On L- Fuzzy Sets

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Abstract

It is found that the interval [0, 1] of real numbers is insufficient to have the truth values of general fuzzy statements. In this paper it is argued that a complete lattice L satisfying the infinite meet distributive law is a best candidate to assume the truth values of fuzzy statements. Such a lattice is called a frame. A thorough discussion is made of fuzzy subsets of a set having truth values in an abstract frame.

Keywords: Lattice, complete lattice, Infinite meet distributive law, Frame, Fuzzy statements.

Introduction

Ever since Zadeh [10] introduced the notion of a fuzzy subset of a set X as function X into the unit interval [0, 1] of real numbers, several mathematicians took interest in the study of fuzzy subsets, in particular, on fuzzy sub algebras of several algebraic structures. Fuzzy statements usually take truth values in the interval [0, 1] of real numbers, while the ordinary (or conventional or crisp) statements take truth vales in the two-element set {F, T } or {0, 1}where F and O stand for 'false' and T and 1 stand for 'true'. However, the interval [0, 1] is found to be insufficient to have the truth values of general fuzzy statements [2and9]. For example, consider the statement 'India is a good country'. The truth value of this statement may not a real number in [0, 1]. Being good country may have several components: good in educational facilities, good in public transport system, good in political awareness among the citizens, good in medical facilities, good for tourism, etc. The truth value

corresponding to each component may be a real number in [0, 1]. In n is the number of such components under consideration, then the truth value of the above fuzzy statement is a n-tuple of real numbers in [0, 1]; that is, an element of $[0, 1]^n$. If C is the collection of all countries on this earth and G is the collection of good countries, the G is not a subset of C, but it is a fuzzy subset of C, since 'being good' is fuzzy. That is, G can be considered as a function of C into a set like $[0, 1]^n$, for a positive integer n. Such a G is called an L-fuzzy subset of C where L= $[0, 1]^n$.

The usual ordering of real numbers makes [0, 1] as a totally ordered set But $[0, 1]^n$ is not a totally ordered set when n > 1, under the usual coordinate-wise ordering. However, $[0, 1]^n$ satisfies certain rich lattice theoretic properties such as the infinite meet distributivity, namely a ∧ (supX) = sup{a ∧ x | x ∈ X} for any element a and any subset X. This distributivity satisfies almost all the major requirements to develop the theory of general fuzzy subsets.

PRELIMINARIES

We briefly recall certain elementary concepts and notations from the theory of partially ordered sets and lattices [1]. A binary relation ≤ on a set X is called a partial order on X if it is reflexive, transitive and anti-symmetric. A pair (X, ≤) is called a partially ordered set or, simply, poset if X is a nonempty set and ≤ is a partial order on X. A poset (X, ≤) is called a lattice (complete lattice) if every nonempty finite subset (respectively, every arbitrary subset) of X has greatest lower bound and least upper bound in X which are respectively called infimum and supremum also; for any subset A of X, we write inf A or glb A or $\wedge A$ or $\bigwedge_{a \in A} a$ for the greatest lower bound (or infimum) of A and supA or lubA or $\vee A$ or $\bigvee_{a \in A} a$ for the least upper bound (or supremum) of A. If A= {a_1, …., a_n}, then we write for the inf A and $\bigvee_{i=1}^{n} a_i$ or $a_1 \vee a_2 \vee ... \vee a_n$ for the supA.

If (L, ≤) is a lattice, then a∧b = inf {a, b} and a∨b = sup{a, b} give two binary operations ∧ and ∨ on L which are both associative, commutative∨ and idempotent and satisfy the absorption laws a ∧ (a ∨ b) = a = a conversely if ∧and ∨ are binary operations on a nonempty set L satisfying all the above properties and if the partial order ≤ on L is defined by a ≤ b ⟺ a = a ∧ b (⟺ a ∨ b = b), then (L, ≤) is a lattice in which a ∧ b and a ∨ b are respectively the infimum and supremum of {a, b}. An element a in a poset (L, ≤) is called the smallest (greatest) element if a ≤ x (respectively x ≤ a) for all x ∈ L . The smallest and greatest elements, if they exist, are usually denoted by 0 and 1 respectively. A poset is called bounded if it has both smallest and greatest elements. A complete lattice is necessarily bounded. Logically, the infimum and supremum of the empty subset of a poset, if they exist, are respectively the greatest element and smallest element. A complete lattice (L, ≤) is called a frame, if it satisfies the infinite meet distributive law; that is, a ∧ (supX) = sup{a ∧ x | x ∈ X} for all a ∈ L and X ≤ L. It is known that a complete lattice (L, ≤) is a frame if and only if, for any a and b ∈ L, there exist a largest element, denoted by

a → b, in L such that x ∧ a ≤ b ⇔ x ≤ a → b for all x ∈ L[9]. A poset (P, ≤) is called a totally ordered set if, for any a and b ∈ P, either a ≤ b or b ≤ a. A subset C of a poset (P, ≤) is called a chain in P if (C, ≤) is totally ordered.

