LAX-NIJENHUIS OPERATORS FOR INTEGRABLE SYSTEMS

Y. Kosmann-Schwarzbach and F. Magri

Y. Kosmann-Schwarzbach
Centre de Mathématiques, URA 169 du CNRS
Ecole Polytechnique, F-91128 Palaiseau, France
e.mail yks@math.polytechnique.fr

F. Magri
Dipartimento di Matematica, Università degli Studi di Milano
Via Saldini 50, I-20133 Milano, Italy
e.mail magri@vmimat.mat.unimi.it

Abstract *The relationship between Lax and bihamiltonian formulations of dynamical systems on finite- or infinite-dimensional phase spaces is investigated. The Lax-Nijenhuis equation is introduced and it is shown that every operator that satisfies that equation satisfies the Lenard recursion relations, while the converse holds for an operator with a simple spectrum. Explicit higher-order Hamiltonian structures for the Toda system, a second Hmailtonian structure of the Euler equation for a rigid body in* n*-dimensional space, and the quadratic Adler-Gelfand-Dickey structure for the KdV hierarchy are derived using the Lax-Nijenhuis equation.*

Résumé *On étudie la relation entre formalisme de Lax et formalisme bihamiltonien sur des espaces de phases de dimension finie ou infinie. On introduit l'équation de Lax-Nijenhuis et l'on montre que tout opérateur qui satisfait cette équation satisfait les relations de récurrence de Lenard, tandis que la réciproque est valable pour un opérateur à spectre simple. On calcule des structures hamiltoniennes d'ordre supérieur pour le système de Toda, une deuxième structure hamiltonienne pour les équations d'Euler d'un corps solide dans l'espace à* n *dimensions, et la deuxième structure de Adler-Gelfand-Dickey pour la hiérarchie KdV en utilisant l'équation de Lax-Nijenhuis.*

PACS numbers: 02.20 +b, 02.40 +m

Keywords: integrable systems, Hamiltonian systems, Lax equations, symmetries, recursion operators. Toda lattice, KdV hierarchy.

AMS classification: 58F05, 58F07, 35Q53

I. Introduction

We present an explanation of a long-standing problem, what is the relationship between the Lax formulation of an integrable system and the existence of a bihamiltonian structure?

When considering differential equations in Lax form[1] on a finite- or infinite-dimensional phase-space manifold, according to the problem at hand, one introduces either a matrix of a given size, or a differential operator of a given degree, or, more generally, a pseudo-differential operator whose coefficients are functions of the phase-space coordinates. In other words, the "Lax operator" is an A-valued map on phase space, where A is an associative algebra that has to be determined in each problem.

We study the case where the phase-space manifold admits a pair of compatible Poisson structures, *i.e.*, has a bihamiltonian structure[2−4]. (See also Refs. 5-7, and Ref. 8 for a complete exposition and further references.) The term "Poisson structure" is most frequently used for finite-dimensional manifolds such as the phase space of dynamical systems defined by evolution ordinary differential equations, while the term "Hamiltonian structure" is commonly used in the case of infinite-dimensional manifolds, *e.g.*, manifolds of functions, such as the phase spaces of systems described by evolution partial differential equations. When two Poisson structures satisfying a compatibility condition are present, the term "bihamiltonian structure" will be applied. We shall mainly consider the finite-dimensional case, but the extension to the infinite-dimensional situation is straightforward, in the setting of the formal calculus of variations in the sense of Gelfand, Dickey and Dorfman. (See Refs. 9, 10 and 8.)

As the defining property of a matrix-valued Lax operator, L, in the presence of a bihamiltonian structure (P, Q) we take the so-called Lenard recursion relations,

$$Q(d\ tr \frac{L^k}{k}) = P(d\ tr \frac{L^{k+1}}{k+1}) \ . \tag{1.1}$$

When L has distinct eigenvalues, λ_i, these relations imply that

$$Q(d\lambda_i) = \lambda_i P(d\lambda_i) \ ,$$

and in all cases they imply the pairwise involutivity of the eigenvalues of L with respect to both Poisson brackets. Moreover, the traces of powers of L, and hence the eigenvalues of L, are conserved along the flow of each evolution equation in Lax form, $\dot{L} = [L, B]$.

