which subtends an angle $(\pi - 2\vartheta)$ to x-axis. The resultant vector **OQ** gives the point Q of the caustic, which corresponds to the point P of the initial curve. It is shown from Fig. 2 that the shape of the caustic is like a curved-triangle, whose sides are parallel to the sides of the triangular plate. We can also see that the caustic presents cusp-points along the lines $\vartheta = 60°, 180°$, and $300°$.

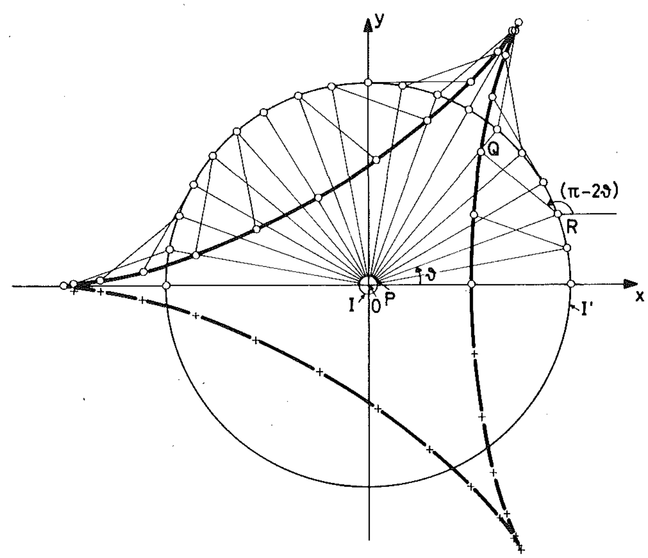

Fig. 2 Caustic and its geometrical construction for a simply supported equilateral triangular plate bent by moments uniformly distributed along the boundary.

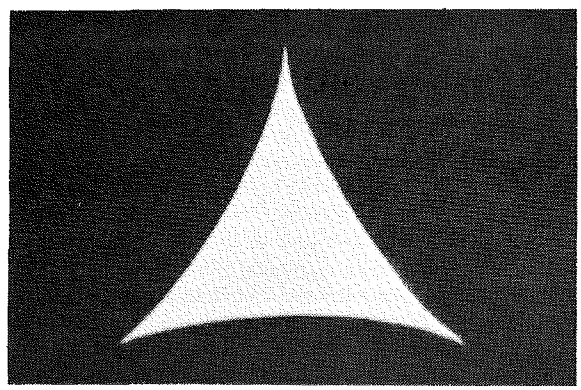

Fig. 3 Experimentally obtained caustic for the case of Fig. 2

For an experimental verification of the foregoing theoretical predictions, and in order to avoid the difficulty involved for the application of uniformly distributed moments along the boundary of the plate, use was made of the analogous problem of a uniformly stretched membrane, the deflections of which are governed by equation (13) if the proportionality factor $(M_n/4aD)$ is replaced by $(q/4aS)$, where q is the uniform pressure applied to the membrane and S is the uniform tension per unit length of its boundary. This membrane was experimentally simulated by a thin plexiglas sheet ($E = 3.2 \times 10^4$ kp/cm^2) of a thickness $t = 0.3$ mm and a height $a = 5$ cm. A uniformly distributed load equal to $q = 0.96$ kp/cm^2 was applied to membrane which was stretched by $S = 6$ kp/cm. For this purpose an airtight box was constructed whose one side was occupied by the thin plate and a uniform load was applied by removing the air from the box. These loading conditions of the membrane correspond to a bending of the triangular plate made of plexiglas with $a = 5$ cm, thickness $t = 0.216$

cm and loaded by uniformly distributed moments equal to $M_n = 5$ kp cm. A He-Ne laser was used to illuminate the plate. The reflected light rays were received on a ground glass screen placed at a distance $z_0 = 103$ cm from the plate. Fig. 3 presents the caustic on the screen. The coincidence of both theoretical and experimental caustics is satisfactory. This can be concluded by comparing Figs. 2 and 3 and taking into account that these figures have been drawn with a scale factor equal to 1:0.65. This proves the potentiality of the method for the solution of flexed plates.

The Axisymmetric Plate

(a) Determination of Bending Moments. For the special case of an axisymmetric plate, whose deflection curve is of the form $z = f(r)$ relations (6) and (8) reduce to

$$W(r) = \lambda_m r - 2z_0 \frac{df(r)}{dr} \tag{16}$$

and

$$\frac{d^2f(r)}{dr^2} = \frac{\lambda_m}{2z_0} \tag{17}$$

Relation (17) indicates that the curvature at the point of the initial curve on the plate, which creates the caustic on the reference screen placed at some distance z_0 from the plate is equal to the reciprocal of the double of the distance between the plate and the reference screen multiplied by the magnification factor λ_m.

