Extremal Dependence of Copulas: A Tail Density Approach

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Abstract

The extremal dependence of a random vector describes the tail behaviors of joint probabilities of the random vector with respect to that of its margins, and has been often studied by using the tail dependence function of its copula. A tail density approach is introduced in this paper to analyze extremal dependence of the copulas that are specified only by densities. The relation between the copula tail densities and regularly varying densities are established, and the tail densities of Archimedean and t copulas are derived explicitly. The tail density approach becomes especially effective for extremal dependence analysis on a vine copula, for which the tail density can be written recursively in the product form of tail densities of bivariate baseline copulas and densities of bivariate linking copulas.

Key words and phrases: Tail dependence, regularly varying density, multivariate extremes, tail risk, vine copula.

1 Introduction

The dependence among multivariate extremes can be described by the relative decay rate of joint tail probabilities of a random vector with respect to that of tail probabilities of its margins, which, in turn, can be rephrased precisely by using multivariate regular variation

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[33], or alternatively, tail dependence functions of copulas [29, 24]. In this paper, we develop a method based on *copula tail densities* for extremal dependence analysis. Our motivation is two-fold: (1) Tail estimates of risk measures, such as Value-at-Risk (VaR), of aggregate dependent losses often boil down to evaluating integrals of tail densities of copulas (see, e.g., [3, 4, 2, 6, 5, 17]), and (2) some important copulas, such as the t copula and vine copulas, are specified only by densities.

Let $X = (X_1, \dots, X_d)$ be a random vector with distribution F and continuous, univariate margins $F_1, \dots, F_d$. Without loss of generality, we may assume that X is non-negative component-wise. Consider the standard case in which the survival functions $\overline{F}_i(x) := 1 - F_i(x)$, $1 \le i \le d$, of the margins are right tail equivalent; that is,

$$\frac{\overline{F}_i(x)}{\overline{F}_1(x)} = \frac{1 - F_i(x)}{1 - F_1(x)} \to 1, \text{ as } x \to \infty, \ 1 \le i \le d. \tag{1.1}$$

The distribution F or random vector X is said to be (multivariate) regularly varying (MRV) with intensity measure ν if

$$\lim_{t\to\infty} \frac{\mathbb{P}(X \in tB)}{\mathbb{P}(X_1 > t)} = \nu(B), \ \forall \text{ relatively compact sets } B \subset \overline{\mathbb{R}}^d_+ \backslash \{0\}, \tag{1.2}$$

satisfying that $\nu(\partial B) = 0$. The extremal dependence information of X is encoded in the intensity measure ν, which is a Radon measure with homogeneous property $\nu(tB) = t^{-\alpha}\nu(B)$, for all relatively compact subsets B that are bounded away from the origin, where $\alpha > 0$ is known as the tail index. Observe from (1.1) and (1.2) that for any MRV random vector X, and $1 \le i \le d$,

$$\lim_{t\to\infty} \frac{\mathbb{P}(X_i > ts)}{\mathbb{P}(X_i > t)} = \nu((s, \infty] \times \overline{\mathbb{R}}^{d-1}) = s^{-\alpha}\nu((1, \infty] \times \overline{\mathbb{R}}^{d-1}), \ \forall \ s > 0.$$

That is, univariate (non-degenerate) margins have regularly varying right tails. In general, a Borel-measurable function $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is *regularly varying with tail index* $\alpha \in \mathbb{R}$ if and only if

$$g(x) = x^{-\alpha} L(x), \ x \ge 0, \tag{1.3}$$

where $L(t)$ is a slowly varying function with $L(xt)/L(t) \to 1$ as $t \to \infty$ for any $x > 0$. The detailed discussions on univariate and multivariate regular variations can be found in [33]. The extension of MRV beyond the non-negative orthant can be done by using the tail probability of $||X||$, where $||\cdot||$ denotes a norm on $\mathbb{R}^d$, in place of the marginal tail probability in (1.2) (see [33], Section 6.5.5). The case that the limit in (1.1) is any non-zero constant can be easily converted into the standard tail equivalent case by properly rescaling margins.

If the limit in (1.1) is zero or infinity, then some margins have heavier tails than others. One way to overcome this problem is to standardize the margins via marginal monotone transforms, such as the copula method.

A copula C is a multivariate distribution with uniformly distributed margins on $[0,1]$. Sklar's theorem (see, e.g., [23], Section 1.6) states that every multivariate distribution F with margins $F_1, \dots, F_d$ can be written as $F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d))$ for some d-dimensional copula C. In fact, in the case of continuous margins, C is unique and

$$C(u_1, \dots, u_d) = F(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d))$$

where $F_i^{-1}(u_i)$ denotes the quantile function of the i-th margin, $1 \le i \le d$. Let $(U_1, \dots, U_d)$ denote a random vector with $U_i, 1 \le i \le d$, being uniformly distributed on $[0,1]$. The survival copula $\hat{C}$ is defined as follows:

$$\hat{C}(u_1, \dots, u_n) = \mathbb{P}(1 - U_1 \le u_1, \dots, 1 - U_n \le u_n) = \overline{C}(1 - u_1, \dots, 1 - u_n) \tag{1.4}$$

where $\overline{C}$ is the joint survival function of C. The lower and upper tail dependence functions, introduced in [21, 26, 32, 24], are defined as follows,

$$\begin{aligned} b^L(w; C) &:= \lim_{u \to 0^+} \frac{C(uw_i, 1 \le i \le d)}{u}, \\ b^U(w; C) &:= \lim_{u \to 0^+} \frac{\overline{C}(1 - uw_i, 1 \le i \le d)}{u}, \ \forall w = (w_1, \dots, w_d) \in \mathbb{R}_+^d \end{aligned} \tag{1.5}$$

provided that the limits exist. The tail dependence functions are also called the tail copulas in other works (see, e.g., [34, 16]). Obviously, $b^L(w; \hat{C}) = b^U(w; C)$, and thus the results on upper tail dependence can be easily translated into the similar results for lower tail dependence. There exists a close relation between the tail dependence functions of a copula C and its extreme value copulas [24]. The upper extreme value copula C^{UEV} is given by

$$C^{\mathrm{UEV}}(u_1, \dots, u_d) := \lim_{n \to \infty} C^n(u_1^{1/n}, \dots, u_d^{1/n}) = \exp\{-a^U(-\log u_1, \dots, -\log u_d; C)\},$$

where a^U is known as the upper exponent function, and if exists, it is related to the upper tail dependence function as follows, for $w = (w_1, \dots, w_d) \in \mathbb{R}_+^d$,

$$a^U(w; C) := \lim_{u \to 0^+} \frac{\mathbb{P}(U_i > 1 - uw_i, \exists\, i \in \{1, \dots, d\})}{u} = \sum_{\emptyset \ne S \subseteq \{1, \dots, d\}} (-1)^{|S|-1} b_S^U(w_S), \tag{1.6}$$

and here $b_S^U(w_S)$ denotes the upper tail dependence function of the margin C_S of C with indexes in S. Note that if the exponent function $a^U(\cdot; C)$ exists for a d-dimensional copula C, then the exponent function of any multivariate margin $C_S(u_i, i \in S)$ of C,

$$a^U(w_S; C_S) = a^U((w_S, 0_{S^c}); C), \ \emptyset \ne S \subset \{1, \dots, d\}$$

also exists, where 0_{S^c} is the $|S^c|$-dimensional vector of zeros. Therefore, the existence of the exponent function $a^U(\cdot; C)$ guarantees that the upper tail dependence function $b^U(\cdot; C_S)$ of any multivariate margin $C_S(u_i, i \in S)$ of C exists. There are close connections between these tail dependence functions and classical notions in multivariate extreme value theory; for example, the upper exponent function is the so called stable tail dependence function (see, e.g., [9], page 257, or [12], section 6.1.5).

With the copula approach, the intensity measure ν can be decomposed into the scale invariant tail dependence and tail index [30].

Theorem 1.1. Let $X = (X_1, \dots, X_d)$ be a random vector with distribution F and copula C, satisfying (1.1).

1. If F is MRV as defined in (1.2) with intensity measure ν, then

$$b^U(w; C) = \nu\Big(\prod_{i=1}^{d}(w_i^{-1/\alpha}, \infty]\Big), \text{ and } a^U(w; C) = \nu\Big(\big(\prod_{i=1}^{d}[0, w_i^{-1/\alpha}]\big)^c\Big).$$

2. If the limit (1.6) exists and marginal distributions $F_1, \dots, F_d$ are regularly varying, then $F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d))$ is MRV.