The recursion relations for differential equations in Lax form first appeared in the context of evolution partial differential equations, and they are actually due to Lax[1]. (It is surprising that it has become customary to call them the Lenard recursion relations, probably because, in his paper of 1976, Lax[11] refers to Lenard's contribution to the derivation of the infinite family of higher-order Korteweg-de Vries equations as reported in the 1974 article by Gardner, Greene, Kruskal and Miura[12]. Actually, Gardner *et al.* derive this "infinite family of equations that leave the eigenvalues of the

Schrődinger equation invariant in time" and they also give "an alternate derivation of this family due to Lenard", and both derivations reveal the recursion operator explicitly but do not relate it to any Hamiltonian property. In that same paper, Lax ascribes the involutivity property of the conserved quantities to Gardner (Ref. 13), where it is not explicit at all! Actually the factorization of the recursion operator as the composition of a Poisson and a symplectic operator is in Lax[11], Magri[2], Gelfand and Dorfman[3], and Fokas and Fuchssteiner[14].)

In this paper, we show that, under suitable conditions on its spectrum, a Lax operator on a bihamiltonian manifold satisfies a universal equation which we call the Lax-Nijenhuis equation because the vanishing of the Nijenhuis torsion of the recursion operator of a bihamiltonian structure appears as a particular case of this property. Conversely, if L satisfies the Lax-Nijenhuis equation, then L is a Lax operator. We then study the converse problem of determining compatible Hamiltonian structures from Lax-Nijenhuis equations. We treat the Toda system (see Refs. 15-20) and the KdV hierarchy (see Refs. 21, 22, 16), and, more generally, Lax equations that are Hamiltonian with respect to a Poisson bracket defined by an R-matrix (see Ref. 23).

In section 2 we recall the definition of Hamiltonian and bihamiltonian structures and we prove that functions that satisfy recursion relations (1.1), where P and Q are compatible Poisson structures, are pairwise in involution with respect to either Hamiltonian structure. Nijenhuis operators appear in the theory of bihamiltonian structures (P, Q) when one considers the (1,1)-tensor $N = QP^{-1}$, where the first Poisson structure is assumed to be invertible, *i.e.*, symplectic. (See Refs. 24-27.) We analyze the properties of Nijenhuis operators and we observe that the vanishing of the Nijenhuis torsion[28] of a (1,1)-tensor implies the fundamental equation (2.5) that is the prototype of the Lax-Nijenhuis equation that we introduce in (3.6).

Section 3 contains the main results concerning the relationship between Lax and bihamiltonian formulations of dynamical systems. It is natural to require that the traces of powers of the Lax operator satisfy the Lenard recursion relations (definition 3.1). It then follows that, under the assumption of the simplicity of its spectrum, such a Lax operator satisfies the Lax-Nijenhuis equation (3.6). Conversely, we show in proposition 3.5 that, if an operator satisfies the Lax-Nijenhuis equation, it satisfies the Lenard recursion relations, and therefore the traces of its powers are in involution. In addition, we prove that this property remains valid for negative and fractional powers, when they are defined.

The hereditary properties of Nijenhuis operators and of Lax operators compatible with a bihamiltonian structure are derived in section 4. We show that the Lax formulation exists for all vector fields,

$$X_k = Q\alpha_k = P\alpha_{k+1} \ ,$$

where α_k is a sequence of differential 1-forms satisfying the Lenard recursion relations, provided that X_0 admits a Lax formulation. When all α_k's are closed, the X_k's constitute a bihamiltonian hierarchy, *i.e.*, a sequence of commuting bihamiltonian vector fields, and we state a further commutation property in proposition 4.3.

The fifth and last section is devoted to examples. For the non-periodic Toda system, we explicitly determine a sequence of skewsymmetric higher-order bivectors satisfying Lax-Nijenhuis equations. The first three elements of this sequence coincide with the known linear, quadratic and cubic Poisson structures. We treat the case of the Euler equations for the n-dimensional rigid body rotating about a fixed point, and some generalizations of it. We then study the Lax-Nijenhuis equation for the KdV equation, and, more generally, for the first equation in the n-th KdV hierarchy, where the Lax operator takes values in a manifold of n-th order differential operators in the graded, associative algebra of formal pseudodifferential operators, and we obtain the second Adler-Gelfand-Dickey Hamiltonian structure from the first one. Finally, this construction is further generalized to determine the quadratic bracket associated with the linear Poisson bracket defined by an R-matrix, *i.e.*, a solution of the modified classical Yang-Baxter equation.

