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SOME OPEN PROBLEMS IN THE THEORY OF SINGULARITIES OF MAPPINGS

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Abstract. This paper surveys some open problems in the theory of singularities of mappings. It does not claim to be comprehensive or fair. The problems are those whose answers I would most like to see.

1. Vanishing homology of parameterisations of hypersurfaces

1.1. μ **versus** τ**.** Germs of mappings from n-space to $n+1$-space show some of the same features as isolated complete intersection singularities. I'm thinking in particular of the relation between the rank of the vanishing homology ("μ") and the $\mathcal{A}_e$–codimension ("τ"). This relation, which I will describe in detail in a moment, can be seen already in the three Reidemeister moves of knot theory. The three moves are those unavoidably present when we deform one plane knot diagram to another.

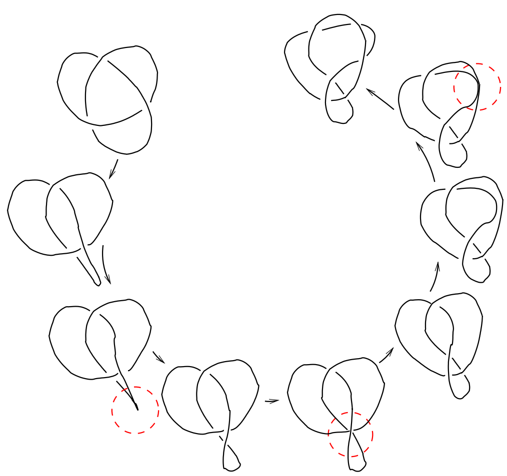

Figure 1: Deforming a planar projection of a trefoil, passing through moves I, III and II

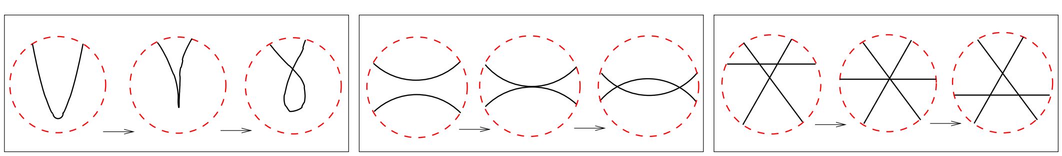

Figure 2: Reidemeister moves I, II and III, isolated in their Milnor balls

Of course, all three moves are really equivalence classes of germs of mappings: we allow arbitrary diffeomorphisms in the source and target. This equivalence relation is known as $\mathcal{A}$-*equivalence*.

I begin with the codimension. Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a multi-germ (with S a finite set). We define the $\mathcal{A}_e$-codimension of f as the dimension of the quotient

$$\frac{\{\frac{d}{dt} f_t|_{t=0} : f_0 = f\}}{\{\frac{d}{dt}\left(\psi_t \circ f \circ \varphi_t\right)|\psi_0 = \mathrm{id}_{\mathbb{C}^p}, \varphi_0 = \mathrm{id}_{\mathbb{C}^n}\}} \tag{1.1}$$

Both numerator and denominator here can be expressed more explicitly.

Clearly, for each $x \in (\mathbb{C}^n, S)$,

$$\frac{d}{dt} f_t(x)|_{t=0} \in T_{f(x)}\mathbb{C}^p.$$

Thus $x \mapsto \frac{d}{dt} f_t(x)|_{t=0}$ is a map from $(\mathbb{C}^n, S) \to T\mathbb{C}^p$ over f: it gives the diagonal arrow in a commutative diagram

$$\begin{tikzcd} T\mathbb{C}^n \arrow[r, "df"] \arrow[d] & T\mathbb{C}^p \arrow[d] \\ \mathbb{C}^n \arrow[r, "f"'] \arrow[ur] & \mathbb{C}^p \end{tikzcd} \tag{1.2}$$

in which the vertical maps are bundle projections. If $\hat{f}$ is *any* diagonal map fitting in the diagram, then

$$f_t(x) = f(x) + t\hat{f}(x)$$

is a 1-parameter deformation whose derivative is $\hat{f}$. Thus the numerator in (1.1) is the free $\mathcal{O}_{\mathbb{C}^n,S}$ module on generators $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_p}$. We denote it by $\theta(f)$.

In particular, the expressions $\frac{\partial \varphi_t}{\partial t}|_{t=0}$ and $\frac{\partial \psi_t}{\partial t}|_{t=0}$, in the denominator of (1.1), define germs of vector fields on $(\mathbb{C}^n, S)$ and $(\mathbb{C}^p, 0)$ respectively. Denoting these by ξ and η we have

$$\frac{d(\psi_t \circ f \circ \psi_t)}{dt}|_{t=0} = df \circ \xi + \eta \circ f.$$

Once again, every germ of vector field ξ and η can appear in this way, so the denominator in (1.1) is equal to

$$\{df \circ \xi : \xi \in \theta_{\mathbb{C}^n,S}\} + \{\eta \circ f : \eta \in \theta_{\mathbb{C}^p,0}\}$$

We write the operators $\xi \mapsto df \circ \xi$ and $\eta \mapsto \eta \circ f$ as tf and ωf respectively, so finally the denominator in (1.1) takes the form

$$tf(\theta_{\mathbb{C}^n,S}) + \omega f(\theta_{\mathbb{C}^p,0}).$$

We call it the *extended tangent space* to the orbit of f, and denote it by $T\mathcal{A}_e f$.

The $\mathcal{A}_e$-codimension of f is the complex vector space dimension of the quotient (1.1). If this dimension is 0 then f is "infinitesimally stable"; in fact from this it follows, by Martinet's versality theorem (1.2 below) that f is stable: every parametrised deformation is trivial.

Example 1.1.