Proof. (1) The relations between the intensity measure and tail dependence function are obtained in Theorem 2.3 of [30] with an intensity measure μ that depends on the norm $||\cdot||$ on $\mathbb{R}^d$. Note that all the intensity measures corresponding to an MRV distribution F are equivalent in the sense that any two of them differ only by a constant scaling factor. Clearly μ in Theorem 2.3 of [30] and ν in (1.2) are related as follows

$$\nu(B) = \frac{\mu(B)}{\mu\left((1, \infty] \times \overline{\mathbb{R}}_+^{d-1}\right)}, \ \forall \text{ relatively compact sets } B \subset \overline{\mathbb{R}}_+^d \backslash \{0\}, \text{ with } \nu(\partial B) = 0.$$

The relations among ν, b^U and a^U now follow immediately from Theorem 2.3 of [30].

(2) If the limit (1.6) exists and marginal distributions $F_1, \dots, F_d$ are regularly varying with tail index α, then it follows from the proof of Theorem 2.3 of [30] that

$$\lim_{t \to \infty} \frac{\mathbb{P}\left(X \in t(\prod_{i=1}^d [0, w_i])^c\right)}{\mathbb{P}(X_1 > t)} = a^U((w_1^{-\alpha}, \dots, w_d^{-\alpha}); C) \tag{1.7}$$

exists for all $(w_1, \dots, w_d) \in \mathbb{R}_+^d \backslash \{0\}$. Define the Radon measure $\nu(\cdot)$ on $\mathbb{R}_+^d \backslash \{0\}$ generated by $\nu((\prod_{i=1}^d [0, w_i])^c) := a^U((w_1^{-\alpha}, \dots, w_d^{-\alpha}); C)$. Using the standard approximation procedure (see, e.g., Lemma 6.1 in [33]), (1.2) follows from (1.7) and thus F is MRV. □

The tail dependence function and intensity measure are equivalent in extremal dependence analysis in the sense that the Radon measure generated by the tail dependence function is a marginally rescaled version of the intensity measure. Note, however, that the tail dependence function and intensity measure are cumulative in nature. A notion that describes extremal dependence locally is the tail density of multivariate regular variation studied in [13]. Consider again a distribution F with tail equivalent margins (1.1) and a norm $||\cdot||$ on $\mathbb{R}^d$.

Theorem 1.2. (de Haan and Resnick, [13]) Assume the density f of F exists and the margins F_i, $1 \le i \le d$, are regularly varying with tail index $\alpha > 0$. If $\frac{f(tx)}{t^{-d}\overline{F}_1(t)} \to \lambda(x) > 0$, as $t \to \infty$, on $\overline{\mathbb{R}}^d_+ \backslash \{0\}$ and uniformly on $\{x > 0 : \ ||x|| = 1\}$ where $\lambda(\cdot)$ is bounded, then, for any $x \in \mathbb{R}^d_+ \backslash \{0\}$,

$$\lim_{t\to\infty} \frac{1 - F(tx)}{\overline{F}_1(t)} = \nu([0, x]^c) = \int_{[0,x]^c} \lambda(y)dy,$$

with homogeneous property that $\lambda(tx) = t^{-\alpha-d}\lambda(x)$ for $t > 0$.

The tail density $\lambda(\cdot)$ in Theorem 1.2 is especially tractable for the distributions that are specified by densities. The goal of this paper is to introduce the tail densities for copulas and derive explicitly the tail densities of the copulas that are specified only by densities, such as the t copula, and vine copulas that are built from bivariate linking copulas using local dependence properties. The rest of this paper is organized as follows. Section 2 introduces the copula tail density, and discusses its properties and the relation with the tail density of multivariate regular variation. An application on the asymptotic expressions of VaR in terms of tail densities is also highlighted in Section 2. Section 3 discusses tail densities for various copulas, including t and D-vine copulas, and finally, some remarks in Section 4 conclude the paper.

2 Tail Densities of Copulas

Let C be a copula with lower and upper tail dependence functions (1.5) and density function c that is continuous on its support. As in [13, 24], we need to impose the uniform convergence condition to ensure the exchanges of limits. Assume that any partial derivative of order d or less for the ratios

$$\frac{C(uw_i, 1 \le i \le d)}{u}, \text{ and } \frac{\overline{C}(1 - uw_i, 1 \le i \le d)}{u} \tag{2.1}$$

converges uniformly on $\mathbb{R}^d_+\backslash\{0\}$ as $u \to 0$. In fact, it is sufficient to assume the uniformity condition for the first ratio. Note that most copulas that are specified by densities satisfy this technical condition on uniform convergence.

For $w = (w_1, \dots, w_d) \in \mathbb{R}^d_+$, let $D_w = \frac{\partial^d}{\partial w_1 \cdots \partial w_d}$ denote the d-order partial differentiation operator with respect to $w_1, \dots, w_d$. Consider the generalized density of the upper tail dependence function $b^U(w; C)$:

$$\frac{\partial^d b^U(w;C)}{\partial w_1 \cdots \partial w_d} = D_w \left(\lim_{u\to 0} \frac{\overline{C}(1-uw_i, 1 \le i \le d)}{u} \right), \quad w = (w_1, \dots, w_d) \in \mathbb{R}^d_+.$$

Since $\overline{C}(1-uw_i, 1 \le i \le d) = 1 - \sum_{\emptyset \neq S \subseteq \{1,\dots,d\}} (-1)^{|S|-1} C_S(1-uw_i, i \in S)$, where C_S denotes the margin of C with indexes in S, we have

$$D_w\overline{C}(1-uw_i, 1 \le i \le d) = (-1)^d D_w C(1-uw_i, 1 \le i \le d) = u^d c(1-uw_i, 1 \le i \le d).$$

Under the uniform convergence assumption (2.1), we can exchange the order of limit and derivative as follows,

$$\begin{aligned} \frac{\partial^d b^U(w;C)}{\partial w_1 \cdots \partial w_d} &= \lim_{u\to 0} \frac{D_w\overline{C}(1-uw_i, 1 \le i \le d)}{u} \\ &= \lim_{u\to 0} u^{d-1} c(1-uw_i, 1 \le i \le d), \quad w = (w_1, \dots, w_d) \in \mathbb{R}^d_+. \end{aligned} \tag{2.2}$$

The limiting function $\lim_{u\to 0} \frac{D_w\overline{C}(1-uw_i, 1\le i\le d)}{u}$ is called the *tail density function* for copula C. More precisely, the lower and upper tail density functions, denoted by $\lambda^L(\cdot; C)$ and $\lambda^U(\cdot; C)$ respectively, are defined as follows:

$$\lambda^L(w;C) := \lim_{u\to 0} \frac{D_w C(uw_i, 1 \le i \le d)}{u}, \quad w = (w_1, \dots, w_d) \in \mathbb{R}^d_+ \tag{2.3}$$

$$\lambda^U(w;C) := \lim_{u\to 0} \frac{D_w\overline{C}(1-uw_i; 1 \le i \le d)}{u}, \quad w = (w_1, \dots, w_d) \in \mathbb{R}^d_+, \tag{2.4}$$

provided that the limits exist.

2.1 Properties of Tail Densities

First of all, the following expressions for tail densities are immediate from the uniform convergence condition (2.1) and exchanging limits as in (2.2).

Proposition 2.1. Let C be a copula with lower and upper tail dependence functions (1.5) and continuous density function c, satisfying (2.1).

1. $\lambda^L(w;C) = \lim_{u\to 0} u^{d-1} c(uw_i, 1 \le i \le d) = \frac{\partial^d b^L(w;C)}{\partial w_1 \cdots \partial w_d}$.

2. $\lambda^U(w;C) = \lim_{u\to 0} u^{d-1} c(1-uw_i, 1\le i\le d) = \frac{\partial^d b^U(w;C)}{\partial w_1\cdots\partial w_d}$.

Under the assumption (2.1), any partial derivative of order d or less for tail dependence functions are continuous. In addition, the tail dependence functions are grounded (Proposition 2.1, [24]); that is, these tail dependence functions are equal to zero if some variables take zero. Thus Proposition 2.1 implies that for any $w = (w_1, \dots, w_d)$,

$$b^L(w;C) = \int_0^{w_1} \cdots \int_0^{w_d} \lambda^L(x;C)dx, \text{ and } b^U(w;C) = \int_0^{w_1} \cdots \int_0^{w_d} \lambda^U(x;C)dx.$$

The tail density functions describe the densities of multivariate extremes. Most frequently used copulas have explicit densities, and using Proposition 2.1, their tail densities can be obtained from copula densities with relative ease. Observe from (1.4) that $\lambda^L(w;\hat{C}) = \lambda^U(w;C)$ for any copula C, where $\hat{C}$ is the survival copula. Since any result regarding $\lambda^U(\cdot;C)$ can be translated via this duality into a similar result for $\lambda^L(\cdot;C)$ and vice versa, we hereafter only discuss one case in details and state the main results involving the other case without proof. We also use frequently the simplified notations $\lambda^L(w)$ and $\lambda^U(w)$, $a^L(w)$ and $a^U(w)$, or $b^L(w)$ and $b^U(w)$ when no confusion arises.

