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Profinite Heyting Algebras

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Abstract For a Heyting algebra A, we show that the following conditions are equivalent: (i) A is profinite; (ii) A is finitely approximable, complete, and completely join-prime generated; (iii) A is isomorphic to the Heyting algebra $\mathrm{Up}(X)$ of upsets of an image-finite poset X. We also show that A is isomorphic to its profinite completion iff A is finitely approximable, complete, and the kernel of every finite homomorphic image of A is a principal filter of A.

Keywords Profinite algebras · Heyting algebras · Duality theory

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1 Introduction

An algebra A is called *profinite* if A is isomorphic to the inverse limit of an inverse family of finite algebras. It is well-known (see, e.g., [8, Sec. VI.2 and VI.3]) that a Boolean algebra is profinite iff it is complete and atomic, and that a distributive lattice is profinite iff it is complete and completely join-prime generated. In [2], a dual description of the profinite completion of a Heyting algebra was given, and a connection between profinite and canonical completions of a Heyting algebra was investigated. On the other hand, no characterization of profinite Heyting algebras

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was known. In this note we fill in this gap by providing several equivalent conditions for a Heyting algebra A to be profinite. In particular, we prove that the following conditions are equivalent: (i) A is profinite; (ii) A is finitely approximable, complete, and completely join-prime generated; (iii) A is isomorphic to the Heyting algebra $\mathrm{Up}(X)$ of upsets of an image-finite poset X. We also provide a dual description of profinite Heyting algebras, and show that a Heyting algebra A is isomorphic to its profinite completion iff A is finitely approximable, complete, and the kernel of every finite homomorphic image of A is a principal filter of A. These characterizations of profinite Heyting algebras have many consequences, some known, but with new simpler proofs, and some new. For example, the description of profinite Boolean algebras is an easy consequence of our results. We also show that a Boolean algebra, or more generally, a Heyting algebra which belongs to a finitely generated variety of Heyting algebras is isomorphic to its profinite completion iff it is finite. Although similar results for distributive lattices are not immediate consequences of our results for Heyting algebras, they are obtained by a simple modification of our proofs. Finally, we describe profinite linear Heyting algebras, prove that a finitely generated Heyting algebra is profinite iff it is finitely approximable and complete, and show that the free 1-generated Heyting algebra (also known as the Rieger-Nishimura lattice) is up to isomorphism a unique profinite free finitely generated Heyting algebra.

2 Complete and Completely Join-Prime Generated Heyting Algebras

We recall that a *Heyting algebra* is a bounded distributive lattice $(A, \wedge, \vee, 0, 1)$ with a binary operation $\to : A^2 \to A$ such that for each $a, b, c \in A$ we have

$$a \wedge c \leq b \text{ iff } c \leq a \to b\,.$$

We also recall that a *Priestley space* is a pair $(X, \leq)$ such that X is a compact space, $\leq$ is a partial order on X, and for each $x, y \in X$, whenever $x \not\leq y$, there exists a clopen upset U of X such that $x \in U$ and $y \notin U$. A Priestley space is an *Esakia space* if for each open subset U of X we have $\downarrow U$ is open in X.

The same way Priestley spaces serve as duals of bounded distributive lattices [10], Esakia spaces serve as duals of Heyting algebras [4]. In fact, every bounded distributive lattice or Heyting algebra A can be represented as the algebra $\mathrm{Up}_\tau(X)$ of clopen upsets of the dual space X of A. The construction of X is well-known: X is the set of prime filters of A ordered by inclusion; for $a \in A$, let

$$\phi(a) = \{x \in X : a \in x\},$$

and generate a topology on X by the basis $\{\phi(a) - \phi(b) : a, b \in A\}$. Then X becomes a Priestley space and ϕ becomes a bounded lattice isomorphism from A to $\mathrm{Up}_\tau(X)$; moreover, whenever A is a Heyting algebra, we have

$$\phi(a \to b) = (\downarrow(\phi(a) - \phi(b)))^c.$$

Let **DL** denote the category of bounded distributive lattices and bounded lattice homomorphisms, and let **HA** denote the category of Heyting algebras and Heyting algebra homomorphisms. Let also **PS** denote the category of Priestley spaces and continuous order-preserving maps. For posets X and Y, we recall that an order-preserving map $f : X \to Y$ is a *bounded morphism* if for each $x \in X$ and $y \in Y$

with $f(x) \leq y$, there exists $z \in X$ such that $x \leq z$ and $f(z) = y$. Let **ES** denote the category of Esakia spaces and continuous bounded morphisms.

Theorem 2.1

(1) (Priestley [10]) **DL** *is dually equivalent to* **PS**.
(2) (Esakia [4]) **HA** *is dually equivalent to* **ES**.

For a Priestley space X and a subset U of X, let $\mathbf{J}U$ denote the largest open upset of X contained in U, and let $\mathbf{D}U$ denote the smallest closed upset of X containing U. The following lemma, established in [7, Lemmas 3.1 and 3.6], will be useful subsequently.

Lemma 2.2 *Let A be a bounded distributive lattice, X be the dual Priestley space of A, and $Y \subseteq X$.*

(1) $\mathbf{J}Y = (\downarrow(\mathrm{Int}(Y))^c)^c = \bigcup\{\phi(a) : \phi(a) \subseteq Y\}$.
(2) $\mathbf{D}Y = \uparrow\overline{Y} = \bigcap\{\phi(a) : Y \subseteq \phi(a)\}$. *Moreover, if X is an Esakia space* (*that is, A is a Heyting algebra*) *and Y is an upset of X, then* $\mathbf{D}Y = \overline{Y}$.

In order to give a dual characterization of complete distributive lattices and complete Heyting algebras, we need the following lemma.

Lemma 2.3 *Let A be a bounded distributive lattice, B be a subset of A, and X be the dual Priestley space of A.*

(1) *$\bigvee B$ exists in A iff $\mathbf{D}(\bigcup_{b\in B}\phi(b))$ is clopen in X.*
(2) *If A is a Heyting algebra, then $\bigvee B$ exists in A iff $\overline{\bigcup_{b\in B}\phi(b)}$ is clopen in X.*
(3) *$\bigwedge B$ exists in A iff $\mathbf{J}(\bigcap_{b\in B}\phi(b))$ is clopen in X.*

