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Technical Briefs

The Application of Curvature Theory to the Trajectory Generation Problem of Robot Manipulators

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This paper illustrates a new application of planar curvature theory to the geometric problem of trajectory generation by a two-link manipulator. The theory yields the instantaneous speed ratio, and the rate of change of the speed ratio, which correspond to the geometry of a desired point trajectory. Separate from the purely geometric speed ratio problem (i.e., the coordination problem) is the time based problem of controlling the joint rates in order to move with the specified path variables.

1 Introduction

A rigid body, constrained to move in a plane with N independent degrees of freedom, where $1 \leq N \leq 3$, is known as a planar N-parameter motion, denoted by M_N. This paper discusses the arm-subassembly of a planar manipulator, whose terminal link is a planar M_2 motion. The purpose is to illustrate a new application of classical curvature theory to the problem of trajectory generation by a multi-degree-of-freedom rigid body system, in this case a planar robot manipulator. Previously, curvature theory has been applied to the *dimensional synthesis* of planar mechanisms. The new application of the theory illustrated here is a problem of *motion synthesis* by industrial manipulators.

2 Two Parameter Plane Motion

The results of planar M_2 motion theory, developed by Bottema and Roth (1979), Chap. 10, pp. 380–385, are presented here in terms of a speed ratio. The purpose is to show how second-order geometric properties of the desired trajectory (i.e., the path radius of curvature), determine second-order geometric M_1 motion properties, namely, the location of the pole and the diameter of the inflection circle. The M_1 motion properties in turn determine the required speed ratio and rate of change of the speed ratio. The geometric results presented here are such that for a given radius of curvature of a desired trajectory, the speed ratio, and its rate of change can be found from purely geometric considerations, and in a canonical form.

A general planar motion, see Fig. 1, can be represented mathematically by,

$$X = x\cos\theta - y\sin\theta + a \tag{1}$$

$$Y = x\sin\theta + y\cos\theta + b, \tag{2}$$

where in Fig. 1, frame E represents the moving rigid body, the frame Σ represents the fixed rigid body and vectors $\overline{\mathbf{q}} = (x, y)^T$ and $\overline{\mathbf{Q}} = (X, Y)^T$ are the coordinates of an arbitrary point Q belonging to the moving rigid body, relative to the frames E and Σ, respectively. The coordinates $\overline{\mathbf{q}} = (x, y)^T$ are constant. The vector $\overline{\mathbf{a}} = (a, b)^T$ represents the position of the origin of E relative to the origin of Σ, expressed in the coordinates of the Σ frame, Bottema and Roth (1979).

Since this is a description of planar M_2 motion, the variables θ, a and b are functions of two dimensionless motion parameters, λ and μ. Choosing $\theta = \lambda$ and $a = \mu L$ (where L is the unit length or scale) and substituting into Eqs. (1) and (2) yields,

$$X = x\cos\lambda - y\sin\lambda + \mu L \tag{3}$$

$$Y = x\sin\lambda + y\cos\lambda + b(\lambda, \mu) \tag{4}$$

Choosing E and Σ to be coincident at the instant under consideration and taking the motion parameters to have the values $\lambda = 0$ and $\mu = 0$, results in $b(0, 0) = 0$, referred to as the zero position. The zero position represents the known instantaneous configuration of the manipulator.

The time derivatives of a variable, such as λ, evaluated in the zero position are denoted by a numeric subscript indicating the order of the time derivative, i.e.,

$$\left.\frac{d\lambda}{dt}\right|_{\substack{\lambda=0\\ \mu=0}} = \lambda_1 \quad \left.\frac{d^2\lambda}{dt^2}\right|_{\substack{\lambda=0\\ \mu=0}} = \lambda_2.$$