It follows from (1.6) that $\frac{\partial^d a^U(w)}{\partial w_1\cdots\partial w_d} = (-1)^{d-1}\frac{\partial^d b^U(w)}{\partial w_1\cdots\partial w_d}$, which, together with Proposition 2.1, implies that

$$\lambda^U(w) = \frac{\partial^d b^U(w)}{\partial w_1\cdots\partial w_d} = (-1)^{d-1}\frac{\partial^d a^U(w)}{\partial w_1\cdots\partial w_d}, \quad w = (w_1,\dots,w_d) \in \mathbb{R}^d_+. \tag{2.5}$$

Similarly,

$$\lambda^L(w) = \frac{\partial^d b^L(w)}{\partial w_1\cdots\partial w_d} = (-1)^{d-1}\frac{\partial^d a^L(w)}{\partial w_1\cdots\partial w_d}, \quad w = (w_1,\dots,w_d) \in \mathbb{R}^d_+, \tag{2.6}$$

where $a^L(w) = \sum_{\emptyset\ne S\subseteq\{1,\dots,d\}} (-1)^{|S|-1} b^L_S(w_S)$ and $b^L_S(w_S)$ denotes the lower tail dependence function of the margin C_S of C with indexes in $S \subset \{1,\dots,d\}$.

Proposition 2.2. Let C be a copula with lower and upper tail dependence functions (1.5) and continuous density function c, satisfying (2.1).

1. The tail density functions are homogeneous of order $1-d$; that is, $\lambda^U(tw) = t^{1-d}\lambda^U(w)$ and $\lambda^L(tw) = t^{1-d}\lambda^L(w)$ for any $t > 0$ and $w = (w_1,\dots,w_d) \in \mathbb{R}^d_+$.

2. If a d-dimensional tail density ($d > 1$) is non-zero and differentiable, then it is directionally decreasing and directionally convex, and it reaches ∞ at the origin and goes down to zero only at ∞.

Proof. We prove the results for λ^U only.

(1) For any number $t \geq 0$ and $w = (w_1, \dots, w_d) \in \mathbb{R}^d_+$, we have

$$\begin{aligned}\lambda^U(tw) &= \lim_{u\to 0} u^{d-1} c(1 - tuw_i, 1 \leq i \leq d) = \lim_{u\to 0} \frac{(tu)^{d-1}}{t^{d-1}} c(1 - tuw_i, 1 \leq i \leq d) \\ &= t^{1-d} \lim_{v\to 0} v^{d-1} c(1 - vw_i, 1 \leq i \leq d) = t^{1-d}\lambda^U(w).\end{aligned}$$

(2) A direct consequence of the homogeneity property is the Euler representation. Since $\lambda^U(\cdot)$ is differentiable, then the well-known Euler's homogeneous theorem implies that

$$(1-d)\lambda^U(w) = \sum_{j=1}^{d} w_j \frac{\partial \lambda^U}{\partial w_j}, \quad \forall w = (w_1, \dots, w_d) \in \mathbb{R}^d_+. \tag{2.7}$$

That is, for all $w \in \mathbb{R}^d_+$, along the ray through w originated from 0, the directional derivative of $\lambda^U(w)$ is non-positive.

It is shown in [24] that the tail dependence function $b^U(\cdot)$ is either identically zero or positive everywhere. We now show that this is also true for $\lambda^U(\cdot)$. If $\lambda^U(w) = 0$ for some $w \in \mathbb{R}^n_+$, then $\lambda^U(tw) = t^{1-d}\lambda^U(w) = 0$ for any $t > 0$. Since $\lim_{t\to 0} \lambda^U(tw) = 0$, then $\lambda^U(x) = 0$ for all $x \geq 0$. That is, $\lambda^U(\cdot)$ is either identically zero or positive everywhere.

For $\lambda^U(\cdot) > 0$, (2.7) implies that $\lambda^U(\cdot)$ is strictly directionally decreasing and directionally convex along all the rays originated from 0. Since $d > 1$, the homogeneity property implies that $\lambda^U(\cdot)$ reaches ∞ at the origin and goes down to zero only at ∞. □

A copula C is said to be upper (lower) *tail dependent* if its upper (lower) tail density is non-zero.

Theorem 2.3. Assume that F is a distribution with tail equivalent, continuous margins F_i, $1 \leq i \leq d$. If the marginal density f_i of F_i, $1 \leq i \leq d$, is regularly varying with tail index $\alpha + 1$, $\alpha > 0$, and the copula C of F satisfies the condition (2.1), then F is multivariate regularly varying with tail density $\lambda(\cdot)$ that is related to the upper tail density $\lambda^U(\cdot)$ of C as follows:

$$\begin{aligned}\lambda(w_1, \dots, w_d) &= \alpha^d (w_1 \cdots w_d)^{-\alpha-1} \lambda^U(w_1^{-\alpha}, \dots, w_d^{-\alpha}) \\ &= \lambda^U(w_1^{-\alpha}, \dots, w_d^{-\alpha}) |J(w_1^{-\alpha}, \dots, w_d^{-\alpha})|,\end{aligned} \tag{2.8}$$

where $J(w_1^{-\alpha}, \dots, w_d^{-\alpha})$ is the Jacobian determinant of the homeomorphic transform $y_i = w_i^{-\alpha}$, $1 \leq i \leq d$.

Proof. Let c denote the density of copula C, and then the density f of F is given by

$$f(x) = c(F_1(x_1), \dots, F_d(x_d)) \prod_{i=1}^{d} f_i(x_i), \; x = (x_1, \dots, x_d) \in \mathbb{R}^d_+.$$

Consider

$$f(tx) = c(F_1(tx_1), \dots, F_d(tx_d)) \prod_{i=1}^{d} f_i(tx_i), \ t > 0, \ x = (x_1, \dots, x_d) \in \mathbb{R}^d_+. \qquad (2.9)$$

Because of the regularly varying property of the tail equivalent margins, for sufficiently large $t > 0$, we have

$$f_i(tx_i) = t^{-\alpha-1}(x_i^{-\alpha-1} L_i(tx_i)) \approx t^{-\alpha-1} L_1(t) x_i^{-\alpha-1}, \ 1 \le i \le d.$$

Due to Karamata's theorem (see Theorem 2.1 in [33]), the margin F_i, $1 \le i \le d$, is regularly varying with tail index α and

$$F_i(tx_i) \approx 1 - \alpha^{-1}(tx_i) f_i(tx_i) \approx 1 - \alpha^{-1} t^{-\alpha} L_1(t) x_i^{-\alpha}, \ 1 \le i \le d.$$

Plug these tail estimates into (2.9) with $u = \alpha^{-1} t^{-\alpha} L_1(t) \to 0$ as $t \to \infty$, and we have

$$\begin{aligned} \frac{f(tx)}{t^{-d}\overline{F}_1(t)} &= \frac{\alpha^d t^{-d} u^d (\prod_{i=1}^d x_i^{-\alpha-1}) c(1 - u x_1^{-\alpha}, \dots, 1 - u x_d^{-\alpha})}{t^{-d} u} \\ &= \alpha^d \Big(\prod_{i=1}^{d} x_i^{-\alpha-1} \Big) u^{d-1} c(1 - u x_1^{-\alpha}, \dots, 1 - u x_d^{-\alpha}), \end{aligned}$$

which, via (2.1) and (2.2), converges uniformly on $\mathbb{R}^d_+ \backslash \{0\}$ as $t \to \infty$ or equivalently $u \to 0$. By Theorem 1.2, F is regularly varying with intensity measure ν and tail density λ, and for any $x \in \overline{\mathbb{R}}^d_+ \backslash \{0\}$,

$$\lim_{t\to\infty} \frac{1 - F(tx)}{\overline{F}_1(t)} = \nu([0, x]^c) = \int_{[0,x]^c} \lambda(y) dy.$$

Since ν is a Radon measure, we have that $\nu((x, \infty]) = \int_{(x,\infty]} \lambda(y) dy$. It follows from Theorem 1.1 that for any $w = (w_1, \dots, w_d) \in \mathbb{R}^d_+$,

$$b^U(w_1^{-\alpha}, \dots, w_d^{-\alpha}) = \int_{(w,\infty]} \lambda(y) dy,$$

By taking the derivatives on both sides with respect to $w_1, \dots, w_d$, (2.8) follows from Proposition 2.1. □

Remark 2.4. 1. It was shown in [13] that the condition in Theorem 1.2 implies that $\frac{f(tx)}{t^{-d}\overline{F}_1(t)}$ converges, as $t \to \infty$, uniformly on $\{x : ||x|| > \delta\}$ for any small $\delta > 0$, where $|| \cdot ||$ denotes any norm on $\mathbb{R}^d_+$. In contrast, the assumption in Theorem 2.3 is slightly stronger and implies that $\frac{f(tx)}{t^{-d}\overline{F}_1(t)}$ converges, as $t \to \infty$, uniformly on $\mathbb{R}^d_+ \backslash \{0\}$.