For background and many results on integrable systems, we refer to Refs. 29 and 30. See Ref. 31 for a discussion closely connected with ours, but undertaken from a different point of view.

II. Lenard recursion relations on a bihamiltonian manifold

In this section, we recall some well-known results on Lenard recursion relations which we shall need for the study of Lax operators in section 3. First we introduce the concept of a bihamiltonian manifold.

A Poisson manifold (also called a Hamiltonian manifold) is a manifold equipped with a Poisson bracket. We recall that a Poisson bracket can be defined in terms of a field of bivectors (a bivector for short) called the Poisson bivector. If P is a bivector on a manifold M, we identify P with the linear bundle map,

$$P : T^*M \to TM,$$

defined by $\langle \beta, P\alpha \rangle = P(\alpha, \beta)$, for $\alpha, \beta \in T^*M$. We set $X_f = Pdf$, for any function $f \in C^\infty(M)$, and we call X_f the Hamiltonian vector field with Hamiltonian f. We also define the Poisson bracket,

$$\{f, g\}_P = X_f \cdot g,$$

for f and $g \in C^\infty(M)$. Recall that a bivector P on M is a Poisson bivector if and only if one of the following equivalent conditions is satisfied:

(1) $[P, P] = 0$, where $[\ ,\ ]$ is the Schouten bracket,
(2) the Poisson bracket $\{\ ,\ \}_P$ satisfies the Jacobi identity,
(3) $[X_f, X_g] = X_{\{f,g\}_P}$, for $f, g \in C^\infty(M)$.

These conditions are equivalent because, by the definition of the Schouten and Poisson brackets,

$$-\frac{1}{2}[P, P](df, dg, dh) = \{f, \{g, h\}_P\}_P + \{g, \{h, f\}_P\}_P + \{h, \{f, g\}_P\}_P$$
$$= ([Pdf\ , Pdg] - Pd(P(df, dg))).h = ([X_f, X_g] - X_{\{f,g\}_P}).h,$$

for $f, g, h \in C^\infty(M)$.

Definition 2.1.— *A bihamiltonian manifold (M, P, Q) is a manifold M equipped with Poisson structures, P and Q, which are compatible, i.e., such that any linear combination of P and Q is a Poisson structure. A (locally) bihamiltonian vector field on (M, P, Q) is a vector field leaving P and Q invariant.*

Thus, on a bihamiltonian manifold there exists a pencil of Poisson structures, $P_\lambda = Q - \lambda P$, for $\lambda \in \mathbb{R} \cup \{\infty\}$. A sufficient condition for a vector field X to be (locally) bihamiltonian is that there exist closed differential 1-forms α and β such that $X = P\beta = Q\alpha$. In particular, if there exist functions f and g such that $X = P(dg) = Q(df)$, then X is bihamiltonian.

Lemma 2.2.— *Let P and Q be Poisson structures on M. Then P and Q are compatible if and only if one of the following equivalent conditions is satisfied:*

(i) $[P, Q] = 0$,

(ii) $\circlearrowleft (\{f, \{g, h\}_P\}_Q + \{f, \{g, h\}_Q\}_P) = 0$, *where* $\circlearrowleft$ *denotes the sum over the circular permutations of* f, g, h,

(iii) $[X_f, Y_g] + [Y_f, X_g] = X_{\{f,g\}_Q} + Y_{\{f,g\}_P}$,

for $f, g \in C^\infty(M)$, *where* $X_f = Pdf$, $Y_f = Qdf$.

Proof. In fact, each of these conditions is the polarization of the corresponding condition for a single Poisson structure, and each is obtained by bilinearity from the corresponding conditions for P, Q and $P + Q$. ∎

For a Hamiltonian system on a symplectic manifold - the phase space - to be completely integrable in the sense of Liouville and Arnold[32], there must exist a number of independent conserved quantities, equal to half the dimension of the symplectic manifold, which are pairwise in involution. Here we consider the case where the phase space is a bihamiltonian manifold, and we show that when a sequence of functions defined on it satisfies the Lenard recursion relations, these functions are pairwise in involution. We shall denote the positive integers by $\mathbb{N}^*$.