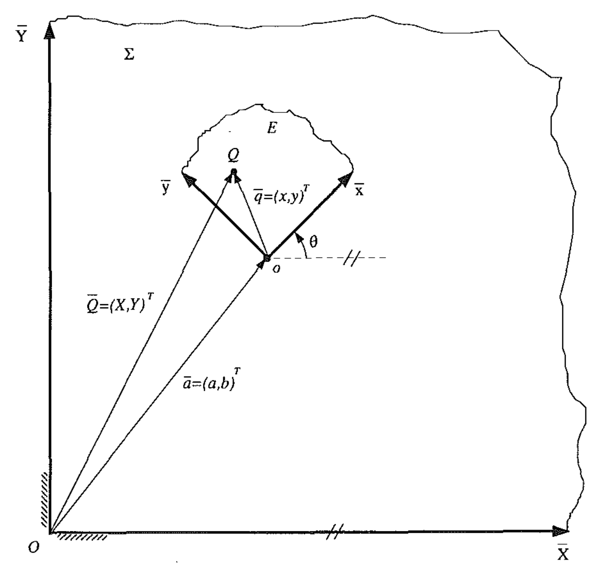

Fig. 1 A general plane motion

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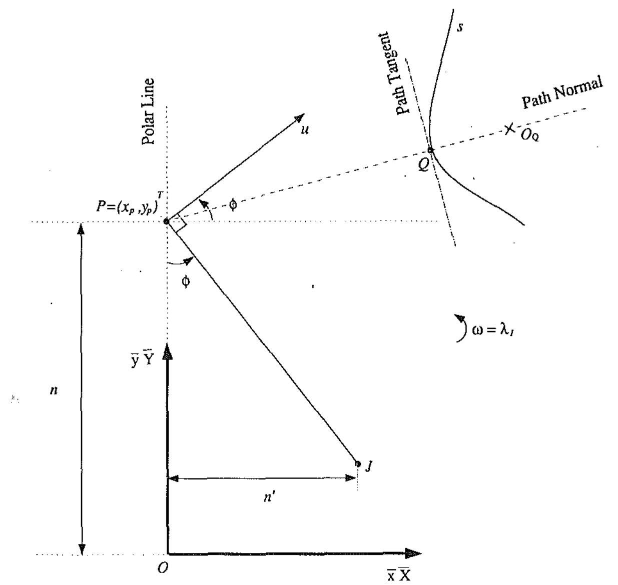

Fig. 2 The point Q on a moving body following an arbitrary path

The partial derivatives of a variable such as X, with respect to λ or μ, evaluated in the zero position are denoted by subscripts λ and μ, i.e.,

$$\left.\frac{\partial X}{\partial \lambda}\right|_{\substack{\lambda=0\\ \mu=0}}=X_\lambda \quad \left.\frac{\partial X}{\partial \mu}\right|_{\substack{\lambda=0\\ \mu=0}}=X_\mu \quad \left.\frac{\partial^2 X}{\partial \lambda^2}\right|_{\substack{\lambda=0\\ \mu=0}}=X_{\lambda\lambda}$$

$$\left.\frac{\partial^2 X}{\partial \mu^2}\right|_{\substack{\lambda=0\\ \mu=0}}=X_{\mu\mu} \quad \left.\frac{\partial^2 X}{\partial \lambda \partial \mu}\right|_{\substack{\lambda=0\\ \mu=0}}=X_{\lambda\mu}$$

The first time derivative of Eqs. (3) and (4) evaluated in the zero position yields the instantaneous velocity of the arbitrary point Q, i.e., the instantaneous velocity distribution of the moving body,

$$X_1=-y\lambda_1+\mu_1 L \tag{5}$$

$$Y_1=x\lambda_1+b_\lambda\lambda_1+b_\mu\mu_1 \tag{6}$$

Setting Eqs. (5) and (6) equal to zero yields the location of the pole (a point on the moving body with zero velocity),

$$x_P=-(b_\lambda+b_\mu n) \tag{7}$$

$$y_P=Ln \tag{8}$$

where n is the instantaneous speed ratio and is defined as,

$$n=\frac{\mu_1}{\lambda_1}=\left.\frac{d\mu}{d\lambda}\right|_{\substack{\lambda=0\\ \mu=0}} \tag{9}$$

Clearly n is a desired M_1 motion property. Eliminating n from Eqs. (7) and (8) yields the locus of all possible poles for the M_2 motion, which is a straight line referred to as the Polar Line, Bottema and Roth (1979). Equations (1)–(8) have previously been derived in Bottema and Roth (1979).