L-FUZZY SUBSETS OF A SET

It is well known that if A is an algebraic structure and X is any nonempty set X, then the set A^X of all mappings of X into A can be made as an algebraic structure of the same type as A by defining the fundamental as an algebraic structure of the same type as A by defining the fundamental type as A by defining the fundamental operationally definable algebra (like a group or a ring or a module or a lattice), the A^X is an algebra belonging to the variety generated by A. In particular, if 2 is the two element lattice {0, 1} with 0 < 1, then 2^X is a Boolean algebra for any nonempty set X, since 2 is a Boolean algebra. Also, recall that 2^X is isomorphic to the Boolean algebra P(X) of all subsets of X, under the mapping which maps any A ⊆ X with its characteristic map χ_A defined by $\chi_A(x) = 1$ or 0 according as x is in A or not in A. Therefore, the usual (or crisp) subsets of X can be identified with mappings of X into 2. With this background, we define the following.

Definition 3.1. Let X be a nonempty set and L = (L, ≤) be a frame. Any mapping of X into L is called an L-fuzzy subset of X. The set of all L-fuzzy subsets of X is denoted by $FS_L(X)$.

Since L is a complete lattice, it has smallest element 0 and greatest element 1 and hence 2 can be treated as a subset of L. This facilitates us to treat the usual subsets of X as L- fuzzy subsets of X. For the sake of distinguishing the subsets of X from the L-fuzzy subsets of X, the subsets of X are usually called the crisp subsets of X. The lattice structure on L can be extended to $FS_L(X)(= L^X)$ as given below.

Definition 3.2. For any L-fuzzy subsets A and B of X, define A ≤ B if and only if A(x) ≤ B(x) for all x ∈ X. Clearly ≤ is a partial order on $FS_L(X)$. Also, for any crisp subsets S and T of X, we have $\chi S ≤ \chi T ⇔ S ⊆ T$.

The following is an easy verification.

Theorem 3.3. $(FS_L(X), ≤)$ is a frame in which, for any $\{A_i\}_{i \in I}$, the infimum and supremum of $\{A_i\}_{i \in I}$ are respectively given by

$$(\inf_{i \in I}\{A_i\}(x)) = \inf_{i \in I} A_i(x) \text{ and } (\sup_{i \in I}\{A_i\})(x) = \sup_{i \in I} A_i(x), \text{ for any x ∈ X.}$$

Also, for any A and B in $FS_L(X)$, (A → B)(x) = A(x) → B (x) for all x ∈ X.

Definition 3.4. For any L- fuzzy subset A of X and for any α ∈ L, define

$A_\alpha = A^{-1}([\alpha, 1]) = \{x \in X : \alpha ≤ A(x)\}$. Then Aα is called the α - cut of A .

The following is a straight forward verification.

Theorem 3.5. (1) For any A, B ∈ $FS_L(X)$, $A \leq B \Leftrightarrow A_\alpha \subseteq B_\alpha$ for all $\alpha \in L$

(2) For any $\{A_i\}_{i \in I} \subseteq FS_L(X)$ and $A = \underset{i \in I}{\wedge} A_i$, $A_\alpha = \underset{i \in I}{\bigcap} A_{i\alpha}$ for all $\alpha \in L$.

Even though the α-cut of the infimum of A_i's is simply the set intersection of the α-cuts of A_i's, the α-cut of the supremum of A_i's may not be the set union of the α-cuts of A_i's. However the α-cuts of $\underset{i \in I}{\vee} A_i$ can be can be expressed in terms of the α-cuts of Ai's. First, let us recall that, for any $\alpha \in L$ and $M \subseteq L$, M is said to be a cover of α (or α is said to be covered by M) if $\alpha \leq \sup M$. The following can be easily proved.