Proof

(1) First assume that $\bigvee B$ exists in A. Then $b \leq \bigvee B$, and so $\phi(b) \subseteq \phi(\bigvee B)$ for each $b \in B$. Therefore, $\bigcup_{b\in B}\phi(b) \subseteq \phi(\bigvee B)$, and so $\mathbf{D}(\bigcup_{b\in B}\phi(b)) \subseteq \phi(\bigvee B)$ since $\phi(\bigvee B)$ is a closed upset. Now suppose that $x \notin \mathbf{D}(\bigcup_{b\in B}\phi(b))$. By Lemma 2.2, there is $a \in A$ such that $x \notin \phi(a)$ and $\bigcup_{b\in B}\phi(b) \subseteq \phi(a)$. Therefore, $b \leq a$ for each $b \in B$, and so $\bigvee B \leq a$. It follows that $\phi(\bigvee B) \subseteq \phi(a)$, and so $x \notin \phi(\bigvee B)$. Thus, $\phi(\bigvee B) \subseteq \mathbf{D}(\bigcup_{b\in B}\phi(b))$. Consequently, $\phi(\bigvee B) = \mathbf{D}(\bigcup_{b\in B}\phi(b))$, and so $\mathbf{D}(\bigcup_{b\in B}\phi(b))$ is clopen in X. Now let $\mathbf{D}(\bigcup_{b\in B}\phi(b))$ be clopen in X. Then there exists $a \in A$ such that $\mathbf{D}(\bigcup_{b\in B}\phi(b)) = \phi(a)$. But then a is the least upper bound of B, so $a = \bigvee B$, and so $\bigvee B$ exists in A.
(2) If A is a Heyting algebra, then X is an Esakia space, and so, by Lemma 2.2, $\mathbf{D}(U) = \overline{U}$ for each upset U of X. Now apply (1).
(3) can be proved using an argument dual to (1). □

As an immediate consequence of Lemma 2.3 we obtain the following dual characterization of complete distributive lattices and complete Heyting algebras (see [11, Sec. 8] and [7, Remark after Thm. 3.8]).

Theorem 2.4 *Let A be a bounded distributive lattice and X be its dual Priestley space.*

(1) *The following conditions are equivalent:*

 (a) *A is complete.*
 (b) *For every open upset U of X, we have $\mathbf{D}(U)$ is clopen in X.*
 (c) *For every closed upset V of X, we have $\mathbf{J}(V)$ is clopen in X.*

(2) *If A is a Heyting algebra, then A is complete iff for every open upset U of X, its closure $\overline{U}$ is clopen in X.*

Proof

(1) We prove that (a) is equivalent to (b). That (a) is equivalent to (c) can be proved similarly. First suppose that A is complete. If U is an open upset of X, then $\mathbf{J}U = U$, and so, by Lemma 2.2, $U = \bigcup\{\phi(a) : \phi(a) \subseteq U\}$. Let $B = \{a \in A : \phi(a) \subseteq U\}$. Since A is complete, $\bigvee B$ exists in A. Therefore, by Lemma 2.3, $\mathbf{D}(U)$ is clopen. Now suppose $\mathbf{D}(U)$ is clopen for each open upset U of X. For a subset B of A, $\bigcup_{b\in B}\phi(b)$ is an open upset of X. Therefore, $\mathbf{D}(\bigcup_{b\in B}\phi(b))$ is clopen in X. This, by Lemma 2.3, implies that $\bigvee B$ exists. Thus, A is complete.
(2) follows from (1) and Lemma 2.2. □

Definition 2.5 (Priestley [11]) We call a Priestley space X extremally order-disconnected if $\mathbf{D}(U)$ is clopen for each open upset U of X.

Remark 2.6 In view of Definition 2.5, Theorem 2.4 states that a bounded distributive lattice A is complete iff its dual space X is extremally order-disconnected. Moreover, if A is a Heyting algebra, then X is extremally order-disconnected iff $\overline{U}$ is clopen for each open upset U of X.

Let A be a bounded distributive lattice. We recall that an element $a \neq 0$ of A is *join-prime* if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$ for all $b, c \in A$. We also recall that $0 \neq a \in A$ is *completely join-prime* if for each $B \subseteq A$ such that $\bigvee B$ exists in A we have $a \leq \bigvee B$ implies there exists $b \in B$ with $a \leq b$. Let $J(A)$ denote the set of join-prime elements of A and $J^{\infty}(A)$ denote the set of completely join-prime elements of A.

Theorem 2.7 *Let A be a Heyting algebra and let X be its dual space.*

(1) *$a \in J(A)$ iff there exists $x \in X$ such that $\phi(a) = \uparrow x$.*
(2) *$a \in J^{\infty}(A)$ iff there exists an isolated point $x \in X$ such that $\phi(a) = \uparrow x$.*

Proof

(1) First suppose that $\phi(a) = \uparrow x$ for some $x \in X$. If $a \leq b \vee c$, then $\uparrow x = \phi(a) \subseteq \phi(b) \cup \phi(c)$, so $x \in \phi(b)$ or $x \in \phi(c)$, and so $\uparrow x \subseteq \phi(b)$ or $\uparrow x \subseteq \phi(c)$. Therefore, $a \leq b$ or $a \leq c$, and so $a \in J(A)$. Now suppose that a is join-prime. Let $\min(\phi(a))$ denote the set of minimal points of $\phi(a)$. By [6, p. 54, Thm. 2.1], for every closed upset U of an Esakia space, we have $U = \uparrow\min(U)$. Therefore, $\phi(a) = \uparrow\min(\phi(a))$. We show that $\min(\phi(a))$ is a singleton. Suppose not. Fix two distinct elements $x, y \in \min(\phi(a))$. Obviously, for every $z \in \min(\phi(a))$ with

$x \neq z$ we have $z \not\leq x$. Thus, there exists a clopen upset U_z such that $z \in U_z$ and $x \notin U_z$. Also, $x \not\leq y$ implies there exists a clopen upset U_x such that $x \in U_x$ and $y \notin U_x$. Then $\min(\phi(a)) \subseteq U_x \cup \bigcup\{U_z : z \in \min(\phi(a))$ and $z \neq x\}$, and so $\phi(a) = \uparrow\min(\phi(a)) \subseteq U_x \cup \bigcup\{U_z : z \in \min(\phi(a))$ and $z \neq x\}$. Since $\phi(a)$ is compact, there exist $U_{z_1}, \ldots, U_{z_n}$ such that $\phi(a) \subseteq U_x \cup U$, where $U = U_{z_1} \cup \cdots \cup U_{z_n}$. As both U_x and U are clopen upsets of X, there exist $b, c \in A$ such that $U_x = \phi(b)$ and $U = \phi(c)$. Therefore, $\phi(a) \subseteq \phi(b) \cup \phi(c)$, but $\phi(a) \not\subseteq \phi(b)$ (as $y \in \phi(a)$ but $y \notin \phi(b)$) and $\phi(a) \not\subseteq \phi(c)$ (as $x \in \phi(a)$ but $x \notin \phi(c)$). Thus, $a \leq b \vee c$, but $a \not\leq b$ and $a \not\leq c$, which contradicts to a being join-prime. Therefore, $\min(\phi(a))$ is a singleton, and so $\phi(a) = \uparrow x$ for some $x \in X$.