The coincident frames, still arbitrarily in position and orientation are further restricted by choosing the coincident $\bar{y}$, $\bar{Y}$ axes to be along the Polar Line. This implies that $x_P=0$, and from Eq. (7) we have $b_\mu=0$ and $b_\lambda=0$, since n is arbitrary for the M_2 motion. The equation of the Polar Line is now reduced to its simplest (or canonical) form, namely the $\bar{y}$, $\bar{Y}$ axes. The coincident frames, still arbitrary in their position on the Polar Line are further restricted by choosing to locate the coincident origins such that $b_{\lambda\lambda}=0$ and then directing the coincident $\bar{y}$, $\bar{Y}$ axes along the Polar Line such that $b_{\mu\mu}>0$. This completely defines the coincident frames E and Σ which are now referred to as the canonical coordinate systems, Bottema and Roth (1979), since they yield the algebraically simplest instantaneous kinematic equations of motion of the moving body E.

For a desired trajectory of point Q, (i.e., a desired M_1 motion) the intersection of the path normal (the line O_QQ), with the Polar Line defines the location of P (y_p), see Fig. 2. Thus in order to achieve the desired path tangent and path normal, the required speed ratio, from Eq. (8) is

$$n=\frac{y_P}{L} \tag{10}$$

Note that the speed ratio is simply the coordinate of the pole on the coincident axes, i.e., *a geometric quantity which is the distance from O to P represents the speed ratio required to achieve the path tangent of the desired trajectory.* (Note that L represents a unit of length, i.e., the scale).

Differentiating Eqs. (3) and (4) with respect to λ and μ, evaluating in the zero position, and using the canonical coordinate system yields,

$$X_\lambda=-y \quad X_\mu=L \quad Y_\lambda=x \quad Y_\mu=0 \tag{11}$$

$$X_{\lambda\lambda}=-x \quad X_{\lambda\mu}=0 \quad X_{\mu\mu}=0$$
$$Y_{\lambda\lambda}=-y \quad Y_{\lambda\mu}=b_{\lambda\mu} \quad Y_{\mu\mu}=b_{\mu\mu}. \tag{12}$$

From Eqs. (11) and (12), the coordinates of the point Q in the fixed reference frame may be represented by the second-order Taylor series expansion,

$$X=x+X_\lambda\lambda+X_\mu\mu+\frac{1}{2}(X_{\lambda\lambda}\lambda^2+2X_{\lambda\mu}\lambda\mu+X_{\mu\mu}\mu^2)$$
$$=x-y\lambda+L\mu-\frac{1}{2}x\lambda^2 \tag{13}$$

$$Y=y+Y_\lambda\lambda+Y_\mu\mu+\frac{1}{2}(Y_{\lambda\lambda}\lambda^2+2Y_{\lambda\mu}\lambda\mu+Y_{\mu\mu}\mu^2)$$
$$=y+x\lambda-\frac{1}{2}y\lambda^2+b_{\lambda\mu}\lambda\mu+\frac{1}{2}b_{\mu\mu}\mu^2 \tag{14}$$