Theorem 3.6. Let $\{A_i\}_{i \in I}$ be a nonempty class of L-fuzzy subsets of a set X and $A = \cup A_i$, Then the α-cut of A is given by $A_\alpha = \bigcup \{ \underset{\beta \in M}{\bigcap} (\underset{i \in I}{\bigcup} A_{i\beta}) / M \text{ is a cov } er \text{ of } \alpha \}$ For any L-fuzzy subset A of X, $\{A\alpha / \alpha \in L\}$ is a class of crisp subsets of X such that $\underset{\alpha \in M}{\bigcap} A_\alpha = A_{\sup M}$ for all $M \subseteq L$.

The converse of this is also true, in the following sense.

Theorem 3.7. For any class $\{S\alpha / \alpha \in L\}$ of crisp subsets of X such that $\alpha \in M$ Sα = SsupM for all $M \subseteq L$, there exists a unique L-fuzzy subset A of X whose α-cut is precisely Sα for all $\alpha \in L$.

The following is an useful tool in working with L-fuzzy subsets or crisp subsets of a set X. A class $\{Ai\}_{i \in I}$ of L-fuzzy subsets of X is said to be directed above if, for any i and $j \in I$, there exists $k \in I$ such that $Ai \leq A_k$ and $A_j \leq A_k$.

Theorem 3.8. Let $\{Ai\}_{i \in I}$ be a directed above class of L- fuzzy subsets of a set X and $x_1, x_2, ..., x_n \in X$, then

$$\underset{i=1}{\overset{n}{\wedge}} (\underset{i \in I}{\vee} A_i(x_r)) = \underset{i \in I}{\vee} (\underset{r=1}{\overset{n}{\wedge}} A_i(x_r))$$

Proof. Let α and β denote respectively the left side and right side of the above required equation. It is clear that $\beta \leq \alpha$. Also, by the infinite meet distributivity in L, we have $\alpha = \underset{i_1, i_2, ..i_n \in I}{\vee} (A_{i1}(x_1) \wedge A_{i2}(x_2) \wedge ... \wedge A_{in}(x_n)) ..... (*)$

Now, for any $i_1, i_2, ..., i_n \in I$, there exists $j \in I$ such that $Air \leq Aj$ for all $1 \leq r \leq n$ and hence $A_{i1}(x_1) \wedge A_{i2}(x_2) \wedge ... \wedge A_{in}(x_n) \leq A_j(x_1) \wedge A_j(x_2) \wedge ... \wedge A_j(x_n) \leq \beta$.

From this and (*) above, we get that $\alpha \leq \beta$. Thus $\alpha = \beta$.

For any $\alpha \in L$, the L-fuzzy subset A of X defined by $A(x) = \alpha$ for all $x \in X$ is called a constant L-fuzzy subset of X and is denoted by $\bar{\alpha}$.

Note that $0 (= \chi_\Phi)$ and T $(= \chi_X)$ are respectively the smallest and greatest elements in $FS_L(X)$. For any $\alpha \in L$ and a crisp subset Y of X, we define $\alpha_Y : X \rightarrow L$ by

$$\alpha_Y(x) = \begin{cases} 1, & \text{if } x \in Y \\ \alpha, & \text{if } x \notin Y. \end{cases}$$

The following is an easy verification.

Theorem 3.9. Let X be a nonempty set and L a frame. For any $1 \neq \alpha \in L$, $Y \to \alpha_Y$ is an embedding of the lattice P (X) of all crisp subsets of X into the lattice $FS_L(X)$ of L- fuzzy subsets of X. Also, for any proper crisp subset Y of X, $\alpha \to \alpha_Y$ is an embedding of L into the lattice $FS_L(X)$ of L-fuzzy subsets of X. Also, for any proper crisp subset Y of X, $\alpha \to \alpha_Y$ is an embedding of L in to the lattice $FS_L(X)$.

Note that $FS_L(X)$ is precisely the set L^X of all mappings of X into Y . Since L is a frame, L^X is also a frame. L being a complete lattice satisfying the infinite meet distributivity, we have, for any α and β in L, the largest element $\alpha \to \beta$ with the property that $\alpha \wedge \gamma \leq \beta \Longleftrightarrow \gamma \leq \alpha \to \beta$ for any $\gamma \in L$. Infact, $\alpha \to \beta = \sup\{\gamma \in L \mid \alpha \wedge \gamma \leq \beta\}$. Therefore, for any A and B in $FS_L(X)$, we have $A \to B$ in $FS_L(X)$ given by $(A \to B)(x) = A(x) \to B(x)$ for all $x \in X$ and hence we have the following.

Theorem 3.10. For any nonempty set X and for any frame L, The set $FS_L(X)$ of all L fuzzy subsets of X is a frame under the point wise ordering.

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