(2) First suppose that $\phi(a) = \uparrow x$ for an isolated point $x \in X$. If $a \leq \bigvee B$, then by Lemma 2.3, $\uparrow x = \phi(a) \subseteq \overline{\bigcup_{b \in B} \phi(b)}$. Therefore, $x \in \overline{\bigcup_{b \in B} \phi(b)}$. Since x is an isolated point, we obtain $x \in \bigcup_{b \in B} \phi(b)$. Thus, $x \in \phi(b)$ for some $b \in B$. It follows that $\phi(a) = \uparrow x \subseteq \phi(b)$, so $a \leq b$, and so $a \in J^{\infty}(A)$. Now suppose that a is completely join-prime. Since every completely join-prime element is also join-prime, by (1) we get that $\phi(a) = \uparrow x$ for some $x \in X$. We show that x is an isolated point. Because X is an Esakia space, $\{x\}$ is closed, and so $U = \phi(a) - \{x\}$ is an open upset. By Lemma 2.2, $U = \mathbf{J}(U) = \bigcup\{\phi(b) : \phi(b) \subseteq U\}$. If x is not an isolated point, then $\overline{U} = \phi(a)$. Therefore, $\phi(a) = \overline{\bigcup\{\phi(b) : \phi(b) \subseteq U\}}$. By Lemma 2.3 this means that $a = \bigvee B$. But since $x \notin U$, we have that $x \notin \phi(b)$ for each $b \in B$. Therefore, $a \not\leq b$ for each $b \in B$, implying that a is not completely join-prime. The obtained contradiction proves that x is an isolated point. □

We note that Theorem 2.7.1 is also true for bounded distributive lattices. On the other hand, there exist bounded distributive lattices in which Theorem 2.7.2 is not true, as follows from the following example.

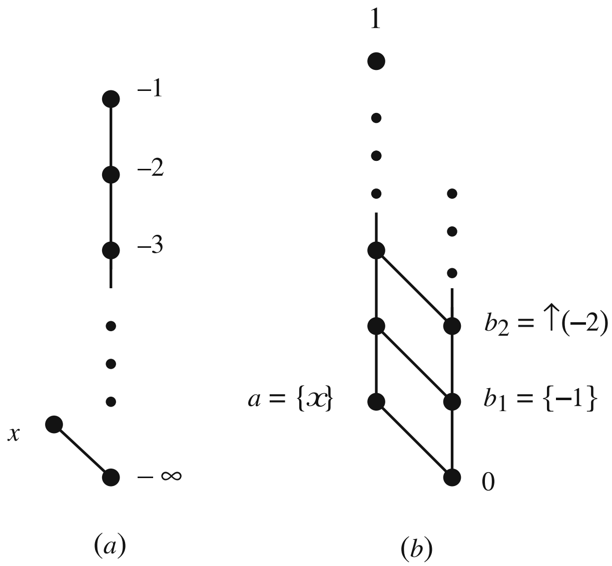
Example 2.8 Let $\mathbb{Z}^-$ denote the set of negative integers with the discrete topology, and let $\alpha(\mathbb{Z}^-) = \mathbb{Z}^- \cup \{-\infty\}$ be the one-point compactification of $\mathbb{Z}^-$. Let also X be the disjoint union of $\alpha(\mathbb{Z}^-)$ with a one-point space $\{x\}$. Define a partial order $\leq$ on X as it is shown in Fig. 1a. It is easy to check that X is a Priestley space. The distributive lattice A whose dual Priestley space is X is shown in Fig. 1b. Clearly x is an isolated point of X and $\uparrow x$ is clopen in X. Let $a \in A$ be such that $\uparrow x = \phi(a)$. Then $a \leq 1 = \bigvee_{n \in \mathbb{N}} b_n$, but $a \not\leq b_n$ for each $n \in \mathbb{N}$ (see Fig. 1b). Therefore, $a \notin J^{\infty}(A)$, but $\phi(a) = \uparrow x$ for an isolated point $x \in X$. Obviously, A is *not* a Heyting algebra!

We recall that a Heyting algebra A is *well-connected* if $a \vee b = 1$ implies $a = 1$ or $b = 1$, and that A is *subdirectly irreducible* if there exists a smallest filter properly containing the filter $\{1\}$. Since A is well-connected iff 1 is join-prime and A is subdirectly irreducible iff 1 is completely join-prime, the following theorem, first established in [5, p. 152], is an immediate corollary of Theorem 2.7.

Theorem 2.9 *Let A be a Heyting algebra with dual space X.*

(1) *A is well-connected iff $X = \uparrow x$ for some $x \in X$.*
(2) *A is subdirectly irreducible iff $X = \uparrow x$ for some isolated point $x \in X$.*

Fig. 1 **a** Partial order $\leq$ on X. **b** Distributive lattice A



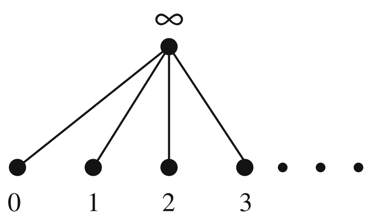
We call a Heyting algebra A *completely join-prime generated* if every element of A is a join of completely join-prime elements of A. Equivalently, A is completely join-prime generated if for each $a, b \in A$ with $a \not\leq b$, there is $p \in J^{\infty}(A)$ such that $p \leq a$ and $p \not\leq b$. Let X be the dual space of A. We let X_{iso} denote the set of isolated points of X, and set

$$X_0 = \{x \in X_{\text{iso}} : \uparrow x \in \text{Up}_{\tau}(X)\}.$$

Clearly $X_0 \subseteq X_{\text{iso}}$, but in general, X_0 is not equal to X_{iso} as the following example shows.

Example 2.10 Let $X = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of the set of natural numbers with the discrete topology. Define a partial order on X as it is shown in Fig. 2. Then it is easy to verify that X is an Esakia space, that $X_{\text{iso}} = \mathbb{N}$, and that $X_0 = \emptyset$. Therefore, $X_0 \neq X_{\text{iso}}$.

Fig. 2 Partial order on X



As a consequence of Theorem 2.7, we obtain the following characterization of completely join-prime generated Heyting algebras.

Theorem 2.11 *Let A be a Heyting algebra and X be its dual space. Then A is completely join-prime generated iff X_0 is dense in X.*

Proof First suppose that A is completely join-prime generated. We show that X_0 is dense in X. Let $\phi(a) - \phi(b)$ be a nonempty basic open set. Then $a \not\leq b$. Since A is completely join-prime generated, there is $p \in J^\infty(A)$ such that $p \leq a$ and $p \not\leq b$. As $p \in J^\infty(A)$, by Theorem 2.7.2, there is $x \in X_0$ such that $\phi(p) = {\uparrow}x$. Therefore, $x \in \phi(a)$ and $x \notin \phi(b)$. Thus, $(\phi(a) - \phi(b)) \cap X_0 \neq \emptyset$, and so X_0 is dense in X.