(1) *The germ in the centre of the first Reidemeister move can be parametrised by* $f(x) = (x^2, x^3)$. *Every power of* x, *except for* x^1, *can be written as a monomial in* x^2 *and* x^3, *so*

$$\omega f(\theta_{\mathbb{C}^2,0}) + Sp_{\mathbb{C}}\left\{\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}\right\} = \theta(f).$$

Now $\begin{pmatrix}0\\x\end{pmatrix}$ *is not in* $T\mathcal{A}_ef$*, since the order of the coefficient of* $\partial/\partial y_2$ *in every member of* $T\mathcal{A}_ef$ *is at least* 2*. On the other hand,*

$$tf\left(\frac{\partial}{\partial x}\right) = \begin{pmatrix}2x\\3x^2\end{pmatrix}$$

and it follows that

$$T\mathcal{A}_ef + Sp_{\mathbb{C}}\left\{\begin{pmatrix}0\\x\end{pmatrix}\right\} = \theta(f) \tag{1.3}$$

and f *has* $\mathcal{A}_e$*-codimension 1.*

(2) *For a multi-germ* $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ *with* $S = \{s_1, \dots, s_k\}$*, we denote by* f_j*, for* $j = 1, \dots, k$*, the associated mono-germs* $(\mathbb{C}^n, s_j) \to (\mathbb{C}^p, 0)$*. Elements of* $\theta(f)$ *can be represented by* $p \times k$ *matrices, with the* j*'th column representing the elements of* $\theta(f_j)$*. For example, consider the bi-germ*

$$g : \begin{cases} s \mapsto & (s, 0) \\ t \mapsto & (0, t) \end{cases}$$

parameterising a transverse crossing of two immersed branches. It is infinitesimally stable. To see this, observe that if a, b, c *and* d *all vanish at* 0 *then the element*

$$\begin{pmatrix}a(s) & c(t)\\ b(s) & d(t)\end{pmatrix} \tag{1.4}$$

of $\theta(g)$ *is equal to*

$$\omega g\begin{pmatrix}a(y_1) + c(y_2)\\ b(y_1) + d(y_2)\end{pmatrix},$$

while if a_0, b_0, c_0, d_0 *are arbitrary constants then*

$$tg(a_0 - c_0, d_0 - b_0) + \omega g\begin{pmatrix}c_0\\b_0\end{pmatrix} = \begin{pmatrix}a_0 & c_0\\ b_0 & d_0\end{pmatrix}.$$

This completes the proof of infinitesimal stability.

(3) *Consider the perturbation* $f_t : x \mapsto (x^2, x^3 - tx)$ *of the germ* f *in Example (1) above; it is an immersion, and for real* $t > 0$*, or any complex* $t \neq 0$*, it has one double point – the points* $\pm\sqrt{t}$ *have the same image,* $(t, 0)$*. The two branches of the image meet transversely at* $(t, 0)$*, and otherwise* f_t *is an embedding. Thus it is a stable perturbation of* f*. The image has the homotopy type of a circle, as you can see in Figure 2.*

Similar slightly more complicated calculations show that the codimension of Reidemeister moves II and III is also 1, again equal to the rank of their vanishing homology. Other elementary calculations with plane curve singularities register the same coincidence. The curve germ

$$x \mapsto (x^2, x^{2k+1})$$

has $\mathcal{A}_e$-codimension k (this is an easy exercise, mimicking the procedure in Example 1.1). On the other hand one can perturb it [1] to a curve whose only singularities are k transverse crossings

[1]One has to be careful what one means by a "perturbation" of an unstable map-germ. Its singularities must somehow emerge from the unstable point(s) of the original germ, rather than migrating in from somewhere distant. A proper definition requires the selection of a "conical" representative of f ([Fuk82]) – the equivalent for mappings of the well known notion of a conical neighbourhood of a point in an analytic variety. A perturbation is then a map $\tilde{f}_t$ obtained from a conical representative $\tilde{f} : U \to \mathbb{C}^p$ of f, by a parameterised deformation small enough so that during the passage from $\tilde{f}$ to $\tilde{f}_t$, the restriction to a neighbourhood of ∂U remains unchanged,

– indeed, this can even be done in a real perturbation. A disc (real or complex) with k pairs of points identified, is homotopy equivalent to a wedge of k circles, and has first homology $\mathbb{Z}^k$. So the topological complexity of the image of a stable perturbation, as measured by the rank of its first homology, is equal to the $\mathcal{A}_e$-codimension. One of the main unanswered questions is how far does this coincidence extend.

Before going on, I point out that the $\mathcal{A}_e$-tangent space of a map-germ f serves for more than the definition of the $\mathcal{A}_e$-codimension of f. The following *versality theorem* was proved for $\mathcal{A}$-equivalence by Jean Martinet in [Mar77] (and more accessibly published in [Mar82]).

Theorem 1.2. *An unfolding $F : (\mathbb{K}^n \times \mathbb{K}^d, S \times \{0\}) \to (\mathbb{K}^p \times \mathbb{K}^d, (0,0))$ of $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $F(x, t_1, \ldots, t_d) = (f_t(x), u)$, is $\mathcal{A}_e$-versal if and only if the images in $\theta(f)/T\mathcal{A}_e f$ of the initial velocities $\partial f_t/\partial t_i|_{t=0}, i = 1, \ldots, d$, span it as a $\mathbb{K}$-vector space.*

Versality of F means that every unfolding $G(x, u)$ of f is parameterised-equivalent, to an unfolding induced from F by a map of parameters $u \mapsto t(u)$. It follows that every perturbation of f is equivalent to f_t for some t.

Note that from the versality theorem it follows that if f is infinitesimally stable then it is stable. This makes it possible to clarify the notion of stable perturbation. It is simply a perturbation for which every germ is infinitesimally stable.

A versal unfolding contains every possible perturbation of f, up to equivalence; if f has a stable perturbation at all, then for a dense set of parameter values t, f_t, (defined on a suitably small domain) is a stable perturbation of f. The complement of this set of parameter values is an analytic subset of the base space ($\mathbb{R}^d$ or $\mathbb{C}^d$) of the unfolding F, and therefore in the complex case does not separate it. For this reason any two good parameter values t and t' can be joined by a path in the set of good parameter values. From this it follows that f_t and $f_{t'}$ are topologically equivalent, thus proving the (topological) uniqueness of the stable perturbation over $\mathbb{C}$.