Proposition 2.3.— *Let P and Q be Poisson structures on a manifold, M, and let $(f_k), k \in \mathbb{N}^*$, be a sequence of complex-valued functions on M that satisfy the Lenard recursion relation,*

$$Q(df_k) = P(df_{k+1}) \ , \tag{2.1}$$

for $k \in \mathbb{N}^*$. Then the functions, f_k, are pairwise in involution with respect to both Poisson brackets.

Proof. Let m be a nonegative integer, and let (C_m) be the property that, for all $k \geq 1, P(df_k, df_{k+m}) = 0$ and $Q(df_k, df_{k+m}) = 0$. Clearly (C_0) holds. Now for any $k \geq 1, m \geq 0$,

$$P(df_k, df_{k+m+1}) = -\langle df_k, P(df_{k+m+1})\rangle = -\langle df_k, Q(df_{k+m})\rangle \ ,$$

and

$$Q(df_k, df_{k+m+1}) = \langle df_{k+m+1}, Q(df_k)\rangle = \langle df_{k+m+1}, P(df_{k+1})\rangle \ .$$

Thus it is clear that (C_{m+1}) holds if (C_m) holds. Therefore (C_m) is proved for all nonnegative integers, m. Thus $P(df_k, df_\ell) = Q(df_k, df_\ell) = 0$ for any $k, \ell \in \mathbb{N}^*$. ∎

We remark that this proof uses only (2.1) and the skew-symmetry of P and Q. However, the assumption that P and Q are compatible Poisson structures is essential in order to guarantee the existence of functions, f_k, fulfilling the Lenard recursion relations (2.1). The question of the existence of such functions in the case of an arbitrary bihamiltonian structure is a difficult problem which is beyond the scope of the present paper. Here, we shall demonstrate their existence in a special case, that of a bihamiltonian manifold, (M, P, Q), where P is an *invertible* Poisson structure, *i.e.*, a symplectic structure. The field of (1,1)-tensors,

$$N = QP^{-1} \ , \tag{2.2}$$

is called the *recursion operator* or the *Nijenhuis operator* of the bihamiltonian structure. The first name is justified by the fact that N maps symmetries of a bihamiltonian system into symmetries of the same system (see section 4), while the second name is justified by the well-known result proved in lemma 2.5 below. Nijenhuis operators provide the basic examples of the Lax-Nijenhuis operators to be defined in subsection 3.2.

We recall that the Nijenhuis torsion of a field of (1,1)-tensors N on a manifold M is the vector-valued 2-form $T(N)$ on M defined by

$$T(N)(X,Y) = [NX, NY] - N([NX, Y] + [X, NY]) + N^2[X,Y], \tag{2.3}$$

for all vector fields X, Y on M.

Definition 2.4.— *A field of (1,1)-tensors with vanishing Nijenhuis torsion is called a Nijenhuis tensor or Nijenhuis operator.*

Lemma 2.5.— *If (P,Q) is a bihamiltonian structure on M, and $Q = NP$, where N is a (1,1)-tensor on M, then the Nijenhuis torsion, $T(N)$, of N vanishes on the image of P. In particular, if (P,Q) is a bihamiltonian structure, with P invertible, then the recursion operator, $N = QP^{-1}$, is a Nijenhuis operator.*

Proof. Assume that $Q = NP$. It is enough to show that $T(N)$ vanishes on any pair of vectors (Pdf, Pdg) where $f, g \in C^\infty(M)$. In fact, using the notations of lemma 2.2,

$$\begin{aligned} T(N)(Pdf,Pdg) &= [NPdf, NPdg] - N([NPdf, Pdg] + [Pdf, NPdg]) + N^2[Pdf, Pdg] \\ &= [Y_f, Y_g] - N([Y_f, X_g] + [X_f, Y_g]) + N^2[X_f, X_g] \ . \end{aligned}$$