Now suppose that X_0 is dense in X. We show that A is completely join-prime generated. Let $a \not\leq b$. Then $\phi(a) \not\subseteq \phi(b)$, and so $\phi(a) - \phi(b) \neq \emptyset$. Since X_0 is dense and $\phi(a) - \phi(b)$ is nonempty, so is $(\phi(a) - \phi(b)) \cap X_0$. Let $x \in (\phi(a) - \phi(b)) \cap X_0$. By Theorem 2.7.2, there is $p \in J^\infty(A)$ such that $\phi(p) = {\uparrow}x$. Therefore, $\phi(p) \subseteq \phi(a)$ and $\phi(p) \not\subseteq \phi(b)$. Thus, there is $p \in J^\infty(A)$ such that $p \leq a$ and $p \not\leq a$, and so A is completely join-prime generated. □

On the other hand, it may happen that in the dual space X of a Heyting algebra A the set X_{iso} of isolated points of X is dense in X, but nevertheless A is not completely join-prime generated. Indeed, let A be the Heyting algebra of clopen upsets of the space X described in Example 2.10. Then X_{iso} is dense in X, but since $X_0 = \emptyset$, Theorem 2.7.2 implies that $J^\infty(A) = \emptyset$. Therefore, A is not completely join-prime generated.

As a consequence of Theorems 2.4 and 2.7, we obtain the following characterization of complete and completely join-prime generated Heyting algebras.

Theorem 2.12 *Let A be a Heyting algebra and X be its dual space. Then the following conditions are equivalent:*

(1) *A is complete and completely join-prime generated.*
(2) *X is extremally order-disconnected and X_0 is dense in X.*
(3) *There is a poset Y such that A is isomorphic to* $\text{Up}(Y)$.

Proof

(1) ⇒ (2) follows from Theorem 2.4, Remark 2.6, and Theorem 2.11.

(2) ⇒ (3) We show that A is isomorphic to $\text{Up}(X_0)$. Define $\alpha : A \to \text{Up}(X_0)$ by $\alpha(a) = \phi(a) \cap X_0$. If $a \leq b$, then $\phi(a) \subseteq \phi(b)$, and so $\alpha(a) = \phi(a) \cap X_0 \subseteq \phi(b) \cap X_0 = \alpha(b)$. If $a \not\leq b$, then $\phi(a) \not\subseteq \phi(b)$. Therefore, $\phi(a) - \phi(b) \neq \emptyset$, so $(\phi(a) - \phi(b)) \cap X_0 \neq \emptyset$, and so $\alpha(a) = \phi(a) \cap X_0 \not\subseteq \phi(b) \cap X_0 = \alpha(b)$. Consequently, $a \leq b$ iff $\alpha(a) \subseteq \alpha(b)$. To see that α is onto, let U be an upset of X_0. We let $V = \bigcup_{x \in U} {\uparrow}x$. Then V is an open upset of X. Since X is an extremally order-disconnected Esakia space, $\overline{V}$ is a clopen upset of X. Moreover, as $X_0 \subseteq X_{\text{iso}}$, we have $\overline{V} \cap X_0 = U$. Therefore, there exists $a \in A$ such that $\alpha(a) = \phi(a) \cap X_0 = \overline{V} \cap X_0 = U$, and so α is onto. Thus, $\alpha : A \to \text{Up}(X_0)$ is an isomorphism.

(3) ⇒ (1) Suppose there is a poset Y such that A is isomorphic to $\text{Up}(Y)$. It is easy to see that $\text{Up}(Y)$ is complete and that $J^\infty(\text{Up}(Y)) = \{{\uparrow}y : y \in Y\}$. Since

$U = \bigcup_{u \in U} \uparrow u$ for each $U \in \mathrm{Up}(Y)$, it follows that $\mathrm{Up}(Y)$ is completely join-prime generated. Thus, A is complete and completely join-prime generated. □

The equivalence of conditions (1) and (3) of Theorem 2.12 is well-known. It can, in fact, be extended to the dual equivalence of the category $\mathbf{HA}^+$ of complete and completely join-prime generated Heyting algebras and complete Heyting algebra homomorphisms and the category $\mathbf{Pos}^{\mathrm{b}}$ of posets and bounded morphisms. The contravariant functors $(-)_+ : \mathbf{HA}^+ \to \mathbf{Pos}^{\mathrm{b}}$ and $(-)^+ : \mathbf{Pos}^{\mathrm{b}} \to \mathbf{HA}^+$ are constructed as follows (see, e.g, [1, Sec. 7]): If $A \in \mathbf{HA}^+$, then $A_+ = (J^\infty(A), \geq)$, and if $h \in \hom(A, B)$, then $h_+ : J^\infty(B) \to J^\infty(A)$ is given by $h_+(b) = \bigwedge\{a \in A : b \leq h(a)\}$ for each $b \in B$. If $X \in \mathbf{Pos}^{\mathrm{b}}$, then $X^+ = \mathrm{Up}(X)$, and if $f \in \hom(X, Y)$, then $f^+ : \mathrm{Up}(Y) \to \mathrm{Up}(X)$ is given by $f^+ = f^{-1}$. In the next section we define the full subcategory of $\mathbf{HA}^+$ which will turn out to be dually equivalent to the full subcategory of $\mathbf{Pos}^{\mathrm{b}}$ of image-finite posets.

3 Profinite Heyting Algebras

We recall that an algebra A is *finitely approximable* if A is isomorphic to a subalgebra of a product of finite algebras [9, p. 60]. It follows that A is finitely approximable iff A is a subdirect product of its finite homomorphic images. We give a dual characterization of finitely approximable Heyting algebras. Let A be a Heyting algebra and let X be the dual space of A. Set

$$X_{\mathrm{fin}} = \{x \in X : \ \uparrow x \text{ is finite}\}.$$

A version of the next theorem was first established in [5, p. 152]. Our main tool in proving it is the correspondence between homomorphic images of A and closed upsets of X [4, Thm. 4]. In particular, we have that finite homomorphic images of A correspond to finite upsets of X, or equivalently, of X_{fin}.

Theorem 3.1 *Let A be a Heyting algebra and let X be the dual space of A. Then A is finitely approximable iff X_{fin} is dense in X.*

Proof First suppose that A is finitely approximable. Let $\{A_i : i \in I\}$ be the family of finite homomorphic images of A. Since A is finitely approximable, A is a subdirect product of $\{A_i : i \in I\}$. Let $e : A \to \prod_{i \in I} A_i$ be the embedding. We denote by π_j the j-th projection $\prod_{i \in I} A_i \to A_j$. Let X_i be the dual space of A_i. Then, since A_i is a finite homomorphic image of A, by duality it follows that X_i is a finite upset of X. Therefore, $\bigcup_{i \in I} X_i \subseteq X_{\mathrm{fin}}$. We show that $\bigcup_{i \in I} X_i$ is dense in X. Because $\{\phi(a) - \phi(b) : a, b \in A\}$ forms a basis for the topology on X, it is sufficient to show that for each $a, b \in A$ with $\phi(a) - \phi(b) \neq \emptyset$, there exists $i \in I$ such that $(\phi(a) - \phi(b)) \cap X_i \neq \emptyset$. From $\phi(a) - \phi(b) \neq \emptyset$ it follows that $a \not\leq b$. Therefore, there exists $i \in I$ such that $\pi_i(e(a)) \not\leq \pi_i(e(b))$. Thus, $\phi(a) \cap X_i \not\subseteq \phi(b) \cap X_i$, and so there exists $i \in I$ such that $(\phi(a) - \phi(b)) \cap X_i \neq \emptyset$. It follows that $\bigcup_{i \in I} X_i$ intersects every nonempty basis element of X, so $\bigcup_{i \in I} X_i$ is dense in X. Consequently, X_{fin} is also dense in X.