The coordinates of the pole up to the first-order are obtained from the time derivative of Eqs. (13) and (14),

$$x_P=L\lambda\frac{d\mu}{d\lambda}-b_{\lambda\mu}\mu-b_{\lambda\mu}\lambda\frac{d\mu}{d\lambda}-b_{\mu\mu}\mu\frac{d\mu}{d\lambda} \tag{15}$$

$$y_P=L\frac{d\mu}{d\lambda}. \tag{16}$$

Equations (11)–(16) have been derived by Bottema and Roth (1979). We now define the instantaneous rate of change of the speed ratio as,

$$n'=\left.\frac{dn}{d\lambda}\right|_{\substack{\lambda=0\\ \mu=0}}=\left.\frac{d^2\mu}{d\lambda^2}\right|_{\substack{\lambda=0\\ \mu=0}} \tag{17}$$

Taking the time derivative of Eqs. (15) and (16), the x and y components of the pole velocity in the zero position are,

$$(x_p)_1=L\lambda_1 n-b_{\lambda\mu}\mu_1-b_{\lambda\mu}\lambda_1 n-b_{\mu\mu}\mu_1 n \tag{18}$$

$$(y_p)_1=L\lambda_1 n' \tag{19}$$

In the graphical construction of Fig. 2, knowing the desired radius of curvature of the path of Q (i.e., the directed distance O_QQ), one can apply the Euler-Savary Equation (graphically if desired) to find the corresponding diameter of the inflection circle, denoted as the directed distance (PJ). From planar M_1 motion theory, it is well known that

$$u=(PJ)\omega$$

where u is the pole velocity and $\omega=\lambda_1$ is the angular velocity of the moving body, Hall (1986). From the graphical construction in Fig. 2, if we denote the angle made from the coincident $\bar{y}$, $\bar{Y}$ axes to (PJ) as ϕ, and note that u has the direction of (PJ) rotated $\pi/2$ in the direction of ω, Hall (1986), we see that the y component of the pole velocity is given by,

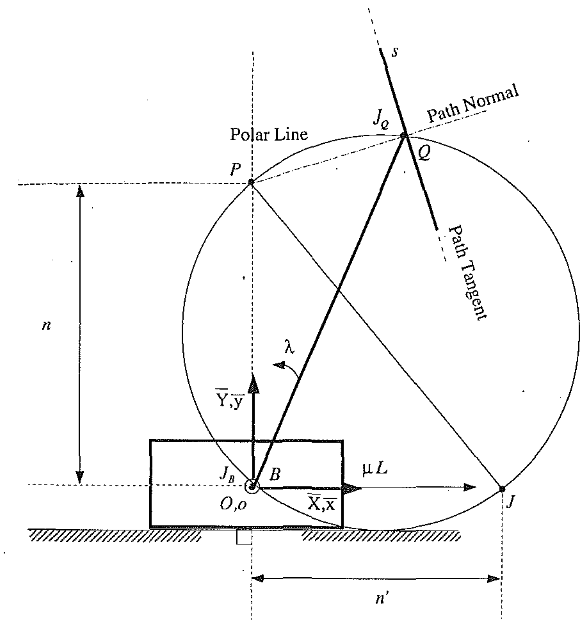

Fig. 3 Prismatic revolute positioner following straight line path

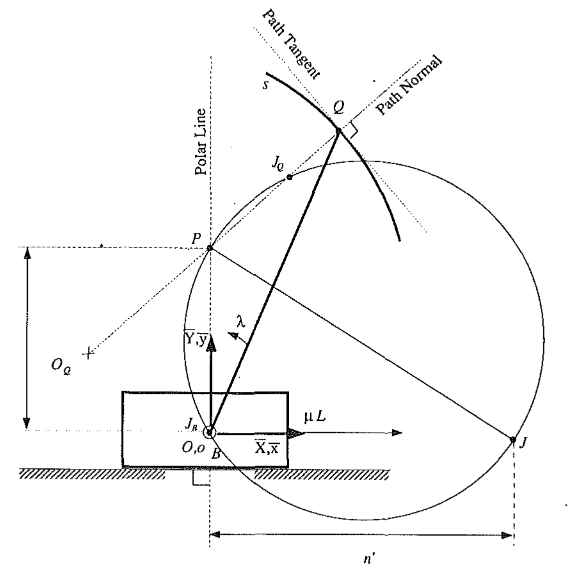

Fig. 4 Prismatic revolute positioner follower a parabolic path

$$(y_P)_1 = (PJ)\ \omega \sin\phi \tag{20}$$

Equating (20) to (19) and solving for the instantaneous rate of change of the speed ratio yields,