2. It follows from Karamata's theorem that the regularly varying property of the density f_i on $[0, \infty)$ implies that the marginal distribution F_i is regularly varying. Conversely, however, the regularly varying property of a marginal distribution F_i on $[0, \infty)$ implies that the density f_i is regularly varying if f_i is monotone on $[0, \infty)$ (see Proposition 2.5 in [33]). In fact, it can be easily seen that if f_i is asymptotically monotone in a left neighborhood of ∞, then the regularly varying property of a marginal distribution F_i ensures the regular variation of the density f_i. Therefore, the regularly varying assumption on marginal densities imposed in Theorem 2.3 is slightly stronger than the regularly varying condition on the margins F_i.

The tail densities of Archimedean copulas follow immediately from Proposition 2.1, and the tail dependence functions of Archimedean copulas. The expressions of these tail dependence functions were derived in [18, 6, 10] (also see Propositions 2.5 and 3.3 in [24]).

Proposition 2.5. Let $C(u; \phi) = \phi(\sum_{i=1}^{d} \phi^{-1}(u_i))$ be an Archimedean copula with strict generator ϕ^{-1}, where ϕ is regularly varying at ∞ with tail index $\theta > 0$. The lower tail dependence function and lower tail density of C are given by

$$b^L(w) = \Big(\sum_{j=1}^{d} w_j^{-1/\theta}\Big)^{-\theta}, \ \lambda^L(w) = \prod_{i=2}^{d}\Big(1 + \frac{i-1}{\theta}\Big)\Big(\prod_{i=1}^{d} w_i\Big)^{-1-1/\theta}\Big(\sum_{i=1}^{d} w_i^{-1/\theta}\Big)^{-\theta-d}.$$

Proposition 2.6. Let $C(u; \phi) = \phi(\sum_{i=1}^{d} \phi^{-1}(u_i))$ be an Archimedean copula where the generator ϕ^{-1} is regularly varying at 1 with tail index $\beta > 1$. The upper exponent function and upper tail density of C are given by

$$a^U(w) = (\sum_{j=1}^{d} w_j^{\beta})^{1/\beta}, \ \lambda^U(w) = \prod_{i=2}^{d}((i-1)\beta - 1)\Big(\prod_{i=1}^{d} w_i\Big)^{\beta-1}\Big(\sum_{i=1}^{d} w_i^{\beta}\Big)^{-d+1/\beta}.$$

Note that the extremal behavior of Archimedean copulas can be deduced from their stochastic representation as the survival copulas of l_1-symmetric distributions, and the extremal behavior of the radial part (or scale mixing) of the representation is determined by its so called Williamson d-transform [31, 15]. An example involving a copula with upper tail dependence is given by the Gumbel copula.

Example 2.7. Consider the bivariate Gumbel copula $C(u_1, u_2; \delta) = \exp\{-[(-\ln u_1)^{\delta} + (-\ln u_2)^{\delta}]^{\frac{1}{\delta}}\}, \delta > 1$. The Gumbel copula is an Archimedean copula with the Laplace transform $\phi(s) = \exp\{-s^{1/\delta}\}$ and generator $\phi^{-1}(t) = (-\log t)^{\delta}$, which is regularly varying at 1 with tail index $\delta > 1$. It follows from Proposition 2.6 that the upper tail density function

$$\lambda^U(w) = \lim_{u \to 0} uc(1 - uw_1, 1 - uw_2) = (\delta - 1)w_1^{\delta-1}w_2^{\delta-1}(w_1^{\delta} + w_2^{\delta})^{\frac{1}{\delta}-2},$$

for any $\delta > 1$. □

An example of the copulas with lower tail dependence is the Clayton copula.

Example 2.8. Consider a bivariate Clayton copula $C(u,v;\theta) = (u^{-\delta} + v^{-\delta} - 1)^{-\frac{1}{\delta}},\ \delta > 0$. This is an Archimedean copula with Laplace transform $\phi(s) = (1+s)^{-1/\delta}$, which is regularly varying at ∞ with tail index $1/\delta$. Therefore the lower tail density function is given via Proposition 2.5 by

$$\lambda(\boldsymbol{w}) \;=\; (1+\delta)(w_1 w_2)^{-\delta-1}(w_1^{-\delta} + w_2^{-\delta})^{-\frac{1}{\delta}-2},$$

for any $\delta > 0$. □

2.2 Approximation of Tail Risk Measures via Tail Densities

The asymptotic analysis of VaR for aggregated dependent losses usually boils down to evaluations of integrals of tail densities of underlying copulas over some upper subsets in $\mathbb{R}^d$. Such an asymptotic analysis was initiated in [35, 3] for aggregated dependent losses with Archimedean copula structure, and VaR estimates were further studied in [2, 6, 5] for loss variables with general copula structures. The tail estimate of VaR under Archimedean dependence was applied in [17] to study the sub- and superadditivity properties of VaR. The tail estimates of VaR for aggregated dependent losses with general multivariate regularly varying distribution can be expressed in terms of the intensity measure $\nu(\cdot)$ [14], which, via Theorems 1.2 and 2.3, can be further expressed as integrals of tail densities of underlying copulas over certain upper subsets in $\mathbb{R}^d$. The asymptotic analysis of tail conditional expectation for dependent losses shares a similar idea (see [25]); that is, tail risks can often be expressed as integrals of tail densities.

We illustrate this idea by deriving the tail asymptotics for the value-at-risk $\mathrm{VaR}_p(||X||)$ (i.e., $100p$th-percentile of $||X||$) of a loss vector X, as $p \to 1$, in terms of tail densities, where $||\cdot||$ denotes any norm on $\mathbb{R}^d_+$ that preserves the component-wise ordering. Consider a non-negative MRV random loss vector $X = (X_1, \dots, X_d)$ with upper tail density λ^U, joint distribution F and continuous margins $F_1, \dots, F_d$ that are tail equivalent with tail index $\alpha > 0$. Consider the following limit:

$$q_{||\cdot||}(\alpha, \lambda^U) := \lim_{t\to\infty} \frac{\mathbb{P}(||X|| > t)}{\overline{F}_1(t)}.$$

This limiting constant depends on the intensity measure ν, which in turn depends on tail index α, tail density λ^U and norm $||\cdot||$. Let G denote the distribution function of $||X||$. As $t \to \infty$,

$$\overline{F}_1(t) \approx [q_{||\cdot||}(\alpha, \lambda^U)]^{-1}\overline{G}(t),$$

hence we have $t \approx \overline{F}_1^{-1}([q_{||\cdot||}(\alpha, \lambda^U)]^{-1}\overline{G}(t))$ as $t \to \infty$. Define $u := \overline{G}(t)$ for sufficiently large t. Then $\overline{G}^{-1}(u) \approx \overline{F}_1^{-1}([q_{||\cdot||}(\alpha, \lambda^U)]^{-1}u)$ for sufficiently small u. Since $\overline{F}_1$ is regularly varying at ∞ with tail index $\alpha > 0$, we have from Proposition 2.6 of [33] that $\overline{F}_1^{-1}(t)$ is regularly varying at 0, or more precisely, $\overline{F}_1^{-1}(uc)/\overline{F}_1^{-1}(u) \to c^{-\frac{1}{\alpha}}$ as $u \to 0^+$ for any $c > 0$. Thus $\overline{F}_1^{-1}([q_{||\cdot||}(\alpha, \lambda^U)]^{-1}u)/\overline{F}_1^{-1}(u) \to q_{||\cdot||}(\alpha, \lambda^U)^{\frac{1}{\alpha}}$. Therefore, $\overline{G}^{-1}(u) \approx q_{||\cdot||}(\alpha, \lambda^U)^{\frac{1}{\alpha}}\overline{F}_1^{-1}(u)$ for sufficiently small u, i.e., $\lim_{u\to 0^+} \overline{G}^{-1}(u)/\overline{F}_1^{-1}(u) = q_{||\cdot||}(\alpha, \lambda^U)^{\frac{1}{\alpha}}$. Replace u by $1 - p$, and we have,

$$\lim_{p\to 1} \frac{\mathrm{VaR}_p(||X||)}{\mathrm{VaR}_p(X_1)} = q_{||\cdot||}(\alpha, \lambda^U)^{\frac{1}{\alpha}}. \tag{2.10}$$

That is, the risk measure $\mathrm{VaR}_p(||X||)$ for a heavy tailed loss vector X can be approximated via $q_{||\cdot||}(\alpha, \lambda^U)^{\frac{1}{\alpha}}\mathrm{VaR}_p(X_1)$ as $p \to 1$, where $\mathrm{VaR}_p(X_1)$ measures the marginal risk, and $q_{||\cdot||}(\alpha, \lambda^U)$ encodes extremal dependence information among losses $X_1, \dots, X_d$ and can be evaluated in terms of the tail density in the next theorem.