Now suppose that X_{fin} is dense in X. Let $\{X_i : i \in I\}$ be the family of finite upsets of X. Then $X_{\mathrm{fin}} = \bigcup_{i \in I} X_i$. Let A_i be the Heyting algebra $\mathrm{Up}(X_i)$ of upsets of X_i. Define $e : A \to \prod_{i\in I} A_i$ by $e(a) = (\phi(a) \cap X_i)_{i\in I}$. That e is a Heyting algebra homomorphism is easy to verify. We show that e is 1-1. If $a \not\leq b$, then $\phi(a) - \phi(b) \neq \emptyset$. Since $\bigcup_{i\in I} X_i$ is dense in X, $(\phi(a) - \phi(b)) \cap \bigcup_{i\in I} X_i \neq \emptyset$. Therefore, there exists $i \in I$ such that $(\phi(a) - \phi(b)) \cap X_i \neq \emptyset$. Thus, $\pi_i(e(a)) \not\leq \pi_i(e(b))$, so $e(a) \not\leq e(b)$, and so e is 1-1. It follows that A is finitely approximable. □

Let A be an algebra, and let I be the set of congruences θ on A such that A/θ is finite. We denote the image of $a \in A$ in A/θ by $[a]_\theta$. If $\psi \subseteq \theta$, then there is a canonical projection $\varphi_{\psi\theta} : A/\psi \to A/\theta$ given by $\varphi_{\psi\theta}([a]_\psi) = [a]_\theta$. Then $(I, \supseteq)$ is a directed set, and $(I, \{A/\theta\}, \{\varphi_{\psi\theta}\})$ is an inverse system of algebras. Let $\widehat{A}$ be the inverse limit of this inverse system. It is well-known that

$$\widehat{A} = \{(a_\theta)_{\theta\in I} \in \prod_{\theta\in I} A/\theta : \varphi_{\psi\theta}(a_\psi) = a_\theta \text{ whenever } \psi \subseteq \theta\}.$$

Following [2, Def. 2.4], we call $\widehat{A}$ the *profinite completion* of A. We define the canonical homomorphism $e : A \to \widehat{A}$ by $e(a) = ([a]_\theta)_{\theta\in I}$.

Proposition 3.2 *The canonical map $e : A \to \widehat{A}$ is* 1-1 *iff A is finitely approximable.*

Proof If e is 1-1, then it follows from the definition of $\widehat{A}$ that A is isomorphic to a subalgebra of a product of finite algebras, thus A is finitely approximable. Conversely, if A is finitely approximable, then A is a subdirect product of the collection $\{A/\theta : \theta \in I\}$ of finite homomorphic images of A. Therefore, $a \neq b$ in A implies that there exists $\theta \in I$ such that $[a]_\theta \neq [b]_\theta$. Thus, the images of a and b in the inverse limit of $(I, \{A/\theta\}, \{\varphi_{\psi\theta}\})$ are different, and so e is 1-1. □

Remark 3.3 Since not every Heyting algebra is finitely approximable, for Heyting algebras the canonical map $e : A \to \widehat{A}$ need not be 1-1. For a simple example, see [2, p. 153].

Definition 3.4 We call an algebra A profinite if it is isomorphic to the inverse limit of an inverse system of finite algebras.

Obvious examples of profinite algebras are profinite completions of algebras. Let A be a Heyting algebra and let X be its dual space. A characterization of the profinite completion $\widehat{A}$ of A was given in [2, Thm. 4.7], where it was shown that $\widehat{A}$ is isomorphic to the Heyting algebra of upsets of X_{fin}. Now we give both the algebraic and dual characterizations of profinite Heyting algebras.

Definition 3.5 We say that a poset X is image-finite if $\uparrow x$ is finite for each $x \in X$.

We recall from [2, Prop. 3.4] that if X is a poset, then the dual space of the Heyting algebra $\text{Up}(X)$ of upsets of X is order-homeomorphic to the Nachbin order-compactification $n(X)$ of X. Thus, for each order-preserving map f from X to an Esakia space Y, there exists a unique extension nf of f to $n(X)$. Moreover, if f is a bounded morphism, then so is nf (see [2, Lem. 4.3]). Furthermore, if X is image-finite, then the canonical order-embedding $j : X \to n(X)$ is a bounded morphism (see [2, Lem. 4.5]).

Theorem 3.6 *Let A be a Heyting algebra and let X be its dual space. Then the following conditions are equivalent:*

(1) *A is profinite.*
(2) *A is finitely approximable, complete, and completely join-prime generated.*
(3) *A is finitely approximable, complete, and $1 = \bigvee J^\infty(A)$.*
(4) *X is extremally order-disconnected and X_{iso} is a dense upset of X contained in X_{fin}.*
(5) *X is extremally order-disconnected and X_0 is a dense upset of X contained in X_{fin}.*
(6) *There is an image-finite poset Y such that A is isomorphic to* $\mathrm{Up}(Y)$.

Proof

(1) ⇒ (2) Clearly if A is profinite, then A is finitely approximable. That A is also complete and completely join-prime generated follows from [2, Lemmas 2.5 and 2.7].

(2) ⇒ (3) is trivial.

(3) ⇒ (4) Since A is a complete Heyting algebra, by Theorem 2.4 and Remark 2.6, X is extremally order-disconnected. Because A is finitely approximable, by Theorem 3.1, X_{fin} is dense in X. Thus, $X_{\text{iso}} \subseteq X_{\text{fin}}$. Let $x \in X_{\text{iso}}$. Since $1 = \bigvee J^\infty(A)$, by Lemma 2.3, $x \in \overline{\bigcup\{\phi(a) : a \in J^\infty(A)\}}$. Because x is an isolated point, we have $x \in \bigcup\{\phi(a) : a \in J^\infty(A)\}$. Therefore, there is $a \in J^\infty(A)$ such that $x \in \phi(a)$. By Theorem 2.7, there is $y \in X_0 \subseteq X_{\text{iso}}$ such that $\phi(a) = {\uparrow} y$. This means that ${\uparrow} y$ is clopen and $x \in {\uparrow} y$. Because $X_{\text{iso}} \subseteq X_{\text{fin}}$, we have that ${\uparrow} y$ is finite. Thus, ${\uparrow} y$ is a finite clopen upset, and so every element of ${\uparrow} y$ is an isolated point. This implies that ${\uparrow} x \subseteq X_{\text{iso}}$. Consequently, X_{iso} is an upset. We show that X_{iso} is dense in X. By Lemma 2.3,

$$X = \phi(1) = \phi\left(\bigvee J^\infty(A)\right) = \overline{\bigcup_{a \in J^\infty(A)} \phi(a)} = \overline{\bigcup_{x \in X_0} {\uparrow} x} \subseteq \overline{\bigcup_{x \in X_{\text{iso}}} {\uparrow} x} = \overline{X_{\text{iso}}}.$$

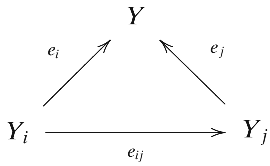
Thus, $\overline{X_{\text{iso}}} = X$, and so X_{iso} is a dense upset of X contained in X_{fin}.