As it has been already pointed out it is a simple matter to determine on the plate the particular point (circle for the axisymmetric plate), which corresponds to a given point of the caustic formed on the screen at a distance z_0 from the plate. This can be done, for example, by tracing a set of concentric circles on the plate and observing their reflections on the reference screen. After the determination of the corresponding points on the plate whose plate curvature is equal to $(\lambda_m/2z_0)$ the distribution of the first derivative on the plate can be easily found by a simple integration of the second derivatives, the constant of integration being calculated from the boundary conditions. After the determination of the first and the second derivatives of the deflection of the plate the radial M_r and the tangential M_t components of moment on the plate can be easily determined by [15]

$$M_r = D\left(\frac{d^2f}{dr^2} + \frac{\nu}{r}\frac{df}{dr}\right), \qquad M_t = -D\left(\nu\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right) \tag{18}$$

Besides the previous outlined procedure, the determination of the point on the plate at which the corresponding curvature is given by relation (17) can be made by substituting the differential quantity dr in relation (16) by the differential quantity dz_0 of the distance z_0 between the plate and the reference screen Sc. Indeed, from relations (16) and (17) it can be derived that

$$r = \frac{1}{\lambda_m}\left(W - z_0\frac{dW}{dz_0}\right) \tag{19}$$

Relation (19) enables the experimental determination of point $P(r)$ on the plate whose curvature is given by relation (17). Indeed, the radius W of the caustic on the reference screen Sc at a distance z_0 from the plate can be directly determined experimentally, since the derivative (dW/dz_0) can be approximated by

$$\frac{dW}{dz_0} = \frac{\Delta W}{\Delta z_0} = \frac{W_1 - W_2}{\Delta z_0} \tag{20}$$

where W_1 and W_2 are the radii of the caustics obtained when the reference screen is moved by $(\Delta z_0/2)$ and $(-\Delta z_0/2)$, respectively, from its position at a distance z_0 from the plate.

The exact location of point $P(r)$ on the plate allows the reference of the first derivative (df/dr) at this point and this derivative can be directly determined from relation (16) as follows:

$$\frac{df}{dr} = \frac{\lambda_m r - W}{2z_0} \tag{21}$$

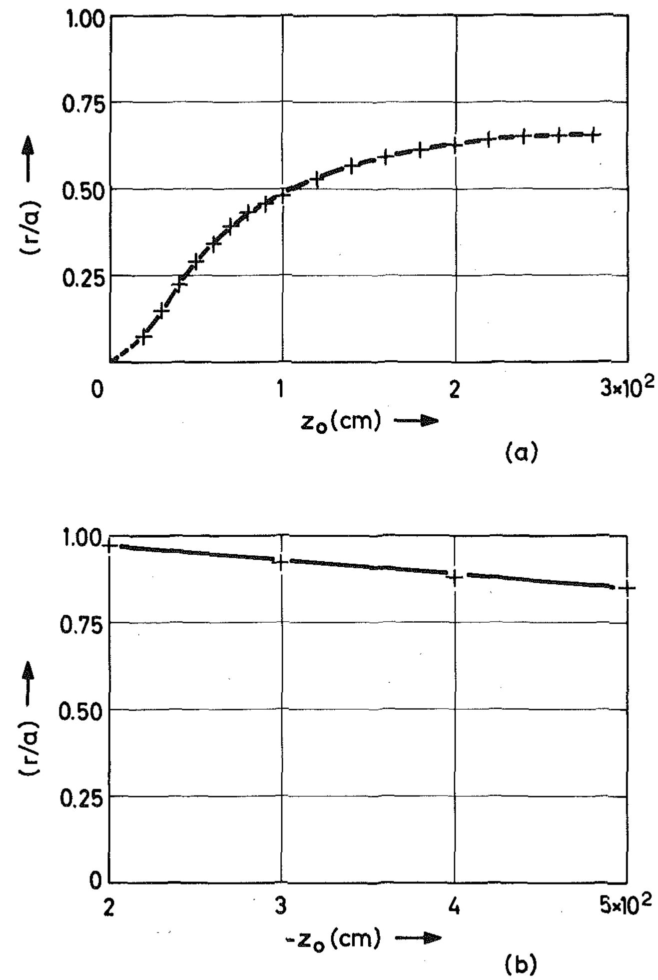

Fig. 4 Variation of the positions (r/a), normalized to the radius a of the plate, on the plate engendering the caustics for various distances z_0 of the reference screen from the plate

From the values of derivatives (df/dr) and (d^2f/dr^2) the bending moments M_r and M_t can be directly calculated from relation (18). The rigidity D of the plate and Poisson's ratio ν can easily be determined by a simple calibration test.

(b) Experimental Evidence. A simply supported circular plate loaded by a concentrated load at its center was used to provide a case of comparison of theoretical and experimental results and assessment of the accuracy of the method. The plate was made from a polymethylmethacrylate (plexiglas) plate of thickness $t = 2.16$ mm. Since plexiglas plates are well polished from the factory, there was no need for further polishing and excellent results were obtained from reflections of the laser light beam on such surfaces. For the eventual case where metallic plates must be studied the polishing of their reflective surface did not necessitate to be very fine since the coherent light beam of laser gave always satisfactory reflections with average hand polishing [13]. The radius a of the plate was $a = 58.1$ mm. The elastic modulus E and Poisson's ratio ν of plexiglas were determined by a tension specimen and found to be: $E = 32{,}000$ kp/cm^2 and $\nu = 0.36$. From these values the rigidity D of the plate, given by relation (10), was found to be $D = 30.88$ kp/cm. An especially designed loading frame was used for supporting the specimen and applying the load.