Theorem 2.9. Assume that F is a distribution with tail equivalent, continuous margins F_i, $1 \le i \le d$. If the marginal density f_i of F_i, $1 \le i \le d$, is regularly varying with tail index $\alpha + 1$, $\alpha > 0$, and the copula C of F satisfies the condition (2.1), then $q_{||\cdot||}(\alpha, \lambda^U)$ has the following representation

$$q_{||\cdot||}(\alpha, \lambda^U) = \alpha^d \int_W \lambda^U(w_1^{-\alpha}, \dots, w_d^{-\alpha})(w_1 \cdots w_d)^{-\alpha-1} dw \tag{2.11}$$

where $W = \{w \ge 0 : ||w|| > 1\}$.

Proof. It follows from (1.2) that

$$\frac{\mathbb{P}(||X|| > t)}{\mathbb{P}(X_1 > t)} = \frac{\mathbb{P}(X \in tW)}{\mathbb{P}(X_1 > t)} \to \nu(W) = q_{||\cdot||}(\alpha, \lambda^U), \text{ as } t \to \infty,$$

where $W = \{w : ||w|| > 1\}$. On the other hand, it follows from Theorem 2.3 that

$$\nu((x, \infty]) = \alpha^d \int_{(x,\infty]} \lambda^U(w_1^{-\alpha}, \dots, w_d^{-\alpha})(w_1 \cdots w_d)^{-\alpha-1} dw, \ \forall x \in \mathbb{R}_+^d. \tag{2.12}$$

Since $||\cdot||$ preserves the component-wise order, W is an upper subset. The standard approximation and (2.12) lead to

$$\nu(W) = \alpha^d \int_W \lambda^U(w_1^{-\alpha}, \dots, w_d^{-\alpha})(w_1 \cdots w_d)^{-\alpha-1} dw$$

and (2.11) follows. □

Note that under the assumptions of Theorem 2.9, the intensity measure $\nu(\cdot)$ is absolutely continuous with respect to the Lebesgue measure and the tail density $\lambda(\cdot)$ is the unique Radon-Nikodym derivative of $\nu(\cdot)$ with respect to the Lebesgue measure.

Example 2.10. Consider a random loss vector $(X_1, \dots, X_d)$ which satisfies the assumptions of Theorem 2.9.

1. Assume that $(X_1, \dots, X_d)$ has an Archimedean copula $C(u_1, \dots, u_d) = \phi(\sum_{i=1}^d \phi^{-1}(u_i))$, where the generator ϕ^{-1} is regularly varying at 1 with tail index $\beta > 1$. Then Theorem 2.9 and Proposition 2.6 yield

$$q_{||\cdot||}(\alpha, \lambda^U) = \alpha^d \prod_{i=2}^d [(i-1)\beta - 1] \int_W \Big(\prod_{i=1}^d w_i^{-\alpha\beta-1}\Big)\Big(\sum_{i=1}^d w_i^{-\alpha\beta}\Big)^{-d+1/\beta} dw_1 \cdots dw_d$$

   where $W = \{w : ||w|| > 1\}$.

2. Assume that the survival copula of $(X_1, \cdots, X_d)$ is Archimedean with $C(u_1, \cdots, u_d) = \phi(\sum_{i=1}^d \phi^{-1}(u_i))$, where the inverse generator ϕ is regularly varying at ∞ with tail index $\theta > 0$. Then Theorem 2.9 and Proposition 2.5 yield

$$q_{||\cdot||}(\alpha, \lambda^U) = \alpha^d \prod_{i=2}^d \Big(1 + \frac{i-1}{\theta}\Big) \int_W \Big(\prod_{i=1}^d w_i^{\frac{\alpha}{\theta}-1}\Big)\Big(\sum_{i=1}^d w_i^{\frac{\alpha}{\theta}}\Big)^{-\theta-d} dw_1 \cdots dw_d$$

   where $W = \{w : ||w|| > 1\}$. □

Example 2.10 (2) is obtained in [17] for the l_1 norm. Also see [2, 3, 6] for the detailed discussions on tail estimates of aggregated dependent risks.

3 Tail Densities of t and Vine Copulas

In this section, we first derive the tail density of the well-known t copula that is specified by the density. We also derive the tail densities of D-vine copulas that are built from the densities of bivariate linking copulas via local dependence properties.

3.1 Tail Density of t Copula

Consider a d-dimensional symmetric t distribution $t_d(\nu, \Sigma)$ with mean 0 and its density function:

$$f_t(x; \nu, \Sigma) = \frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu}{2})(\nu\pi)^{d/2}} |\Sigma|^{-\frac{1}{2}} \Big[1 + \frac{1}{\nu}(x^\top \Sigma^{-1} x)\Big]^{-\frac{\nu+d}{2}} \tag{3.1}$$

where $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $\nu > 0$ is the degree of freedom, and $\Sigma = (\rho_{ij})$ is a $d \times d$ symmetric dispersion matrix. If a random vector X has the t distribution $t_d(\nu, \Sigma)$, then $X \stackrel{d}{=} \sqrt{R}(Z_1, \dots, Z_d)$, where $(Z_1, \dots, Z_d)$ has a multivariate normal distribution $N(0, \Sigma)$, and the scale variable R, independent of $(Z_1, \ldots, Z_d)$, has an inverse Gamma distribution, which is known to be regularly varying with tail index $\nu/2$ [11].

The one dimensional marginal t distribution has the density

$$f_i(x_i) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})(\nu\pi)^{1/2}}\Big(1 + \frac{x_i^2}{\nu}\Big)^{-\frac{\nu+1}{2}},\ x_i \in \mathbb{R},\ 1 \le i \le d. \tag{3.2}$$

Note that f_i has regularly varying tails with tail index $\nu + 1$. Karamata's theorem (see page 25 of [33]) implies that the margin F_i has a regularly varying right tail with $1 - F_i(x_i) \approx \nu^{-1}x_i^{-\nu}L(x_i, \nu)$ as $x_i \to \infty$, where

$$L(x_i, \nu) \approx \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\nu\pi}}\Big(\frac{1}{x_i^2} + \frac{1}{\nu}\Big)^{-(\nu+1)/2} \to \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\nu\pi}}\nu^{(\nu+1)/2} =: \ell,\ \text{as } x_i \to \infty.$$

The limiting constant $\ell > 0$ is an explicit constant only depending on ν. Set $F_i(x_i) = 1 - uw_i$ then we have $\nu^{-1}x_i^{-\nu}\ell \approx uw_i$ as $u \to 0$. Thus we obtain the following estimates:

$$F_i^{-1}(1 - uw_i) \approx \nu^{-\frac{1}{\nu}}\ell^{\frac{1}{\nu}}(uw_i)^{-\frac{1}{\nu}},\ 1 \le i \le d,\ \text{for sufficiently small } u.$$

Plug these estimates into the t copula density $c(1 - uw_1, \ldots, 1 - uw_d)$ with (3.1) and (3.2), and we obtain that as $u \to 0$,

$$\begin{aligned} c(1 - uw_1, \ldots, 1 - uw_d) &= f_t(F_1^{-1}(1 - uw_1), \ldots, F_d^{-1}(1 - uw_d))\prod_{i=1}^{d}[f_i(F_i^{-1}(1 - uw_i))]^{-1} \\ &\approx u^{1-d}|\Sigma|^{-\frac{1}{2}}\nu^{(1-d)(\frac{\nu}{2}+1)}\ell^{d-1}\frac{\Gamma(\frac{\nu+d}{2})\Gamma^{d-1}(\frac{\nu}{2})}{\Gamma^d(\frac{\nu+1}{2})}\frac{[(w^{-\frac{1}{\nu}})^\top\Sigma^{-1}w^{-\frac{1}{\nu}}]^{-\frac{\nu+d}{2}}}{\prod_{i=1}^d w_i^{\frac{\nu+1}{\nu}}} \end{aligned}$$

where $w^{-1/\nu} = (w_1^{-1/\nu}, \ldots, w_d^{-1/\nu})$. It follows from Proposition 2.1 that the upper tail density function of a multivariate t copula is given below:

$$\begin{aligned} \lambda^U(w) &= |\Sigma|^{-\frac{1}{2}}\nu^{(1-d)(\frac{\nu}{2}+1)}\ell^{d-1}\frac{\Gamma(\frac{\nu+d}{2})\Gamma^{d-1}(\frac{\nu}{2})}{\Gamma^d(\frac{\nu+1}{2})}\frac{[(w^{-\frac{1}{\nu}})^\top\Sigma^{-1}w^{-\frac{1}{\nu}}]^{-\frac{\nu+d}{2}}}{\prod_{i=1}^d w_i^{\frac{\nu+1}{\nu}}} \\ &= |\Sigma|^{-\frac{1}{2}}\nu^{1-d}\frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu+1}{2})\pi^{(d-1)/2}}\frac{[(w^{-\frac{1}{\nu}})^\top\Sigma^{-1}w^{-\frac{1}{\nu}}]^{-\frac{\nu+d}{2}}}{\prod_{i=1}^d w_i^{\frac{\nu+1}{\nu}}}. \end{aligned} \tag{3.3}$$

To get $q_{||\cdot||}(\alpha, \lambda^U)$ for a loss vector with multivariate t copula and regularly varying margins, plug $w_i^{-\alpha}$, $1 \le i \le d$, into the tail density and utilize (2.11) as is shown as follows.