(4) ⇒ (5) It follows from the definition of X_0 that $X_0 \subseteq X_{\text{iso}}$. On the other hand, since X_{iso} is an upset, if $x \in X_{\text{iso}}$, then ${\uparrow} x \subseteq X_{\text{iso}}$. Moreover, as $X_{\text{iso}} \subseteq X_{\text{fin}}$, ${\uparrow} x$ is finite. Therefore, ${\uparrow} x$ is a finite subset of X_{iso}, hence ${\uparrow} x$ is clopen. Thus, $x \in X_0$. Consequently, $X_0 = X_{\text{iso}}$, whence the implication follows.

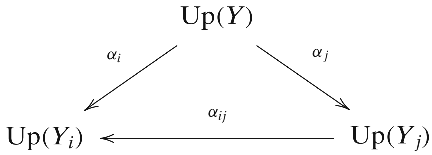
(5) ⇒ (6) Since $X_0 \subseteq X_{\text{fin}}$, X_0 is image-finite. We show that A is isomorphic to $\mathrm{Up}(X_0)$. Define $\alpha : A \to \mathrm{Up}(X_0)$ by $\alpha(a) = \phi(a) \cap X_0$. The proof of the implication (2) ⇒ (3) of Theorem 2.12 shows that $a \leq b$ iff $\alpha(a) \subseteq \alpha(b)$. To see that α is onto is simpler than in the proof of Theorem 2.12. Let U be an upset of X_0. Since X_0 is an open upset of X, so is U. Therefore, $\overline{U}$ is a clopen upset of X as X is an extremally order-disconnected Esakia space. Moreover, $\overline{U} \cap X_0 = U$. Thus, there exists $a \in A$ such that $\alpha(a) = \phi(a) \cap X_0 = \overline{U} \cap X_0 = U$, and so α is onto. Consequently, $\alpha : A \to \mathrm{Up}(X_0)$ is an isomorphism.

(6) ⇒ (1) Suppose that A is isomorphic to $\mathrm{Up}(Y)$ for some image finite-poset Y. We show that $\mathrm{Up}(Y)$ is profinite. Let $\{Y_i : i \in I\}$ be the family of

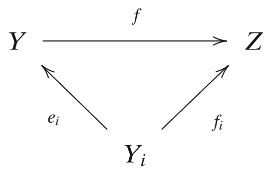
finite upsets of Y. We can order I by $i \leq j$ if $Y_i \subseteq Y_j$. Let $e_{ij} : Y_i \to Y_j$ denote the identity map. Then $(I, \{Y_i\}, \{e_{ij}\})$ forms a directed system of finite posets and $Y = \bigcup_{i \in I} Y_i$ together with the inclusions $e_i : Y_i \to Y$ is the direct limit of $(I, \{Y_i\}, \{e_{ij}\})$. For each $i \in I$, let $\mathrm{Up}(Y_i)$ denote the Heyting algebra of upsets of Y_i. Clearly $\alpha_i : \mathrm{Up}(Y) \to \mathrm{Up}(Y_i)$, given by $\alpha_i(U) = U \cap Y_i$, and $\alpha_{ij} : \mathrm{Up}(Y_j) \to \mathrm{Up}(Y_i)$, given by $\alpha_{ij}(U) = U \cap Y_i$, are Heyting algebra homomorphisms (which are dual to the embeddings $e_i : Y_i \to Y$ and $e_{ij} : Y_i \to Y_j$, respectively). Moreover, since the diagram



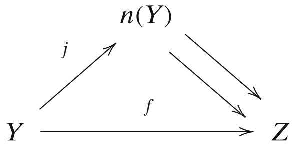
commutes, so does the diagram



We show that $(\mathrm{Up}(Y), \{\alpha_i\})$ is the inverse limit of the inverse system $(I, \{\mathrm{Up}(Y_i)\}, \{\alpha_{ij}\})$ of finite homomorphic images of $\mathrm{Up}(Y)$ by showing that $(\mathrm{Up}(Y), \{\alpha_i\})$ satisfies the universal mapping property of an inverse limit. Let B be a Heyting algebra together with Heyting algebra homomorphisms $\gamma_i : B \to \mathrm{Up}(Y_i)$, such that $i \leq j$ implies $\alpha_{ij} \circ \gamma_j = \gamma_i$. Let Z be the dual space of B, and let $f_i : Y_i \to Z$ be the dual of $\gamma_i : B \to \mathrm{Up}(Y_i)$. Since Y is the direct limit of the Y_i, there is $f : Y \to Z$ such that $f \circ e_i = f_i$ for each $i \in I$.



Because each f_i is a bounded morphism, so is f. Therefore, by identifying B with $\mathrm{Up}_\tau(Z)$, we obtain a Heyting algebra homomorphism $\gamma : B \to \mathrm{Up}(Y)$ (the restriction of f^{-1} to $\mathrm{Up}_\tau(Z)$) such that $\alpha_i \circ \gamma = \gamma_i$ for each $i \in I$. Suppose we had a second Heyting algebra homomorphism $\gamma' : B \to \mathrm{Up}(Y)$ with $\alpha_i \circ \gamma' = \gamma_i$ for each $i \in I$. Since $\mathrm{Up}(Y)$ is isomorphic to the Heyting algebra $\mathrm{Up}_\tau(n(Y))$ of clopen upsets of the Nachbin order-compactification $n(Y)$ of Y, we would have two Heyting algebra homomorphisms $\gamma, \gamma' : B \to \mathrm{Up}_\tau(n(Y))$, and so two continuous bounded morphisms from $n(Y)$ to Z extending f.



However, by the mapping property for $n(Y)$, there is a unique extension of f. Therefore, $\gamma = \gamma'$. Thus, by the universal mapping property for inverse limits, we have that $\mathrm{Up}(Y)$ is isomorphic to the inverse limit of $(I, \{\mathrm{Up}(Y_i)\}, \{\alpha_{ij}\})$. Consequently, $\mathrm{Up}(Y)$ is profinite, implying that A is profinite. □

Remark 3.7 Since every Heyting algebra $(A, \leq)$ isomorphic to the Heyting algebra of upsets of a poset is a bi-Heyting algebra (that is, its order dual $(A, \geq)$ is also a Heyting algebra), we deduce from Theorem 3.6 that every profinite Heyting algebra is a bi-Heyting algebra.

Example 3.8 As follows from Theorem 3.6, a Heyting algebra A is profinite iff A is finitely approximable, complete, and $1 = \bigvee J^\infty(A)$. We sketch a few examples showing that none of these three conditions can be eliminated.