We look at some more examples in two dimensions. It turns out that there are five "Reidemeister moves" for mappings from 2-space to 3-space. They were first described by Victor Goryunov in [Gor91]. I list them here, and in each case describe a 1-parameter versal unfolding, which the reader can check by finding a basis for $\theta(f)/T\mathcal{A}_e f$ and applying Theorem 1.2. They are

(1) The S_1 singularity (birth of two Whitney umbrellas), parameterised by

$$(x, y) \mapsto (x, y^2, y^3 \pm x^2 y).$$

Here, as in (2), the two forms, distinguished by $\pm$ in the third component, are inequivalent over $\mathbb{R}$ but equivalent over $\mathbb{C}$. The unfolding $F(x, y, t) = (f_t(x, y), t),$, with $f_t(x, y) = (x, y^2, y^3 \pm x^2 y + ty)$, is $\mathcal{A}_e$- versal.

(2) The Morse tangency (the surface equivalent of the tacnode RMII), a bi-germ parameterised by

$$\begin{cases} (x_1, y_1) & \mapsto & (x_1, y_1, 0) \\ (x_2, y_2) & \mapsto & (x_2, y_2, x_2^2 \pm y_2^2) \end{cases}$$

A versal unfolding on parameter u is obtained by adding the unfolding parameter t to the third component of f_1 (or of f_2).

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up to diffeomorphism. In the study of singularities of mappings, the notion of stable perturbation plays a role closely analogous to the role of the Milnor fibre in the theory of singular points of analytic varieties.

(3) The degenerate triple point, parameterised by

$$\begin{cases} (x_1, y_1) & \mapsto & (x_1, y_1, 0) \\ (x_2, y_2) & \mapsto & (0, x_2, y_2) \\ (x_3, y_3) & \mapsto & (x_3 - y_3^2, y_3, -x_3 - y_3^2) \end{cases}$$

Here three immersed surfaces meet two-by-two transversely, with each tangent to the curve of intersection of the other two. The unfolding in which f_3 is modified to

$$f_{3,t}(x_3, y_3) = (x_3 - y_3^2 + t, y_3, -x_3 - y_3^2 + t)$$

is $\mathcal{A}_e$-versal.

(4) The umbrella with an immersed plane passing through it, parameterised by

$$\begin{cases} (x_1, y_1) & \mapsto & (x_1, y_1^2, x_1 y_1) \\ (x_2, y_2) & \mapsto & (x_2, -x_2, y_2) \end{cases}$$

A versal unfolding is obtained by adding t to the second component of f_2.

(5) The quadruple point, in which four immersed planes meet, with each three in general position. The three coordinate planes and a fourth plane with equation $u + v + w = 0$ can be parameterised by

$$\begin{cases} (x_1, y_1) & \mapsto & (0, x_1, y_1) \\ (x_2, y_2) & \mapsto & (x_2, 0, y_2) \\ (x_3, y_3) & \mapsto & (x_3, y_3, 0) \\ (x_4, y_4) & \mapsto & (x_4, y_4, -x_4 - y_4) \end{cases}$$

This is versally unfolded by adding (t, t, t) to f_4.

Remarkably, as Goryunov's drawings show, each one (taking the positive variant in the first and second case, where there is a choice of sign) can be perturbed to a mapping whose image is homotopy-equivalent to a 2-sphere.

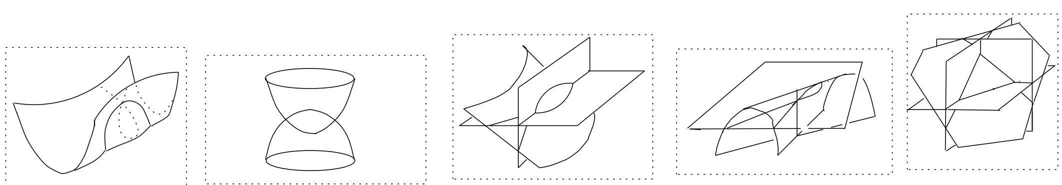

Figure 3: Images of stable perturbations of codimension 1 maps from 2-space to 3-space

The first three can be obtained from the three classical Reidemeister moves, by a procedure known as *augmentation*, introduced by Tom Cooper in his Warwick thesis in 1994 (see also [CMWA02] for a published account). In this, one takes a 1-parameter versal deformation $F(x, t) = (f_t(x), t)$ of a germ of map from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ and defines the augmentation Af of f, a germ from $\mathbb{C}^{n+1}$ to $\mathbb{C}^{n+2}$, by $Af(t, x) = (f_{t^2}(x), t)$. Cooper introduced two further operations by which one constructs new codimension 1 map-germs from codimension 1 map-germs one dimension lower down: they are known as monic and binary concatenation (see [CMWA02]). The effect of monic concatenation is to add the space $\{t = 0\}$ to the image of a versal unfolding F on parameter t. Augmentation and monic concatenation are shown as arrows in Figure 4. It is interesting to note that contained in the image Z_t of a stable perturbation of a monic concatenation of a germ f, one can see the image of a stable perturbation of f, as the intersection of Z_t with the hyperplane $\{t = 0\}$. Similarly, inside the image of a stable perturbation Y_t of an augmentation Af, one can see the image of a stable perturbation of f. Both sub-images are shown, drawn with double thickness, in the bottom row of Figure 4. Note that the middle row of Figure 4 shows the images of the germs rather than of their stable perturbations.