Proposition 3.1. If a non-negative loss vector $X = (X_1, \cdots, X_d)$ has a t copula with degree of freedom ν and dispersion matrix Σ and tail equivalent, regularly varying margins with tail index $\alpha > 0$, then

$$\text{VaR}_p(||X||) \approx q_{||\cdot||}(\nu, \lambda^U)^{\frac{1}{\alpha}}\text{VaR}_p(X_1),\ \text{as } p \to 1,$$

where

$$q_{||\cdot||}(\alpha, \lambda^U) = |\Sigma|^{-\frac{1}{2}} \nu^{1-d} \alpha^d \frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu+1}{2})\pi^{(d-1)/2}} \int_W [(w^{\alpha/\nu})^\top \Sigma^{-1} w^{\alpha/\nu}]^{-\frac{\nu+d}{2}} \prod_{i=1}^{d} w_i^{-1+\alpha/\nu} dw.$$

Here $w^{\alpha/\nu} = (w_1^{\alpha/\nu}, \dots, w_d^{\alpha/\nu})$ and $W = \{w : ||w|| > 1\}$.

If $\alpha = \nu$, then the tail density of the t copula and (2.8) yield the (upper) tail density of the *truncated multivariate t distribution*:

$$\lambda(w) = |\Sigma|^{-\frac{1}{2}} \nu \frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu+1}{2})\pi^{(d-1)/2}} (w^\top \Sigma^{-1} w)^{-\frac{\nu+d}{2}}. \tag{3.4}$$

If a non-negative loss vector $X = (X_1, \cdots, X_d)$ has the truncated multivariate t distribution with degree of freedom ν and dispersion matrix Σ, then

$$\mathrm{VaR}_p(||X||) \approx \mathrm{VaR}_p(X_1) \int_W \lambda(w) dw \text{ as, } p \to 1,$$

where $\mathrm{VaR}_p(X_1)$ is the VaR of the margin with truncated standard t distribution.

Remark 3.2. The tail dependence function of the t copula was derived in [32] using Euler's homogeneity representation. In contrast, the t tail density (3.3) is explicit with nice geometric interpretation. Define $||w||_\Sigma := w^\top \Sigma^{-1} w$, $w \in \mathbb{R}^d$. Clearly $||\cdot||_\Sigma$ is a well-defined norm on $\mathbb{R}^d$. Both tail densities (3.3) and (3.4) share a similar geometric interpretation: the tail density $\lambda(\cdot)$ in (3.4) is a decreasing function of $||w||_\Sigma$, whereas that tail density $\lambda^U(\cdot)$ in (3.3) depends on $||w^{-1/\nu}||_\Sigma$ and the Jacobian determinant of the topologically invariant transform $y_i = w_i^{-1/\nu}$, $1 \le i \le d$. While (3.4) looks simpler than (3.3), $\lambda^U(\cdot)$ captures the scale invariant extremal dependence among multivariate t distributed losses, and, as illustrated in Proposition 3.1, can be applied to the situations with general heavy-tailed margins.

Example 3.3. Consider the bivariate t distribution with identity dispersion matrix and its density function: $f(t_1, t_2) = \frac{1}{2\pi}[1 + \frac{(t_1^2+t_2^2)}{\nu}]^{-(\frac{\nu}{2}+1)}, \nu > 1$, where ν is the degrees of freedom. It follows from (3.3) and (3.4) that

$$\begin{aligned} \lambda^U(w_1, w_2) &= \frac{1}{2}\pi^{-\frac{1}{2}} \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+1}{2})} \frac{(w_1^{-\frac{2}{\nu}} + w_2^{-\frac{2}{\nu}})^{-(1+\frac{\nu}{2})}}{(w_1 w_2)^{1+\frac{1}{\nu}}}, \\ \lambda(w_1, w_2) &= \frac{1}{2}\pi^{-\frac{1}{2}} \nu^2 \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+1}{2})} (w_1^2 + w_2^2)^{-(1+\frac{\nu}{2})}, \end{aligned}$$

for any $w_1 > 0$ and $w_2 > 0$. □

3.2 Tail Densities of Vine Copulas

A vine copula is a copula constructed from a set of $d(d-1)/2$ bivariate copulas by using successive mixing according to a tree structure on finite indexes $1, \dots, d$. The crucial assumption for vine copulas is that given a subset S of variables, where $\emptyset \neq S \subset \{1, \dots, d\}$, the conditional copula that links neighboring variables of S does not depend on the conditioning variables in S. Depending on the types of trees, various vine copulas can be constructed. For example, one boundary case of D-vines are constructed on 1-ary trees and the other boundary case of C-vines are constructed on full $(d-1)$-ary trees. The details of these and other regular vines can be found in [7, 8, 27, 28]. For reasons of simpler notation to show main ideas, we discuss only D-vines here in details, but similar results hold for other vine copulas.

Let C be the d-dimensional copula of a random vector $(U_1, \dots, U_d)$ with density c and uniform margins. We simplify the notations for margins and conditional distributions as follows: Let S and S' be two subsets of $\{1, \dots, d\}$.

1. For any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, denote the S-margin of x by $x_S := (x_j, j \in S)$.
2. Denote the S-marginal density by $c_S := c_S(u_j, j \in S)$ of C with indexes in S.
3. Denote the conditional distribution of U_k conditioning on U_j, $j \in S$, by $C_{k|S} := C_{k|S}(u_k|u_S) = \mathbb{P}(U_k \le u_k \mid U_j = u_j, j \in S)$, $k \notin S$.
4. Denote the conditional density of $U_k, k \in S'$ conditioning on U_j, $j \in S$, by $c_{S'|S} := c_{S \cup S'}/c_S$.

Let $\{K_{ij}, 1 \le i < j \le d\}$ be a set of bivariate linking copulas that constitute basic building blocks. We assume that the density of K_{ij}, denoted by k_{ij}, is continuous, and all K_{ij}s satisfy the uniform convergence properties (2.1). A D-vine copula C of uniform random vector $(U_1, \dots, U_d)$ is constructed recursively in terms of densities as follows.

1. Level 1 (baseline): For any $i = 1, \dots, d-1$, the bivariate margin of (U_i, U_{i+1}) is specified by $C_{i,i+1}$ with density $c_{i,i+1} = k_{i,i+1}$.
2. Level 2: For $i = 1, \dots, d-2$, the conditional distribution of (U_i, U_{i+2}) given the common neighbor U_{i+1} is constructed via copula $K_{i,i+2}$. The marginal distribution of (U_i, U_{i+1}, U_{i+2}) is specified by the density

$$c_{\{i,i+1,i+2\}} = c_{i,i+1} c_{i+1,i+2} k_{i,i+2}\big(C_{i|i+1}, C_{i+2|i+1}\big).$$

3. Level l ($l = 2, \dots, d-1$): Conditioning on $(U_j, i+1 \le j \le i+l-1)$, $i = 1, \dots, d-l$, the conditional distribution of (U_i, U_{i+l}) is constructed via copula $K_{i,i+l}$. The marginal distribution of $(U_j, i \le j \le i+l)$ is specified by the density $c_{\{i,\dots,i+l\}}$ via the following expression:

$$\frac{c_{\{i,\dots,i+l\}}}{c_{\{i+1,\dots,i+l-1\}}} = \frac{c_{\{i,\dots,i+l-1\}}}{c_{\{i+1,\dots,i+l-1\}}} \frac{c_{\{i+1,\dots,i+l\}}}{c_{\{i+1,\dots,i+l-1\}}} k_{i,i+l}\big(C_{i|\{i+1,\dots,i+l-1\}}, C_{i+l|\{i+1,\dots,i+l-1\}}\big).$$

For a D-vine, the linking copula K_{ij} appears at level $(j-i)$. A C-vine, in standard form, is constructed similarly with bivariate linking copulas K_{ij}, $i < j$, at level i. See, for example, [1] for graphical illustrations and a short introduction to D-vines vs. C-vines. A regular vine is more flexible but still has $d-l$ linking copulas at level l, $1 \le l \le d-1$. It is evident that at each level of the construction, the conditional distribution of (U_i, U_{i+l}) given $(U_j, i+1 \le j \le i+l-1)$ has the following simple form:

$$C_{\{i,i+l\}|\{i+1,\dots,i+l-1\}} = K_{i,i+l}\big(C_{i|\{i+1,\dots,i+l-1\}},\, C_{i+l|\{i+1,\dots,i+l-1\}}\big), \tag{3.5}$$

with conditional density

$$\begin{aligned} c_{\{i,i+l\}|\{i+1,\dots,i+l-1\}} = {} & c_{i|\{i+1,\dots,i+l-1\}} c_{i+l|\{i+1,\dots,i+l-1\}} \\ & k_{i,i+l}\big(C_{i|\{i+1,\dots,i+l-1\}},\, C_{i+l|\{i+1,\dots,i+l-1\}}\big), \end{aligned} \tag{3.6}$$

in which the linking copula $K_{i,i+l}$ does not depend on conditioning variables $u_{i+1}, \dots, u_{i+l-1}$. This property of the linking copulas simplifies the dependence structure of vine copulas, leading to recursive expressions for their distributions. Since all the bivariate linking copulas satisfy the uniform convergence properties (2.1), then by induction, the d-dimensional D-vine copula C satisfies (2.1).