(1) Free n-generated Heyting algebras, for $n > 1$, provide examples of Heyting algebras which are finitely approximable, satisfy $1 = \bigvee J^\infty(A)$, but are not complete (see Section 4 for details).
(2) To obtain an example of a complete Heyting algebra A such that $1 = \bigvee J^\infty(A)$, but A is not finitely approximable, let B be an infinite complete and atomic Boolean algebra with dual Stone space X. Let Y be the disjoint union of X with a one-point space $\{y\}$. Set $y \leq x$ for each $x \in X$. Clearly Y is an Esakia space. Let A be the Heyting algebra of clopen upsets of Y. Since X is extremally disconnected, Y is extremally order-disconnected, and so A is complete. Moreover, $Y = \uparrow y$ and y is an isolated point. Therefore, $1 \in J^\infty(A)$, and so $1 = \bigvee J^\infty(A)$. Thus, A is complete and $1 = \bigvee J^\infty(A)$. However, A is not finitely approximable because $Y_{\mathrm{fin}} = X$, so $\overline{Y_{\mathrm{fin}}} = X$, and so Y_{fin} is not dense in Y.
(3) Finally, let A be the Heyting algebra of clopen upsets of the Esakia space X described in Example 2.10. It is easy to see that X is extremally order-disconnected. Therefore, A is a complete Heyting algebra. (In fact, A is isomorphic to the Heyting algebra of cofinite subsets of $\mathbb{N}$ together with $\emptyset$.) We already observed that $J^\infty(A) = \emptyset$. Therefore, $1 \neq \bigvee J^\infty(A)$. Moreover, since $X_{\mathrm{fin}} = X$, A is finitely approximable. Thus, A is a finitely approximable complete Heyting algebra such that $1 \neq \bigvee J^\infty(A)$.

Remark 3.9 As we pointed out at the end of Section 2, the category $\mathbf{HA}^+$ of complete and completely join-prime generated Heyting algebras and complete Heyting algebra homomorphisms is dually equivalent to the category $\mathbf{Pos}^{\mathrm{b}}$ of posets and bounded morphisms. Let **ProHA** denote the category of profinite Heyting algebras and complete Heyting algebra homomorphisms. Clearly **ProHA** is a full subcategory of $\mathbf{HA}^+$. Let also $\mathbf{Im^fPos^b}$ denote the category of image-finite posets and bounded morphisms. Clearly $\mathbf{Im^fPos^b}$ is a full subcategory of $\mathbf{Pos^b}$. As a consequence of Theorem 3.6, we obtain that **ProHA** is dually equivalent to $\mathbf{Im^fPos^b}$.

As another corollary of Theorem 3.6, we give necessary and sufficient conditions for a Heyting algebra A to be isomorphic to its profinite completion $\widehat{A}$.

Theorem 3.10 *Let A be a Heyting algebra and let X be its dual space. Then the following conditions are equivalent:*

(1) *A is isomorphic to its profinite completion.*
(2) *A is finitely approximable, complete, and the kernel of every finite homomorphic image of A is a principal filter of A.*
(3) *X is extremally order-disconnected, $X_0 = X_{\text{iso}} = X_{\text{fin}}$, and they are dense in X.*

Proof

(1) ⇒ (2) Suppose that A is isomorphic to its profinite completion $\widehat{A}$. Then A is profinite. So, by Theorem 3.6, A is finitely approximable and complete. Moreover, every finite homomorphic image A_j of A is a finite homomorphic image of $\widehat{A}$. Therefore, the kernel of this homomorphism is a closed (even clopen) filter in the topology $\widehat{A}$ inherited from the product topology on $\prod_{i\in I} A_i$. Thus, by [2, Lem. 2.6], the kernel of this homomorphism is a principal filter of A.

(2) ⇒ (3) Since A is finitely approximable and complete, by Theorem 2.4, Remark 2.6, and Theorem 3.1, X is extremally order-disconnected and X_{fin} is a dense upset of X. Thus, $X_0 \subseteq X_{\text{iso}} \subseteq X_{\text{fin}}$. To show the converse inclusions, let $x \in X_{\text{fin}}$. Then $\uparrow x$ is a finite upset of X, so the Heyting algebra of upsets of $\uparrow x$ is a finite homomorphic image of A. By our assumption, the kernel of this homomorphism is a principal filter. But dually $\uparrow x$ corresponds to this filter. Thus, $\uparrow x$ is clopen. It follows that x is an isolated point because every point of a finite clopen subset of X is an isolated point of X. Consequently, $X_0 = X_{\text{iso}} = X_{\text{fin}}$.

(3) ⇒ (1) By Theorem 3.6, A is isomorphic to $\text{Up}(X_0) = \text{Up}(X_{\text{iso}})$, and by [2, Thm. 4.7], $\widehat{A}$ is isomorphic to $\text{Up}(X_{\text{fin}})$. Now since $X_0 = X_{\text{iso}} = X_{\text{fin}}$, we obtain that A is isomorphic to $\widehat{A}$. □

Remark 3.11 It follows from Theorems 3.6 and 3.10 that if A is finitely approximable, complete, and the kernel of every finite homomorphic image of A is a principal filter of A, then A is automatically completely join-prime generated.

4 Consequences

In this final section we give several consequences of our two main theorems. First we describe all profinite linear Heyting algebras. We recall that a Heyting algebra A is *linear* if for each $a, b \in A$ we have $a \leq b$ or $b \leq a$. For each $n \geq 1$ let L_n denote the n-element linear Heyting algebra. Clearly all L_n are profinite. Let L_∞ denote the linear Heyting algebra and X_∞ denote its dual space shown in Fig. 3.

Lemma 4.1 *Up to isomorphism, L_∞ is the only infinite profinite linear Heyting algebra.*

Proof Let A be an infinite profinite linear Heyting algebra and let X be its dual space. Since A is linearly ordered, so is X. By [6, p. 54], the set $\max X$ of maximal points of X is nonempty. As X is linearly ordered, $\max X$ consists of a single point, say x_0. If $x_0 \notin X_{\mathrm{iso}}$, then X_{iso} is not an upset of X, which contradicts to Theorem 3.6. Therefore, $x_0 \in X_{\mathrm{iso}}$. Let $X_1 = X - \{x_0\}$. Then X_1 is a clopen subset of X, and because of the same reason as above, $\max X_1$ consists of a single point, say x_1. Since X_1 is clopen and X_{iso} is dense in X, we have that $X_{\mathrm{iso}} \cap X_1$ is nonempty, and the same argument as above guarantees that $x_1 \in X_{\mathrm{iso}}$. Continuing this process infinitely, we obtain a decreasing sequence $x_0 > x_1 > \cdots > x_n > \ldots$ of isolated points of X. Let x_∞ be a limit point of $\{x_0, \ldots, x_n, \ldots\}$. We show that $X = \{x_0, \ldots, x_n, \ldots, x_\infty\}$ and that $X_{\mathrm{iso}} = \{x_0, \ldots, x_n, \ldots\}$. Let X contain another point y. Since $y \neq x_n$, we have $y < x_n$ for each n. If $y > x_\infty$, then there exists a clopen upset U such that $y \in U$ and $x_\infty \notin U$. But then $\{x_0, \ldots, x_n, \ldots\} \subseteq U$ and $x_\infty \notin U$, contradicting to $x_\infty \in \overline{\{x_0, \ldots, x_n, \ldots\}}$. Therefore, $x_\infty > y$. Since X_{iso} is an upset of X, it follows that $y \notin X_{\mathrm{iso}}$. This implies that $X_{\mathrm{iso}} = \{x_0, \ldots, x_n, \ldots\}$. Moreover, from $x_\infty > y$ it follows that there exists a clopen upset V such that $x_\infty \in V$ and $y \notin V$. Therefore, $\{x_0, \ldots, x_n, \ldots\} \subseteq V$ and $y \notin V$. Thus, $X - V$ is a nonempty clopen set having the empty intersection with X_{iso}, which contradicts to density of X_{iso}. Consequently, such a y does not exist, and so $X = \{x_0, \ldots, x_n, \ldots, x_\infty\}$. Therefore, X is order-isomorphic and homeomorphic to X_∞, and so A is isomorphic to L_∞. □