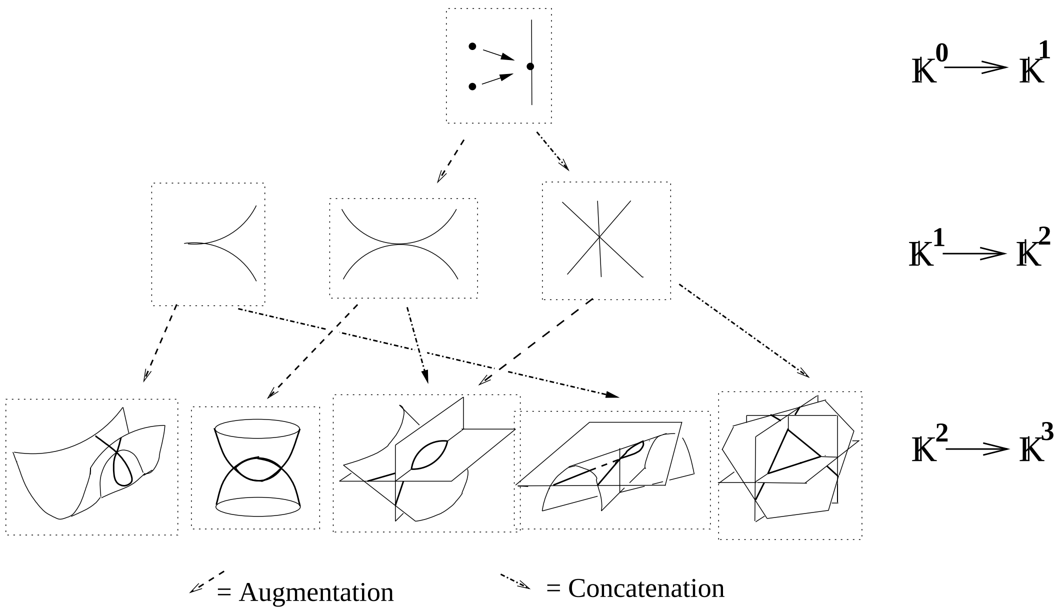

Figure 4: Augmentation and Concatenation generate new codimension 1 germs from old.

The coincidence of $\mathcal{A}_e$-codimension and the rank of the middle homology of the image of a stable perturbation continues to hold here. Indeed it was proved by several authors, beginning with de Jong and Pellikaan (unpublished) and then de Jong and van Straten [dJvS91], later Mond [Mon91b], that the standard relationship between μ and τ(where τ means codimension and μ means the rank of the vanishing homology) holds for germs of maps from surfaces to 3-space. Before stating it we need

Lemma 1.3. *([Sie91]) Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be a map-germ of finite $\mathcal{A}_e$-codimension. Then the image of a stable perturbation of f has the homotopy type of a wedge of n-spheres.*

The number of n -spheres in the wedge is called the *image Milnor number* of f, and denoted by μ_I. Warning: μ_I is not the same as the Milnor number of the image; if $n > 1$ and f is not the germ of an immersion, its image always has non-isolated singularity, so its Milnor number is ∞.

Theorem 1.4. *Let $f : (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ be a map germ of finite $\mathcal{A}_e$–codimension. Then*

(1.5) $\qquad \mu_I \geq \mathcal{A}_e - codim(f),$ *with equality if and only if f is quasihomogeneous.*

An identical result for germs of maps from $\mathbb{C}$ to $\mathbb{C}^2$ was proved in [Mon95]. Abundant evidence supports

Conjecture 1.5. (1.5) *holds for all values of n for which $(n, n+1)$ are in Mather's nice dimensions (cf* [Mat71]*)*[2] .

However, it remains unproved. I summarise the evidence:

(1) There is a comparable result for map germs $(\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ where $n \geq p$ and (n, p) are in Mather's nice dimensions: here it is the discriminant of a stable perturbation that carries the vanishing homology. Denoting the rank of its middle homology by μ_Δ, we have

Theorem 1.6. *(*([DM91])

$$\mu_\Delta \geq \mathcal{A}_e - \textit{codimension},$$

with equality if f is weighted homogeneous.

In fact, as we will see, the proof of Theorem 1.6 very nearly proves Conjecture 1.5, with just one crucial gap.

(2) Kevin Houston ([Hou98]) found a beautiful argument to prove (1.5) for germs of multiplicity (=dimension of the local algebra of the germ) 2: using a normal form for such map-germs, he was able to calculate both the $\mathcal{A}_e$ codimension and the image Milnor number, and show explicitly that they are equal.

(3) Examples of corank 1 germs of maps $(\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$ were described and classified by Houston and Kirk in [HK99]; all satisfied (1.5).

(4) In [Hou02], Kevin Houston generalised Cooper's construction of the augmentation of a germ of codimension 1; in place of the formula $Af(x, t) = (f_{t^2}(x), t)$ used by Cooper, he considers the germ $A_h f(x, t) = (f_{h(t)}, t)$, where $h : (\mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ defines an isolated hypersurface singularity and $F(x, t) = (f_t(x), t)$ is a 1-parameter stable unfolding of a finite codimension map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, which need not have $\mathcal{A}_e$-codimension 1. He shows that when both f and h are weighted homogeneous then $A_h f$ has both $\mathcal{A}_e$-codimension and μ_I equal to the product $\mu_I(f)\mu(h)$.

(5) It was shown in [CMWA02], using the classification of corank 1 stable mono-germs and Cooper's operations of augmentation and concatenation, that all codimension 1 multi-germs for which all constituent mono-germs are of corank ≤ 1 have $\mu_I = 1$ also (and all are quasihomogeneous, and all (modulo choice of real form) have stable pertubations exhibiting the vanishing homology over $\mathbb{R}$).

(6) Ayse Altintas, in [Alt12], gives examples of weighted homogeneous map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ of finite codimension for $n = 3$ and 4, and verifies (1.5) for all of them for which it it possible to calculate μ_I. This includes several infinite series and a sporadic example where $\mathcal{A}_e$-codimension$=\mu_I = 3825$. I return in a moment to a description of her method.