The experimental arrangement was very simple. A light beam emitted from a He-Ne laser after passing through a system of lenses to become parallel was normally impinging on the specimen. The caustics obtained by illuminating the loaded plate were received on a reference screen at some distance z_0 from the plate. Due to the symmetry of loaded plate the corresponding caustics formed on the

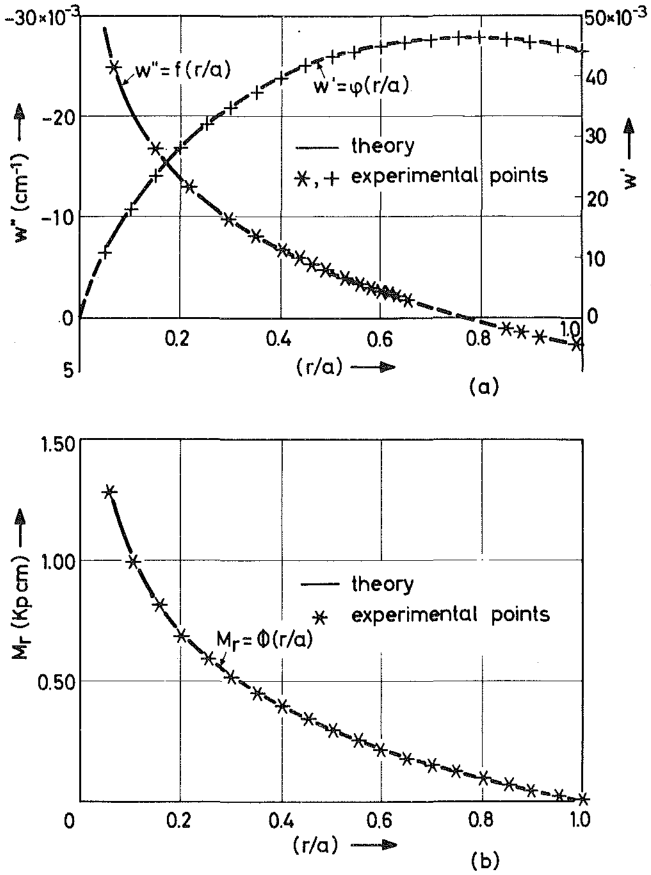

Fig. 5 Variation of the first and second derivatives of the deflection *w* of the plate (*a*) and the radial moment M_r (*b*) along the radius of the circular plate

screen were circles. The radii of these circles however, were varied by moving the screen. The distance z_0 varied between $z_0 = -5 \times 10^3$ and $z_0 = 3 \times 10^3$ mm. At each position of the screen the points on the plate lying on a circle from which the caustic was formed were determined. Very helpful for this determination was the set of concentric circles traced on the specimen before loading whose shadows were depicted on the reference screen. By moving the screen to different distances we can find different points on the plate from which the caustic on the screen was created.

Fig. 4(a) shows the positions on the plate expressed by the normalized radius (r/a) corresponding to different distances z_0 of the reference screen. It can be observed from this figure that by moving the reference screen from the plate, up to infinity, all points of the image of the plate are not covered by the projections of the corresponding initial curves. Thus the reference screen was moved behind the plate. The corresponding points on the plate to various negative positions of the screen are shown in Fig. 4(b). From these two figures it can be concluded that the area of the plate between (r/a) = 0.65 and 0.85 is scanned by placing the reference screen at a distance $z_0 > 300$ cm on the positive position of the screen and $z_0 < -500$ cm on the negative direction. From Fig. 4 the curvature of the flexed plate, expressed by the quantity ($1/2z_0$), can be calculated. Fig. 5(a) shows the values of (d^2w/dr^2) along the radius of the plate. In the same figure the theoretical values of the second derivative of deflection, as these were given in reference [15, p. 68], were also drawn. The comparison of the experimental values of (d^2w/dr^2) with their theoretical values shows a remarkable agreement and proves the accuracy of the method.

By numerical integration of the values of the second derivatives of deflection w the first derivatives were calculated and shown in Fig. 5(a). After the determination of the first and second derivatives of the deflection of the plate the bending moments defined by relations (18) can be determined. The values of M_r along the radius of the plate are shown in Fig. 5(b). In the same figure and for comparison the theoretical values of M_r (formula 90, reference [15, p. 68]) were also drawn.

Conclusions

A simple and potential experimental technique based on the optical method of caustics for the solution of flexed plates was developed. By this method the partial slopes of the flexed plate can be directly calculated. Another alternate possibility is to determine directly by the experiment the coefficients of a power series expansion of the deflection of the plate. Two particular cases were considered: the simply supported triangular plate bent by moments uniformly distributed along the boundary of the plate and the general case of the axisymmetric plate. In the latter case the curvature of the plate can be directly determined by the experiment. In both cases considered experimental evidence showed the potentiality and the high accuracy of the method developed.

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