To obtain the tail densities for vine copulas, we define the lower and upper conditional tail dependence functions, denoted by $t^L_{S'|S}$ and $t^U_{S'|S}$ respectively, as follows, for any $S, S' \subseteq \{1, \dots, d\}$, and all $w = (w_1, \dots, w_d) \in \mathbb{R}^d_+ \backslash \{0\}$,

$$\begin{aligned} t^L_{S'|S}(w_{S'} \mid w_S) &= \lim_{u \downarrow 0} C_{S'|S}(uw_i, i \in S' \mid uw_j, j \in S), \\ t^U_{S'|S}(w_{S'} \mid w_S) &= \lim_{u \downarrow 0} \overline{C}_{S'|S}(1 - uw_i, i \in S' \mid 1 - uw_j, j \in S). \end{aligned} \tag{3.7}$$

Under the uniform convergence assumption (2.1), these limiting functions exist.

Theorem 3.4. Let $\lambda^L_S(w_S)$ and $\lambda^U_S(w_S)$, $S \subseteq \{1, \dots, d\}$, denote respectively the lower and upper tail densities of C_S for a d-dimensional D-vine copula C. Assume that all bivariate linking copulas K_{ij}s have continuous densities and satisfy the uniform convergence properties (2.1).

1. If the baseline linking copulas $K_{i,i+1}$s are all lower tail dependent, then

$$\frac{\lambda^L_{\{1,\dots,d\}}(w)}{\lambda^L_{\{2,\dots,d-1\}}(w_{\{2,\dots,d-1\}})} = \frac{\lambda^L_{\{1,\dots,d-1\}}(w_{\{1,\dots,d-1\}})}{\lambda^L_{\{2,\dots,d-1\}}(w_{\{2,\dots,d-1\}})} \frac{\lambda^L_{\{2,\dots,d\}}(w_{\{2,\dots,d\}})}{\lambda^L_{\{2,\dots,d-1\}}(w_{\{2,\dots,d-1\}})}$$
$$k_{1,d}\left(t^L_{1|2,\dots,d-1}(w_1|w_{\{2,\dots,d-1\}}), t^L_{d|2,\dots,d-1}(w_d|w_{\{2,\dots,d-1\}})\right). \quad (3.8)$$

2. If the baseline linking copulas $K_{i,i+1}$s are all upper tail dependent, then

$$\frac{\lambda^U_{\{1,\dots,d\}}(w)}{\lambda^U_{\{2,\dots,d-1\}}(w_{\{2,\dots,d-1\}})} = \frac{\lambda^U_{\{1,\dots,d-1\}}(w_{\{1,\dots,d-1\}})}{\lambda^U_{\{2,\dots,d-1\}}(w_{\{2,\dots,d-1\}})} \frac{\lambda^U_{\{2,\dots,d\}}(w_{\{2,\dots,d\}})}{\lambda^U_{\{2,\dots,d-1\}}(w_{\{2,\dots,d-1\}})}$$
$$k_{1,d}\left(1 - t^U_{1|2,\dots,d-1}(w_1|w_{\{2,\dots,d-1\}}), 1 - t^U_{d|2,\dots,d-1}(w_d|w_{\{2,\dots,d-1\}})\right).$$

Proof. We prove the lower tail dependence case, and the other case is similar via the duality property (1.4).

It is shown in Theorem 4.1 of [24] that C and its multivariate margins are lower tail dependent, and thus $\lambda^L_S(w_S) > 0$, $S \subseteq \{1, \dots, d\}$, and all the lower conditional tail dependence functions are positive. It follows from (3.5) that the lower conditional tail dependence functions for D-vines are evaluated recursively by:

$$t^L_{\{i,i+l\}|\{i+1,\dots,i+l-1\}}(w_{\{i,i+l\}} \mid w_{\{i+1,\dots,i+l-1\}}) =$$
$$K_{i,i+l}\big(t^L_{i|\{i+1,\dots,i+l-1\}}(w_i \mid w_{\{i+1,\dots,i+l-1\}}), t^L_{i+l|\{i+1,\dots,i+l-1\}}(w_{i+l} \mid w_{\{i+1,\dots,i+l-1\}})\big),$$

for $1 \le i \le d-l$ and $2 \le l \le d-1$. Using (3.6), we have, for any $w = (w_1, \dots, w_d) \in \mathbb{R}^d_+$ and $u > 0$,

$$\frac{u^{d-1}c_{\{1,\dots,d\}}(uw)}{u^{d-3}c_{\{2,\dots,d-1\}}(uw_{\{2,\dots,d-1\}})} = \frac{u^{d-2}c_{\{1,\dots,d-1\}}(uw_{\{1,\dots,d-1\}})}{u^{d-3}c_{\{2,\dots,d-1\}}(uw_{\{2,\dots,d-1\}})} \frac{u^{d-2}c_{\{2,\dots,d\}}(uw_{\{2,\dots,d\}})}{u^{d-3}c_{\{2,\dots,d-1\}}(uw_{\{2,\dots,d-1\}})}$$
$$k_{1,d}\big(C_{1|\{2,\dots,d-1\}}(uw_1 \mid uw_{\{2,\dots,d-1\}}), C_{d|\{2,\dots,d-1\}}(uw_d \mid uw_{\{2,\dots,d-1\}})\big).$$

Since $k_{1,d}$ is continuous and all the lower conditional tail dependence functions are positive, (3.8) follows from Proposition 2.1 by taking the limits as $u \to 0$. □

Remark 3.5. 1. Let $S = \{2, \dots, d-1\}$, and for $i \notin S$,

$$\lambda^L_{i|S}(w_i \mid w_S) := \frac{\lambda^L_{\{i\}\cup S}(w_{\{i\}\cup S})}{\lambda^L_S(w_S)}, \text{ and } \lambda^L_{\{1,d\}|S}(w_1, w_d \mid w_S) := \frac{\lambda^L_{\{1,d\}\cup S}(w_{\{1,d\}\cup S})}{\lambda^L_S(w_S)}.$$

Note that

$$\lambda^L_{i|S}(w_i \mid w_S) = \frac{\partial}{\partial w_i} t^L_{i|S}(w_i|w_S), \ i \notin S$$
$$\lambda^L_{\{1,d\}|S}(w_1, w_d \mid w_S) = \frac{\partial^2}{\partial w_1 \partial w_d} t^L_{\{1,d\}|S}(w_1, w_d|w_{\{1,d\}\cup S})$$

describe the lower *conditional tail densities*, and (3.8) can be rewritten in terms of conditional tail densities as follows,

$$\lambda^L_{\{1,d\}|S}(w_1, w_d \mid w_S) = \lambda^L_{1|S}(w_1 \mid w_S)\lambda^L_{d|S}(w_d \mid w_S)$$
$$k_{1,d}\Big(\int_0^{w_1} \lambda^L_{1|S}(v_1 \mid w_S)dv_1, \int_0^{w_d} \lambda^L_{d|S}(v_d \mid w_S)dv_d\Big),$$

where indexes 1 and d are the two neighbors of the index subset S in the underlying 1-ary tree of the D-vine. The recursion involves only lower dimensional marginal tail densities and perhaps their univariate integrals. The tail dependence function for a d-dimensional D-vine copula obtained in [24] involves $(d-2)$-dimensional integrations, and in contrast, the tail density of a d-dimensional D-vine copula obtained here involves at most one dimensional integrations.