(7) Toru Ohmoto in [Ohm15] has recently developed Thom polynomial techniques which make possible the calculation of μ_I for weighted homogeneous germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ in terms of weights and degrees. He gives formulae for the cases $n = 2$ (already found

[2]The restriction to Mather's nice dimensions is for a curious reason, explained below in the sketch of the proof of Theorem 1.6.

by a different method in [Mon91a]) and $n = 3$ (which is new):

$$\mu_I = \frac{(w_0-d_1)(w_0-d_2)}{24w_0^4w_1w_2}\begin{pmatrix} d_1^2(d_2^2+3d_2w_0+2w_0^2) \\ +d_1w_0(3d_2^2-d_2(19w_0+4(w_1+w_2)) \\ +2w_0(w_0-2(w_1+w_2))) \\ +2w_0^2(d_2^2+d_2(w_0-2(w_1+w_2)) \\ +2(5w_0(w_1+w_2)+3w_1w_2)) \end{pmatrix} \tag{1.6}$$

Here f is assumed to be in linearly adapted form

$$f(x_0,x_1,x_2)\to(x_1,x_2,f_3(x),f_4(x))$$

with weights and degrees

$$(w_0,w_1,w_2)\to(w_1,w_2,d_1,d_2).$$

Ohmoto has checked this against the calculations of Ayse Altintas in [Alt12], with which it agrees. Ohmoto's formula should be compared with formulas derived by Victor Goryunov in [GM93, Section 4]. These are based on a calculation of the homology of the image X_t of a stable perturbation of a corank 1 map-germ : $(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$ in terms of the homology of the *multiple point spaces* $D^k(f_t)$:

$$H_n(X_t:\mathbb{Q})\simeq\bigoplus_{k=2}^{n+1}H^{\mathrm{Alt}}_{n-k+1}(D^k(f_t);\mathbb{Q}) \tag{1.7}$$

(to which I will return) so in the case $n=3$ contain 3 summands.

Here $H^{\mathrm{Alt}}_{n-k+1}(D^k(f_t);\mathbb{Q})$ is the isotypal summand of the representation of the symmetric group S_k on $H_{n-k+1}(D^k(f_t);\mathbb{Q})$ corresponding to the sign representation. In [GM93] there are formulae for the ranks of these modules in terms of weights and degrees, in the case of corank 1 mappings.

In view of Ohmoto's formulae, to verify 1.5 for weighted homogeneous map-germs it would be enough to have a formula for the $\mathcal{A}_e$-codimension of f in terms of weights and degrees. This brings us back to the question of how Altintas checks 1.5 in her examples. Note that the definition of $\mathcal{A}_e$-codimension as the dimension of

$$\frac{\theta(f)}{tf(\theta_{\mathbb{C}^n,0})+\omega f(\theta_{\mathbb{C}^p,0})} \tag{1.8}$$

is not very helpful: $tf:\theta_{\mathbb{C}^n,S}\to\theta(f)$ is a graded inclusion (when $n<p$), but the morphism induced by ωf,

$$\theta_{\mathbb{C}^p,0}\to\frac{\theta(f)}{tf(\theta_{\mathbb{C}^n,0})}$$

(whose cokernel is $\theta(f)/T\mathcal{A}_ef$) has kernel of projective dimension $p-1$ with no known standard projective resolution.

1.2. **Damon's method.** Jim Damon showed in [Dam91] how to calculate $\mathcal{A}_e$-codimension by a completely different method. If $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$ has stable unfolding

$$F:(\mathbb{C}^n\times\mathbb{C}^d,S\times\{0\})\to(\mathbb{C}^p\times\mathbb{C}^d,(0,0))$$

then there is a commutative diagram (from which I omit the base-points)

$$\begin{tikzcd} \mathbb{C}^n\times\mathbb{C}^d \arrow[r, "F"] & \mathbb{C}^p\times\mathbb{C}^d \\ \mathbb{C}^n \arrow[u, "j"] \arrow[r, "f"'] & \mathbb{C}^p \arrow[u, "i"'] \end{tikzcd}$$

in which the vertical arrows are just inclusions $x \mapsto (x, 0)$ and $y \mapsto (y, 0)$. This is in fact a fibre square: the $\mathbb{C}^n$ in the bottom left is the fibre product of the $\mathbb{C}^p$ and $\mathbb{C}^n \times \mathbb{C}^d$ in the bottom right and top left over the $\mathbb{C}^p \times \mathbb{C}^d$ in the top right, and the arrows

(1.9)
$$\begin{tikzcd} {} \\ {} \arrow[u, "j"] \arrow[r, "f"'] & {} \end{tikzcd}$$

are determined by the arrows

(1.10)
$$\begin{tikzcd} {} \arrow[r, "F"] & {} \\ & {} \arrow[u, "i"'] \end{tikzcd}$$

We denote by $i^*(F)$ the germ f in (1.9) resulting from the diagram (1.10). Everything about $i^*(F)$ should be calculable from information about arrows (1.10). It is not hard to check that the quotient (1.8) is isomorphic as $\mathcal{O}_{\mathbb{C}^p}$-module to the quotient

$$\frac{\theta(i)}{ti(\theta_{\mathbb{C}^p,0}) + i^*(\mathrm{Der}(-\log D))} \tag{1.11}$$

Here $\mathrm{Der}(-\log D)$ is the $\mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^d}$-submodule of $\theta_{\mathbb{C}^p\times\mathbb{C}^d}$ consisting of germs of vector fields which are tangent to the discriminant (= image when $n < p$) D of F. Damon showed in [Dam91] that this quotient is isomorphic to (1.8) for *any* germ f obtained by transverse fibre product of i and F with F stable. The argument in [Mon15] is just linear algebra together with the non-trivial but unsurprising fact that $\mathrm{Der}(-\log D)$ is the kernel of the morphism

$$\theta_{\mathbb{C}^p\times\mathbb{C}^d} \to \frac{\theta(F)}{tF(\theta_{\mathbb{C}^n\times\mathbb{C}^d})}.$$

The module (1.11) measures the failure of transversality of the mapping i to the distribution $\mathrm{Der}(-\log D)$; reduced modulo $\mathfrak{m}_{\mathbb{C}^p,0}$ (i.e. evaluating everything at $0 \in \mathbb{C}^p$) it simply becomes

$$\frac{T_{(0,0)}\mathbb{C}^p \times \mathbb{C}^d}{d_0 i(T_0\mathbb{C}^p) + \mathrm{Der}(-\log D)((0,0))}.$$

Stability of f is equivalent to the transversality of i to the distribution $\mathrm{Der}(-\log D)$. To obtain a stable perturbation of f, we perturb i so that it becomes transverse to $\mathrm{Der}(-\log D)$.