$$n' = \frac{(PJ)\sin\phi}{L} \tag{21}$$

which is simply the x component of the inflection circle diameter in the canonical coordinate system as shown in Fig. 2. *This purely geometric quantity represents the rate of change of the speed ratio required to achieve the path curvature of the desired trajectory.*

Equations (10) and (21) represent purely geometric and canonical solutions to the path tracking problem of a planar manipulator arm, and have been developed from the curvature theory of planar M_2 motion.

3 Applications

Consider the prismatic-revolute manipulator arm-subassembly shown in Fig. 3. Point B represents the center of the revolute joint. The Polar Line is the line perpendicular to the axis of the prismatic joint which passes through B. In order to satisfy the conditions on the canonical system the coincident $\overline{y}$, $\overline{Y}$ axes are directed along the Polar Line. The origin of the coincident systems is at point B in order to satisfy the condition $b_{\lambda\lambda} = 0$ and the $\overline{y}$ $\overline{Y}$ axes must be directed as shown in order that $b_{\mu\mu} > 0$. For a detailed development of the canonical coordinate system see Stanišić and Pennock (1986).

Figure 3 shows the manipulator generating a straight line path with the point Q belonging to the robot forearm. The center of curvature of the straight line path, s, is at $\pm\infty$ and the Pole, P, is located by taking the intersection of the path normal with the Polar Line. Since the distance PQ is now known, the corresponding point on the inflection circle, J_Q, is located by using the Euler-Savary equation. In a similar manner the corresponding point for B on the inflection circle, J_B, is located to be at the origin of the moving canonical coordinate system. Now that three points have been located, the inflection circle can be drawn and the diameter (PJ) can be determined. Once the diameter of the inflection circle is determined, the speed ratio, n, and the rate of change of the speed ratio, n', can be determined using Eqs. (10) and (21), respectively.

Figure 4 shows the manipulator generating a parabolic path with point Q fixed in the forearm. Since the equation of the trajectory is known, the radius of curvature of the path, O_Q, is calculated for the point Q. This is a more general case than the straight line path in which the point on the inflection circle, J_Q, was coincident with Q. The pole is located in the same way as for the straight line path. The Euler-Savary equation is applied to find points on the inflection circle, J_Q and J_B, and the inflection circle is drawn. The speed ratio, n, and the rate of change of the speed ratio, n', are calculated from Eqs. (10) and (21) respectively. The procedure can be accomplished graphically or analytically.

4 The Inverse Path Variable Problem - The Time Based Problem

Given a trajectory s, the previous section has shown how curvature theory can be used to determine the speed ratio (n) and the rate of change of the speed ratio (n') that allow a point Q on the forearm to travel on the specified path with the required path tangent and radius of curvature. The two values represent solutions to the basic coordination problem, resulting in only the information needed to generate a specified *trajectory geometry*, i.e., the information needed to track a specified path.

Distinct from this purely geometric path generation problem is the time based problem of moving point Q along the trajectory path with a prescribed instantaneous path speed s_1 and path acceleration s_2. In this section we show how this *time based* problem will determine one of the joint velocities and accelerations, from which the remaining joint velocity and acceleration can be determined through the known speed ratios.

