Note on an Inequality of N. G. de Bruijn

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Abstract

In this note an inequality of N. G. de Bruijn is used to obtain inequalities involving finite sums of sequences of real and complex numbers.

Keywords: Inequalities in the complex plane, multivariate inequalities, finite sums.

1 INTRODUCTION

In 1960 N. G. de Bruijn ([1], [2]) established the following refinement of the classical Cauchy-Buniakowsky-Schwarz inequality:

Theorem A *If $a_1, a_2, \ldots, a_n$ is a sequence of real numbers and $z_1, z_2, \cdots, z_n$ is a sequence of complex numbers, then*

$$\left|\sum_{k=1}^{n} a_k z_k\right|^2 \leq \frac{1}{2}\sum_{k=1}^{n} a_k^2 \left[\sum_{k=1}^{n} |z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right] \tag{1}$$

Equality holds when there is $\alpha \in \mathbb{C}$ such that $\alpha^2 \sum_{k=1}^{n} z_k^2 \geq 0$ and $a_k = Re(\alpha z_k)$ for all k.

In this paper we employ Theorem A to obtain some inequalities for sums of complex numbers similar to those presented in ([3], [4], [5] [6], [7]).

2 THE INEQUALITIES

We now establish the main result.

Theorem 1 *Let $z_1, z_2, \ldots, z_n, (n \geq 2)$ be a sequence of complex numbers. Then,*

$$\sum_{1 \leq i < j \leq n} |z_i + z_j|^2 \leq \left(\frac{3n-4}{2}\right)\sum_{k=1}^{n} |z_k|^2 + \frac{n}{2}\left|\sum_{k=1}^{n} z_k^2\right|.$$

Equality holds when there is $\alpha \in \mathbb{C}$ *such that* $\alpha^2 \sum_{k=1}^n z_k^2 \geq 0$ *and* $a_k = Re(\alpha z_k)$ *for all* k*. In particular, the constants* $(3n-4)/2$ *and* $n/2$ *are the best possible.*

Proof. We begin with an identity.

Lemma 1 *If* $z_1, z_2, \ldots, z_n, (n \geq 2)$ *is a sequence of complex numbers, then*

$$\sum_{1 \leq i < j \leq n} |z_i + z_j|^2 = (n-2) \sum_{k=1}^n |z_k|^2 + \left| \sum_{k=1}^n z_k \right|^2.$$

Proof. The proof is by induction on n. For $n = 2$ the identity is trivial. Assume

$$\sum_{1 \leq i < j \leq n} |z_i + z_j|^2 = (n-2) \sum_{k=1}^n |z_k|^2 + \left| \sum_{k=1}^n z_k \right|^2.$$

Then,

$$\begin{aligned} \sum_{1 \leq i < j \leq n+1} |z_i + z_j|^2 &= (n-2) \sum_{k=1}^n |z_k|^2 + \left| \sum_{k=1}^n z_k \right|^2 + \sum_{k=1}^n |z_k + z_{n+1}|^2 \\ &= (n-1) \sum_{k=1}^{n+1} |z_k|^2 + |z_{n+1}|^2 + \sum_{k=1}^n (z_k \overline{z}_{n+1} + \overline{z}_k z_{n+1}) \\ &= (n-1) \sum_{k=1}^{n+1} |z_k|^2 + \left| \sum_{k=1}^{n+1} z_k \right|^2. \end{aligned}$$

The proof is complete.

□

Setting $a_k = 1, (1 \leq k \leq n)$ into (1), we have

$$\left| \sum_{k=1}^n z_k \right|^2 \leq \frac{n}{2} \left(\sum_{k=1}^n |z_k|^2 + \left| \sum_{k=1}^n z_k^2 \right| \right).$$

On the other hand, from the previous lemma, we have

$$\left| \sum_{k=1}^n z_k \right|^2 = \sum_{1 \leq i < j \leq n} |z_i + z_j|^2 - (n-2) \sum_{k=1}^n |z_k|^2.$$

Substituting the preceding expression into the previous one, and rearranging terms, the statement immediately follows and the proof is complete.

□

Lemma 2 *Let $z_1, z_2, \ldots, z_n, (n \geq 2)$ be complex numbers. Then,*

$$\sum_{1 \leq i < j \leq n} |z_i - z_j|^2 = n \sum_{k=1}^{n} |z_k|^2 - \left| \sum_{k=1}^{n} z_k \right|^2 .$$

Proof. The case $n = 2$ is easily checked. Assume

$$\sum_{1 \leq i < j \leq n} |z_i - z_j|^2 = n \sum_{k=1}^{n} |z_k|^2 - \left| \sum_{k=1}^{n} z_k \right|^2 .$$

Then,

$$\begin{aligned} \sum_{1 \leq i < j \leq n+1} |z_i - z_j|^2 &= n \sum_{k=1}^{n} |z_k|^2 - \left| \sum_{k=1}^{n} z_k \right|^2 + \sum_{k=1}^{n} |z_k|^2 + n|z_{n+1}|^2 \\ &- \sum_{k=1}^{n} (\overline{z}_k z_{n+1} + z_k \overline{z}_{n+1}) \\ &= (n+1) \sum_{k=1}^{n+1} |z_k|^2 - \left| \sum_{k=1}^{n} z_k \right|^2 - |z_{n+1}|^2 \\ &- \overline{z}_{n+1} \left(\sum_{k=1}^{n} z_k \right) - z_{n+1} \left(\sum_{k=1}^{n} \overline{z_k} \right) \\ &= (n+1) \sum_{k=1}^{n+1} |z_k|^2 - \left| \sum_{k=1}^{n+1} z_k \right|^2 . \end{aligned}$$

The statement is proved.

□

Notice that the preceding lemma gives an explicit expression for number $R = n \sum_{k=1}^{n} |z_k|^2 - |\sum_{k=1}^{n} z_k|^2$, which is always positive because of the Cauchy-Buniakowsky-Schwarz inequality, in terms of a finite sum of positive numbers.

Theorem 2 *Let $z_1, z_2, \ldots, z_n, (n \geq 2)$ be complex numbers. Then,*

$$\sum_{1 \leq i < j \leq n} |z_i - z_j|^2 \geq \frac{n}{2} \left(\sum_{k=1}^{n} |z_k|^2 - \left| \sum_{k=1}^{n} z_k^2 \right| \right) .$$

Equality holds when there is $\alpha \in \mathbb{C}$ such that $\alpha^2 \sum_{k=1}^{n} z_k^2 \geq 0$ and $a_k = Re(\alpha z_k)$ for all k.

Proof. Adding up the identities in Lemma 1 and Lemma 2, we get

$$\sum_{1\le i<j\le n} |z_i + z_j|^2 + \sum_{1\le i<j\le n} |z_i - z_j|^2 = 2(n-1)\sum_{k=1}^{n} |z_k|^2.$$

Now, applying Theorem 1, we have

$$2(n-1)\sum_{k=1}^{n} |z_k|^2 - \sum_{1\le i<j\le n} |z_i - z_j|^2 \le \left(\frac{3n-4}{2}\right)\sum_{k=1}^{n} |z_k|^2 + \frac{n}{2}\left|\sum_{k=1}^{n} z_k^2\right|$$

or equivalently

$$\frac{n}{2}\sum_{k=1}^{n} |z_k|^2 \le \sum_{1\le i<j\le n} |z_i - z_j|^2 + \frac{n}{2}\left|\sum_{k=1}^{n} z_k^2\right|$$

from which the statement immediately follows and the proof is complete. □

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