The module (1.11) has the advantage over (1.8) that the numerator is a free module over $\mathcal{O}_{\mathbb{C}^p,0}$ and both modules in the denominator are finitely generated submodules. However its main virtue is that one can extract information about the image Milnor number from the closely related module

$$\frac{\theta(i)}{ti(\theta_{\mathbb{C}^p,0}) + i^*(\mathrm{Der}(-\log h))}, \tag{1.12}$$

where h is an equation for D and $\mathrm{Der}(-\log h)$ means the submodule of $\mathrm{Der}(-\log D)$ consisting of germs of vector fields tangent to all the level sets of h (rather than just $D = \{h = 0\}$). Before proceeding, we note that in general the module in (1.11) is a quotient of the module in (1.12), since $\mathrm{Der}(-\log h) \subset \mathrm{Der}(-\log D)$, and if D and i are weighted homogeneous with respect to the same weights, then (1.12) and (1.11) are the same: $\mathrm{Der}(-\log D)$ is a direct sum of $\mathrm{Der}(-\log h)$ and the $\mathcal{O}_{\mathbb{C}^p,0}$-module generated by the Euler vector field χ_e, and $\chi_e \circ i \in ti(\theta_{\mathbb{C}^p,0})$. By a standard argument involving coherence, one can show that if $I(y,t)=i_t(y)$ is any deformation of $i=i_0$, then

$$\dim_{\mathbb{C}}\frac{\theta(i)}{ti(\theta_{\mathbb{C}^p,0})+i^*(\mathrm{Der}(-\log h))}\geq\sum_y\dim_{\mathbb{C}}\frac{\theta(i_t)_y}{ti_t(\theta_{\mathbb{C}^p,y})+i_t^*(\mathrm{Der}(-\log h))_y}. \tag{1.13}$$

Proposition 1.7. *Provided (n,p) are nice dimensions, the right hand side in (1.13) is the image Milnor number when $p=n+1$, and the discriminant Milnor number when $p\leq n$.*

The proof involves three steps:

(1) For each point $y\notin D(f_t)$, differentiation of a defining equation by vector fields gives rise to an isomorphism
$$\frac{\theta(i_t)_y}{ti_t(\theta_{\mathbb{C}^p,y})+i_t^*(\mathrm{Der}(-\log h))_y}\simeq\frac{\mathcal{O}_{\mathbb{C}^p,y}}{J_{h\circ i_t}}$$
and thus
$$\sum_{y\notin D(f_t)}\dim_{\mathbb{C}}\frac{\theta(i_t)_y}{ti_t(\theta_{\mathbb{C}^p,y})+i_t^*(\mathrm{Der}(-\log h))_y}=\sum_{y\notin D(f_t)}\dim_{\mathbb{C}}\frac{\mathcal{O}_{\mathbb{C}^p,y}}{J_{h\circ i_t}} \tag{1.14}$$

(2) the right hand side in (1.14) is the rank of the middle homology of $D(f_t)$. This is shown by Siersma in [Sie91].

(3) At all points $y\in D(f_t)$,
$$\frac{\theta(i_t)_y}{ti_t(\theta_{\mathbb{C}^p,y})+i_t^*(\mathrm{Der}(-\log D))_y}=0$$
by the isomorphism of (1.11) and (1.8), for we are assuming f_t is stable. In the nice dimensions, all stable germs are quasihomogeneous, and so
$$\frac{\theta(i_t)_y}{ti_t(\theta_{\mathbb{C}^p,y})+i_t^*(\mathrm{Der}(-\log h))_y}=\frac{\theta(i_t)_y}{ti_t(\theta_{\mathbb{C}^p,y})+i_t^*(\mathrm{Der}(-\log D))_y}=0.$$
Thus,
$$\sum_y=\sum_{y\notin D(f_t)}+\sum_{y\in D(f_t)}=\sum_{y\notin D(f_t)}=\mu_\Delta$$

From 1.7 it follows that for a weighted homogeneous germ, $\mu_I=\mathcal{A}_e$-codimension *if and only if* the inequality in (1.13) is an *equality*. So the conjecture is equivalent to *conservation of multiplicity* of the module (1.12). When $n\geq p$, we do have conservation of multiplicity, and this is how Theorem 1.6 is proved. The argument uses a classical theorem of Buchsbaum and Rim, together with the fact that the discriminant of a stable map-germ $F:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$, with $n\geq p$ is a free divisor. Here is a summary:

We obtain f from F by the following fibre square:

$$\begin{tikzcd} \mathbb{C}^N \arrow[r, "F"] & \mathbb{C}^P \\ \mathbb{C}^n \arrow[u] \arrow[r, "f"'] & \mathbb{C}^p \arrow[u, "i"'] \end{tikzcd} \tag{1.15}$$

in which $i\pitchfork F$ and $P-N=p-n$. Let $I(y,t)=i_t(y)$ be a deformation of i. The relative version of the module (1.12),
$$T^1_{\mathrm{rel}}:=\frac{\theta(I)}{tI(\theta_{\mathbb{C}^p\times\mathbb{C}^d/\mathbb{C}^d})+I^*(\mathrm{Der}(-\log D))}$$

has presentation

$$\theta_{\mathbb{C}^p\times\mathbb{C}^d/\mathbb{C}^d}\oplus I^*(\mathrm{Der}(-\log D))\to\theta(I). \tag{1.16}$$

Now $\theta_{\mathbb{C}^p\times\mathbb{C}^d/\mathbb{C}^d}$ is free of rank p, and because $\mathrm{Der}(-\log D)$ is free of rank P, $I^*(\mathrm{Der}(-\log h))$ is free of rank $P-1$; thus 1.16 can be written in the form

$$\mathcal{O}^p\oplus\mathcal{O}^{P-1}\ \to\ \mathcal{O}^P, \tag{1.17}$$

where $\mathcal{O}=\mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^d,0}$. The theorem of Buchsbaum and Rim states that the codimension of the support of the cokernel T^1_{rel} is $\leq p$, and that if equality holds then T^1_{rel} is Cohen Macaulay as $\mathcal{O}$-module. From this it follows that its push-forward $\pi_*(T^1_{\mathrm{rel}})$ to the base space $\mathbb{C}^d$ is free, with rank equal to the dimension of the module (1.12); this implies conservation of multiplicity.