Consider the example prismatic-revolute planar manipulator arm as shown in Fig. 5. The velocity of point Q on the moving body is

$$(\overline{r}_Q)_1 = \begin{bmatrix} \mu_1 L - l\lambda_1 \sin\lambda \\ l\lambda_1 \cos\lambda \end{bmatrix} \tag{22}$$

The desired path speed of Q, s_1, is given and can be expressed from Eq. (22) as,

$$s_1^2 = \mu_1^2 L^2 + l^2\lambda_1^2 - 2\mu_1\lambda_1 L l \sin\lambda \tag{23}$$

Substituting $\mu_1 = n\lambda_1$ from Eq. (9) into Eq. (23) yields,

$$s_1^2 = (n^2L^2 + l^2 - 2nLl\sin\lambda)\lambda_1^2 \tag{24}$$

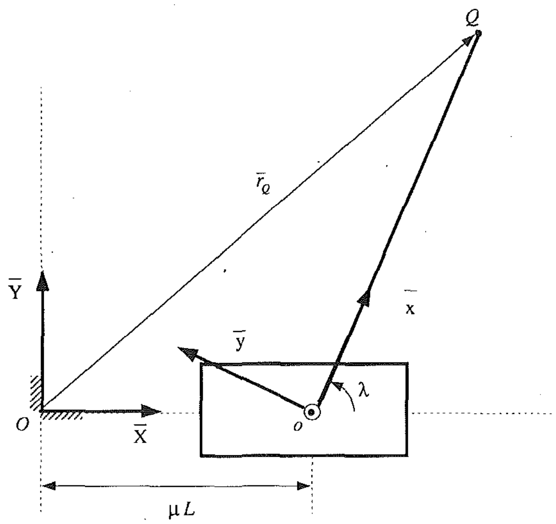

Fig. 5 Positioner displaced relative to fixed canonical system

Solving for λ_1 from Eq. (24) yields,

$$\lambda_1 = \pm \frac{s_1}{\sqrt{(n^2L^2 + l^2 - 2nLl \sin \lambda)}} \tag{25}$$

where the sign of λ_1 is determined by the direction of the desired path speed along the specified path. Differentiating (23) with respect to time yields,

$$s_1 s_2 = \mu_1 \mu_2 L^2 + l^2 \lambda_1 \lambda_2 - \mu_2 \lambda_1 Ll \sin \lambda - \mu_1 \lambda_2 Ll \sin\lambda - \mu_1 \lambda_1^2 Ll \cos \lambda \tag{26}$$

The time derivative of Eq. (9) yields,

$$\mu_2 = n\lambda_2 + n'\lambda_1^2 \tag{27}$$

Substituting (27) into (26) yields,

$$s_1 s_2 = \mu_1 \lambda_2 nL^2 + \mu_1 \lambda_1^2 n' L^2 + \lambda_1 \lambda_2 l^2 - \lambda_1 \lambda_2 nLl \sin \lambda - \lambda_1^3 n' Ll \sin \lambda - \mu_1 \lambda_2 Ll \sin \lambda - \mu_1 \lambda_1^2 \, Ll \cos \lambda$$

and solving for λ_2 gives,

$$\lambda_2 = \frac{s_1 s_2 - \lambda_1^3 L (nn'L - ln' \sin\lambda - ln \cos\lambda)}{\lambda_1 (n^2L^2 + l^2 - 2nLl \sin \lambda)} \tag{28}$$

This completes the solution for λ_1 and λ_2 in order to achieve a desired s_1 and s_2. The corresponding μ_1 and μ_2 can be obtained from Eqs. (9) and (27), respectively, knowing the required speed ratio and the rate of change of the speed ratio.

5 Conclusion

This paper has developed a purely geometric solution to the trajectory generation problem of a planar manipulator using the well known planar curvature theory. Separate from this, the time based path variable problem was also solved. A prismatic-revolute robot positioner was used as an example.

A future paper will extend the concepts developed here to the general case of a planar two-link open-chain and also to the trajectory control problem of a spatial three-link open-chain. These encompass the arm-subassemblies of industrial manipulators. The important problems of singularities arising in the trajectory control will also be addressed.

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