As an immediate consequence of Lemma 4.1, we obtain the following description of profinite linear Heyting algebras.

Theorem 4.2 *The linear Heyting algebras L_∞ and L_n, $n \geq 1$, are up to isomorphism the only profinite linear Heyting algebras.*

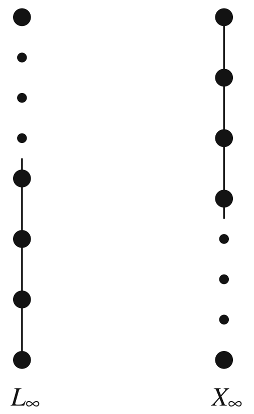


Fig. 3 L_∞ and X_∞

Now we turn our attention to Boolean algebras. It is known (see, e.g., [8, Sec. VI.2 and VI.3]) that a Boolean algebra is profinite iff it is complete and atomic. This we obtain as an immediate corollary of Theorem 3.6.

Theorem 4.3 *Let A be a Boolean algebra and let X be its dual Stone space. Then the following conditions are equivalent:*

(1) *A is profinite.*
(2) *A is complete and atomic.*
(3) *X is extremally disconnected and X_{iso} is a dense subset of X.*
(4) *A is isomorphic to the powerset of some set Y.*

Proof Let A be a Boolean algebra with dual Stone space X. It is enough to notice that every Boolean algebra is finitely approximable; that $a \in A$ is an atom of A iff a is completely join-prime; that upsets of X are simply subsets of X, so every subset of X is image-finite and $X_{\text{fin}} = X$; and that A is complete iff X is extremally disconnected. Now apply Theorem 3.6. □

Since bounded distributive lattice homomorphisms are not necessarily Heyting algebra homomorphisms, an analogue of Theorem 3.6 for bounded distributive lattices requires some adjustments. Firstly, like in the Boolean case, we have that every bounded distributive lattice is finitely approximable. Secondly, since homomorphic images of bounded distributive lattices dually correspond to closed subsets (and do *not* correspond to closed upsets), X_{fin} plays no role in the case of bounded distributive lattices. Consequently, we obtain the following analogue of Theorem 3.6.

Theorem 4.4 *Let A be a bounded distributive lattice and let X be its dual Priestley space. Then the following conditions are equivalent:*

(1) *A is profinite.*
(2) *A is complete and completely join-prime generated.*
(3) *There is a poset Y such that A is isomorphic to* $\text{Up}(Y)$.

Proof The proof of the implication (1) ⇒ (2) is the same as in Theorem 3.6. The equivalence (2) ⇔ (3) is well-known (see, e.g., [8, Sec. VI.2 and VI.3]). For the implication (3) ⇒ (1), observe that finite homomorphic images of A correspond to finite subsets of X and that X is their direct limit. Now use the same idea as in proving the implication (6) ⇒ (1) of Theorem 3.6. □

Now we turn our attention to finitely generated Heyting algebras.

Theorem 4.5 *A finitely generated Heyting algebra is profinite iff it is finitely approximable and complete.*

Proof Let A be a finitely generated Heyting algebra and let X be its dual space. If A is profinite, then it follows from Theorem 3.6 that A is finitely approximable and complete. Conversely, suppose that A is finitely approximable and complete. Since A is finitely generated, it is well-known that $X_{\text{fin}} \subseteq X_0$ (see, e.g., [3, Sec. 3.2]). As A is finitely approximable, by Theorem 3.1, X_{fin} is dense in X. Consequently, $X_{\text{fin}} \subseteq$

$X_0 \subseteq X_{\text{iso}} \subseteq X_{\text{fin}}$, and so $X_0 = X_{\text{iso}} = X_{\text{fin}}$. Therefore, X_0 is dense in X, which by Theorem 2.11 implies that A is completely join-prime generated. Thus, A is finitely approximable, complete, and completely join-prime generated, hence profinite by Theorem 3.6. □

Especially important finitely generated Heyting algebras are the free finitely generated Heyting algebras. Since every free finitely generated Heyting algebra is in addition finitely approximable (see, e.g., [3, Sec. 3.2]), from Theorem 4.5 we obtain that a free finitely generated Heyting algebra is profinite iff it is complete. But the Rieger-Nishimura lattice $\mathfrak{N}$ – the free 1-generated Heyting algebra – is the only complete finitely generated free Heyting algebra (see, e.g., [3, Sec. 3.2]). Thus, $\mathfrak{N}$ is the only profinite Heyting algebra among the free finitely generated Heyting algebras. Since in the dual space of $\mathfrak{N}$ we have in addition that $X_0 = X_{\text{iso}} = X_{\text{fin}}$, from Theorem 3.10 we obtain that $\mathfrak{N}$ is in fact isomorphic to its profinite completion (see [2, Ex. 4.11]).

We conclude the paper by mentioning several applications of Theorem 3.10. We recall that a variety $\mathbf{V}$ of Heyting algebras is *finitely generated* if it is generated by a single finite algebra.

Theorem 4.6 *Let A be a Heyting algebra in a finitely generated variety. Then A is isomorphic to its profinite completion iff A is finite.*

Proof Let A be a Heyting algebra in a finitely generated variety, and let X be the dual space of A. Clearly if A is finite, then $A \simeq \widehat{A}$. Conversely, suppose that $A \simeq \widehat{A}$. Since A belongs to a finitely generated variety, it follows from [2, Sec. 5] that $X_{\text{fin}} = X$. This by Theorem 3.10 implies that $X_{\text{iso}} = X$. Therefore, the topology on X is discrete, and as X is compact, X is finite. Thus, A is finite. □

As a corollary we obtain that a Boolean algebra is isomorphic to its profinite completion iff it is finite. The same result holds also for bounded distributive lattices, but the proof is slightly different.

Remark 4.7 Our main results can also be proved for modal algebras, and more generally, for Boolean algebras with operators.

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