Now for a weighted homogeneous germ, $\mu_I=\mathcal{A}_e$-codimension if and only if $\pi_*(T^1_{\mathrm{rel}})$ is free, and its freeness is equivalent to T^1_{rel} being Cohen Macaulay of grade p; thus conjecture 1.5 is equivalent to the statement that T^1_{rel} is Cohen Macaulay of grade p. When $p=n+1$ then, unlike the case $n\geq p$, no general theorem I know of shows this. It is possible to check Cohen-Macaulayness in examples by using computer algebra packages like *Macaulay* or *Singular*, and this is what Altintas does in her examples. But why should this hold in general?

2. Multiple Point Spaces

The rank, μ_I or μ_Δ, of the vanishing homology of image or discriminant is its crudest topological invariant. There are more subtle topological descriptors. All of the images Y_t of the stable perturbations in Figure 2 have $H_2(Y_t)\simeq\mathbb{Z}$, but the vanishing cycles spring from very different geometrical origins. These can be easily appreciated in the case of two dimensional images, especially when there are good real pictures, in which the real image carries the vanishing homology of the complex image. In higher dimensions they are less evident. The image-computing spectral sequence introduced in [GM93] and [Gor95] computes the homology of the image of a map from the homology of its multiple-point spaces, and reflects these different origins. For mono-germs, the following theorem is proved in [Hou97], generalising an earlier statement in [GM93] (where it is proved for stable perturbations of corank 1 map-germs of finite $\mathcal{A}_e$-codimension).

Theorem 2.1. *Let $f_t:U\to\mathbb{C}^{n+1}$ be a stable perturbation of a map-germ $f:(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$ of finite $\mathcal{A}$-codimension. There is a natural increasing filtration*

$$0=F_1\subseteq F_2\subseteq\ldots\subseteq F_{n+1}=H_n(Y_t;\mathbb{Z})$$

with

$$F_k/F_{k-1}\simeq H^{Alt}_{n-k+1}(D^k(f_t)). \tag{2.1}$$

Here $H^{\mathrm{Alt}}_{n-k+1}(D^k(f_t))$ is the homology of the *alternating chain complex*, introduced by Goryunov in [Gor95]. This is the subcomplex of the singular chain complex consisting of chains on which the symmetric group S_k acts by its sign representation. When integer homology is replaced by rational homology, $H^{\mathrm{Alt}}_{n-k+1}(D^k(f_t))$ is simply the isotypal summand of $H_{n-k+1}(D^k(f_t);\mathbb{Q})$ corresponding to the sign representation, as in the earlier version of the spectral sequence in [GM93].

There is a version of Theorem 2.1 also for the parametrisation of the discriminant of a stable perturbation f_t of a map-germ $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$ with $n\geq p$, given by restricting f_t to its critical set. In this case the filtration begins with $0=F_0\subseteq F_1\subseteq\cdots$ since the critical set of f_t may itself have vanishing cycles.

To highlight the information these descriptions give, consider the case of mono-germs for which μ_I or μ_Δ are equal to 1. According to Conjecture 1.5 and Theorem 1.6, these are the germs of $\mathcal{A}_e$-codimension 1. By 2.1 and its version for discriminants, just one of the multiple point spaces of f_t or $f_t|\Sigma_{f_t}$ has an alternating vanishing cycle, which gives rise to the vanishing cycle in the image or discriminant of f_t.

Question 2.2. *(i) In the case of a stable perturbation of a $\mathcal{A}_e$-codimension 1 mono-germ, how to determine which multiple-point space carries the vanishing alternating cycle?*

(ii) For those stable map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ whose restriction to a generic hyperplane in $(\mathbb{C}^p, 0)$ [3] has $\mathcal{A}_e$-codimension 1, the answer to (i) is determined by the local algebra of the germ, since stable germs are classified by their local algebra. What is this invariant of the algebra?

For example, for the minimal (i.e. not an augmentation of a germ in lower dimensions) codimension 1 map-germ $f : (\mathbb{C}^{2m-1}, 0) \to (\mathbb{C}^{2m}, 0)$ of corank 1, the vanishing homology in the image of a stable perturbation f_t comes from an alternating vanishing cycle in $H_m(D^{m+1}(f_t))$ (see [CMWA02, Section 4]).

Question 2.3. *What is the relation between the cohomological version of the filtration in 2.1 and the weight or Hodge filtrations in the mixed Hodge structure on the vanishing cohomology of images and discriminants?*

Question 2.4. *How to calculate the alternating homology of the multiple point spaces of a stable perturbation for map-germs of corank > 1?*

The examples of corank 2 germs of maps from surfaces to 3-space described in [MNB08] may well provide a useful starting point.

Note that if f has corank > 1, we do not even have explicit generators for the defining ideals of the multiple points spaces $D^k(f)$ for $k > 2$.

3. Fitting ideals

If M is a module over a ring R with presentation $R^q \xrightarrow{\Lambda} R^p \to M \to 0$, the k'th *Fitting ideal* of M, $\mathrm{F}_k^R(M)$, is the ideal in R generated by the minors of size $p-k$ of the matrix Λ provided $q \geq p-k > 0$; $\mathrm{F}_k^R(M)$ is defined to be 0 if $q < p-k$, and R if $p-k \leq 0$. It is not hard to show that this defintion is independent of the choice of presentation. To interpret it, we define $\mu_R(M)$ to be the minimal cardinality of a set of generators for M over R. Then it is easily shown that $V(\mathrm{F}_k(M)) = \{p \in \mathrm{Spec\ R} : \mu_{R_p}(M_p) > k\}$. In analytic geometry, if M is an $\mathcal{O}_X$-module then the Fitting ideal sheaf $\mathcal{F}_k(M)$ is is a sheaf of ideals of $\mathcal{O}_X$ defined analogously, so that its stalk at x is the k'th Fitting ideal of M_x over $\mathcal{O}_{X,x}$.