2. If some baseline linking copulas $K_{i,i+1}$s are tail independent (e.g., $\lambda^L_{i,i+1} = 0$ for some i), then the D-vine copula C is tail independent (i.e., $\lambda^L(w_1, \dots, w_d) = 0$). As illustrated in Proposition 4.3 in [24], however, some margins of the D-vine might still be tail dependent. For example, consider a three-dimensional D-vine copula C with bivariate linking copulas $K_{1,2}$, $K_{2,3}$ and $K_{1,3}$, where baseline linking copulas $K_{1,2}$ and $K_{2,3}$ are lower tail independent. In this situation, C is lower tail independent, but the margin $C_{\{1,3\}}$ can be lower tail dependent if the second level linking copula $K_{1,3}$ is lower tail dependent and the conditional tail probabilities of $K_{1,2}$ and $K_{2,3}$ are regularly varying at 0 with *same tail index*. That is, tail dependence of $C_{\{1,3\}}$ can emerge from tail independence of $K_{1,2}$ and $K_{2,3}$ with synchronized hidden regular variation tail index. This issue of hidden regular variation [19, 22, 20] is still unsolved in the context of vine copulas and other graphical models.

Example 3.6. The lower tail density of the 3-dimensional D-vine is given by:

$$\lambda^L(w_1, w_2, w_3) = \lambda^L_{12}(w_1, w_2) \cdot \lambda^L_{23}(w_2, w_3) \cdot k_{13}(t^L_{1|2}(w_1|w_2), t^L_{3|2}(w_3|w_2)),$$

where $t^L_{1|2}(w_1|w_2) = \int_0^{w_1} \lambda^L_{12}(v_1, w_2)dv_1$, and $t^L_{3|2}(w_3|w_2) = \int_0^{w_3} \lambda^L_{23}(w_2, v_3)dv_3$. The lower tail density of the 4-dimensional D-vine is given by:

$$\begin{aligned}\lambda^L(w_1,w_2, w_3, w_4) = {}& \lambda^L_{12}(w_1, w_2) \cdot \lambda^L_{23}(w_2, w_3) \cdot \lambda^L_{34}(w_3, w_4)\\ & \cdot k_{13}(t^L_{1|2}(w_1|w_2), t^L_{3|2}(w_3|w_2)) \cdot k_{24}(t^L_{2|3}(w_2|w_3), t^L_{4|3}(w_4|w_3))\\ & \cdot k_{14}(t^L_{1|23}(w_1|w_2, w_3), t^L_{4|23}(w_4|w_2, w_3)).\end{aligned}$$

Again, the lower conditional tail dependence functions are just univariate integrals of bivariate and trivariate tail densities. □

4 Concluding Remarks

In this paper, we introduced the notion of the tail density of a copula, and established its basic properties. Coupled with regularly varying margins, the copula tail density is shown to be equivalent to the tail density of multivariate regular variation developed in [13]. Various examples involving Archimedean and t copulas are discussed to illustrate our results.

The usefulness of the copula tail density lies in its ability to analyze extremal dependence properties locally, and such a local extreme value analysis often yields good geometric interpretations, such as in the case of t copulas. When applying the tail density approach to vine copulas, we obtained the recursive expressions of tail densities for D-vine copulas according to the underlying tree structure in terms of lower dimensional tail densities. In contrast to [24], the tail dependence recursions for high dimensional D-vines developed here could only involve one-dimensional integrations. The tail density approach will be used in our future research to characterize the multivariate regular variation properties for vine copulas according to underlying finite tree structures.

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References

[1] Aas, K., Czado, C., Frigessi A. and Bakken, H. (2009). Pair-copula constructions of multiple dependence. Insurance: Mathematics and Economics, 44(2):182-198.

[2] Albrecher, H., Asmussen, S. and Kortschak, D. (2006). Tail asymptotics for the sum of two heavy-tailed dependent risks. Extremes, 9:107–130.

[3] Alink, S., Löwe, M. and Wüthrich, M. V. (2004). Diversification of aggregate dependent risks. Insurance: Mathematics and Economics, 35:77-95.

[4] Alink, S., Löwe, M. and Wüthrich, M. V. (2005). Analysis of the expected shortfall of aggregate dependent risks. ASTIN Bulletin, 35(1): 25-43.

[5] Alink, S., Löwe, M. and Wüthrich, M. V. (2007). Diversification for general copula dependence. Statistica Neerlandica 61(4): 446-465.

[6] Barbe, P., Fougères, A.L., Genest, C. (2006). On the tail behavior of sums of dependent risks. ASTIN Bulletin 36(2):361-374.

[7] Bedford, T. and Cooke, R. M. (2001). Probability density decomposition for conditionally dependent random variables modeled by vines. Annals of Mathematics and Artificial Intelligence, 32:245-268.

[8] Bedford, T. and Cooke, R. M. (2002). Vines - a new graphical model for dependent random variables. The Annals of Statistics, 30:1031-1068.

[9] Beirlant, J., Goegebeur, Y., Teugels, J., and Segers, J. (2004). Statistics of extremes, Wiley Series in Probability and Statistics, John Wiley & Sons Ltd., Chichester,

[10] Charpentier, A. and Segers, J. (2009). Tails of multivariate Archimedean copulas. Journal of Multivariate Analysis, 100:1521-1537.

[11] Chen, Y. and Li, H. (2008). Tail dependence for multivariate t-copulas and its monotonicity. Insurance: Mathematics and Economics, 42:763-770.

[12] de Haan, L. and Ferreira, A. (2006). Extreme value theory. Springer Series in Operations Research and Financial Engineering. Springer, New York.

[13] de Haan, L. and Resnick, S.I. (1987). On regular variation of probability densities, Stochastic Process. Appl., 25:83-95.

[14] Kortschak, D. and Albrecher, H. (2009). Asymptotic results for the sum of dependent non-identically distributed random variables. Methodology and Computing in Applied Probability, 11:279-306.

[15] Larsson, M. and Nešlehová, J. (2011). Extremal behavior of Archimedean copulas. Advances in Applied Probability, 43(1):195-216.

[16] Embrechts, P., Lindskog, F., and McNeil, A. (2003). Modelling dependence with copulas and applications to risk management. In Rachev, S., editor, Handbook of heavy tailed distributions in finance, pages 329-384. Elsevier, Amsterdam.

[17] Embrechts, P., Nešlehová, J. and Wüthrich, M. V., (2009). Additivity properties for value-at-risk under Archimedean dependence and heavy-tailedness. Insurance: Mathematics and Economics, 44(2):164-169.

[18] Genest, C. and Rivest, L.-P. (1989). A characterization of Gumbel's family of extreme value distributions. Statistics and Probability Letters, 8:207-211.

[19] Heffernan, J.E. and Resnick, S.I. (2005). Hidden regular variation and the rank transform. Adv. Appl. Prob., 37(2):393-414.

[20] Hua, L. and Joe, H. (2011). Tail order and intermediate tail dependence of multivariate copulas. Journal of Multivariate Analysis, in press.

[21] Jaworski, P. (2006). On uniform tail expansions of multivariate copulas and wide convergence of measures. Applicationes Mathematicae, 33(2):159-184.

[22] Jaworski, P. (2010). Tail Behaviour of Copulas, Chapter 8 in Copula Theory and Its Applications, Eds. P. Jaworski, F. Durante, W. Härdle, and T. Rychlik, Lecture Notes in Statistics 198, Springer, New York.

[23] Joe, H. (1997). Multivariate Models and Dependence Concepts. Chapman & Hall, London.

[24] Joe, H., Li, H. and Nikoloulopoulos, A.K. (2010), Tail dependence functions and vine copulas. Journal of Multivariate Analysis, 101:252-270.

[25] Joe, H., and Li, H. (2011). Tail risk of multivariate regular variation. Methodology and Computing in Applied Probability, 13:671-693.

[26] Klüppelberg, C., Kuhn, G. and Peng, L. (2008). Semi-parametric models for the multivariate tail dependence function – the asymptotically dependent. Scandinavian Journal of Statistics, 35(4):701-718.

[27] Kurowicka, D. and Cooke, R. (2006). Uncertainty Analysis with High Dimensional Dependence Modelling. Wiley, New York.

[28] Kurowicka, D. and Joe, H. (2011). Dependence Modeling: Vine Copula Handbook. World Scientific, Singapore.

[29] Li, H. (2009). Orthant tail dependence of multivariate extreme value distributions. Journal of Multivariate Analysis, 100:243-256.

[30] Li, H. and Sun, Y. (2009). Tail dependence for heavy-tailed scale mixtures of multivariate distributions. J. Appl. Prob. 46 (4):925-937.

[31] McNeil, A. and Nešlehová, J. (2009). Multivariate Archimedean copulas, d-monotone functions and l_1-norm symmetric distributions. Ann. Statist., 37(5B): 3059-3097.

[32] Nikoloulopoulos, A.K., Joe, H. and Li, H. (2009). Extreme value properties of multivariate t copulas. Extremes, 12:129-148.

[33] Resnick, S. (2007). Heavy-Tail Phenomena: Probabilistic and Statistical Modeling. Springer, New York.

[34] Schmidt, R. and Stadtmüller, U. (2006). Nonparametric estimation of tail dependence. Scandinavian Journal of Statistics, 33:307-335.

[35] Wüthrich, M. V. (2003). Asymptotic Value-at-Risk estimates for sums of dependent random variables. ASTIN Bulletin 33(1):75-92.