If $f : X \to Y$ is a finite analytic map then it follows that $\mathcal{F}_k^{\mathcal{O}_Y}(f_*(\mathcal{O}_X))$ defines the set $M_{k+1}(f)$ of points in Y with $k+1$ or more preimages, counting multiplicity. When X is Cohen-Macaulay of dimension n and Y is a complex manifold of dimension $n+1$ respectively, then a minimal presentation of $f_*(\mathcal{O}_X)$ as $\mathcal{O}_Y$-module is a square matrix. In particular, its determinant generates $\mathcal{F}_0^{\mathcal{O}_Y}((f_*(\mathcal{O}_X))$ and so defines the image of f. We continue to denote the size of this (square) matrix by p. This application of Fitting ideals has been studied by Gruson and Peskine in [GP82], by Mond and Pellikaan in [MP89] and by Kleiman, Lipman and Ulrich in [KLU92], [KLU96] and [KU97], and by Altintas and Mond in [AM13]. When $k > 0$, the expected codimension of $M_k(f)$ in $\mathbb{C}^{n+1}$, k, is different from the codimension of the variety of zeros of the ideal of $(p-k+1)$ minors of a generic $p \times p$ matrix, and so standard structure theorems on generic determinantal varieties give no information on the spaces $M_k(f)$.

Nevertheless, a series of refinements of the description of the ideals $\mathcal{F}_k(f_*(\mathcal{O}_X))$, based on the fact that $f_*(\mathcal{O}_X)$ is an $\mathcal{O}_Y$-algebra, shows that for $k = 0, 1$ and 2, $\mathcal{O}_Y / \mathcal{F}_k(f_*(\mathcal{O}_X))$ is

[3] That is, the germ $i^*(F)$ resulting from the diagram (1.10) where i parametrises a generic hyperplane.

Cohen Macaulay provided it has the expected dimension. In particular, $\mathcal{O}_X$ has a distinguished generator 1 and therefore there is a distinguished row in the matrix Λ of any presentation. The $(p-1)$ minors of the matrix obtained by deleting the distinguished row of Λ were shown in [MP89] to generate $\mathcal{F}_1(f_*(\mathcal{O}_X))$; it follows that as a codimension 2 variety defined by the maximal minors of a $(p-1)\times p$ matrix, $V(\mathcal{F}_1)$ is Cohen-Macaulay. When X is Gorenstein, then $\mathcal{O}_X$ is presented by a symmetric matrix Λ over $\mathcal{O}_Y$ ([MP89]), and [MP89] goes on to show that $\mathcal{F}_2(\mathcal{O}_X)$ is generated by the $(p-2)$ minors of the matrix obtained by deleting the distinguished row and column of Λ. Again, Cohen-Macaulayness of $\mathcal{O}_Y/\mathcal{F}_2$ follows, this time by a theorem on the minors of a generic symmetric matrix due to Józefiak in [Józ78] .

In a similar vein, Gruson and Peskine showed in [GP82] that if f is a map of corank 1 (a "curvilinear map" in the language of Kleiman *et al*), then for each k, if $V(F_k)$ has codimension $k+1$ in Y, then $\mathcal{F}_k(f_*\mathcal{O}_X)$ defines a Cohen-Macaulay space. The result was reproved in [MP89]. Here the fact that $\mathcal{O}_X$ is cyclic as $\mathcal{O}_Y$-algebra – generated over $\mathcal{O}_Y$ by powers of a primitive element – provides a nested family of $(p-k)\times p$ submatrices of the matrix Λ of a presentation of $\mathcal{O}_X$ as $\mathcal{O}_Y$-module with respect to these powers. It turns out that $\mathcal{F}_k$ is generated by the maximal minors of the $(p-k)\times p$ submatrix; this makes it dimensionally correct, and now Cohen Macaulayness follows from the theorem of Buchsbaum and Rim for generic matrices.

In all of these cases, progress is made by using the fact that $\mathcal{O}_X$ is an $\mathcal{O}_Y$ algebra of a certain type (Cohen Macaulay, Gorenstein, cyclic, ...) to show that the relevant Fitting ideal is in fact generated by the minors of a suitable submatrix Λ' of Λ. Whereas the codimension of $V(F_k)$ is different from the codimension of the variety of zeros of the ideal of $(p-k)\times(p-k)$ minors of a generic $p\times p$ matrix, in each case the Cohen-Macaulayness of $\mathcal{O}_Y/F_k$ is proved by showing that this codimension is the right one for the minors of a generic matrix of the size of Λ'.

Let us refer to this as the *submatrix method*.

Work on proving Cohen-Macaulayness for these target multiple point spaces defined by Fitting ideals seems to have come largely to a stop after the 1997 paper of Kleiman and Ulrich. The submatrix method had exhausted its potential. It seems that an approach is needed which engages with the $\mathcal{O}_Y$-algebra stucture of $\mathcal{O}_X$ more deeply. The recent development of computer algebraic geometry packages such as *Macaulay* and *Singular*, together with increases in computational power, have brought the calculation of more Fitting ideal multiple-point spaces within reach. Calculations now suggest that there is more to be proved. Here are two rather tendentious conjectures, which are supported by all the calculations I have been able to do.

Conjecture 3.1. *Let $f:(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$ be finite and generically 1-1, and suppose that $\dim M_j(f)=n+1-j$ for $0\le j\le k$. Then $\mathcal{O}_{Y,0}/F_k(f_*\mathcal{O}_X))$ is Cohen Macaulay.*

Conjecture 3.2. *With f as in Conjecture 3.1, let Λ be a symmetric presentation matrix for $f_*(\mathcal{O}_X)$ over $\mathcal{O}_Y$, with respect to generators $f_1=1,g_2,\dots,g_p$. Let Λ' be the $(p-1)\times(p-1)$ matrix obtained from Λ by deleting its first row and column. Then $F_k(f_*(\mathcal{O}_X))$ is generated by the $(p-k)$ minors of Λ'.*

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