Using the results of lemma 2.2, we obtain

$$T(N)(Pdf, Pdg) = Y_{\{f,g\}_Q} - N(X_{\{f,g\}_Q} + Y_{\{f,g\}_P}) + N^2 X_{\{f,g\}_P} \ ,$$

which vanishes since $NX_h = Y_h$, for $h \in C^\infty(M)$. ∎

The condition that $T(N) = 0$ is equivalent to the condition that

$$\mathcal{L}_{NX}N - N\mathcal{L}_X N = 0 \ , \tag{2.4}$$

for all vector fields X on M, where $\mathcal{L}_X$ denotes the Lie derivative with respect to X. In fact,

$$T(N)(X,Y) = \mathcal{L}_{NX}(NY) - N\mathcal{L}_{NX}Y - N(\mathcal{L}_X(NY) - N(\mathcal{L}_X Y))$$

$$= (\mathcal{L}_{NX}N)Y - N(\mathcal{L}_X N)Y \ .$$

Whence

Proposition 2.6.— *Let N be a Nijenhuis tensor on a manifold M. Then*

$$\mathcal{L}_{NX}N - \mathcal{L}_X(\frac{N^2}{2}) = [N, \frac{1}{2}\mathcal{L}_X N] \ , \tag{2.5}$$

for all vector fields, X, on M.

Proof. Relation (2.5) follows from the preceding expression of $T(N)$ and the assumption that $T(N) = 0$. ∎

For the Nijenhuis operator, $N = QP^{-1}$, of a bihamiltonian structure, (P, Q), where P is invertible, equation (2.5) becomes

$$\mathcal{L}_{Q\alpha}N - \mathcal{L}_{P\alpha}(\frac{N^2}{2}) = [N, \widehat{N}(\alpha)] \tag{2.6}$$

for all differential forms α on M, where $\widehat{N}(\alpha) = \frac{1}{2}\mathcal{L}_{P_\alpha}N$. This property is the prototype of that of Lax operators on bihamiltonian manifolds.

III. Lax operators

We first describe the development of the notion of Lax operator from the simplest case to that of Lax operators on Hamiltonian and bihamiltonian phase spaces. We then motivate our definition of Lax-Nijenhuis operators.

3.1 Lax operators on Hamiltonian and bihamiltonian phase spaces

A dynamical system, $\frac{dx}{dt} = X(x)$, on a manifold M is said to admit a Lax formulation if there exist square matrices L and B, by no means unique, whose coefficients depend on x, such that the given dynamical system is equivalent to

$$\frac{dL}{dt} = [L, B] \ , \tag{3.1}$$

where $[L, B] = LB - BL$ is the usual commutator. Usually, L is called the Lax operator or the Lax matrix. In fact, both L and B are maps from the manifold M (the space of dependent variables) to the associative algebra of square matrices of a given size. The existence of a Lax formulation for a given dynamical system is important because it implies the existence of a sequence of conserved quantities,

$$J_k = \frac{1}{k} tr L^k \ , \tag{3.2}$$

for $k \in \mathbb{N}^*$, where tr denotes the trace of a matrix. (These conserved quantities need not be functionnally independent.) In fact,

$$\frac{dJ_k}{dt} = \frac{1}{k} tr L^{k-1} \frac{dL}{dt} = \frac{1}{k} tr L^{k-1}[L, B] = \frac{1}{k} tr(L^k B - L^{k-1} BL) = 0 \ .$$

If, moreover, the Lax mapping L is defined on a phase space with a Hamiltonian structure, *i.e.*, on a Poisson manifold, then it is natural to require that the traces of powers of L, which are conserved quantities, be pairwise in involution. In this case, this requirement becomes part of the definition of a Lax operator.

Let us now consider the case where the phase space is a bihamiltonian manifold (M, P, Q). We have seen in section 2 that, on a bihamiltonian manifold, recursion relations (2.1) for functions f_k imply the pairwise involutivity of these functions. It is natural to require that a Lax operator L defined on a bihamiltonian phase space (M, P, Q) be such that quantities J_k defined by (3.2), proportional to the traces of powers of L, satisfy the so-called Lenard recursion relations

$$Q(dJ_k) = P(dJ_{k+1}) \ , \tag{3.3}$$

for $k \in \mathbb{N}^*$. So, we are led to introduce the following definition of Lax operators on a bihamiltonian phase space (M, P, Q).

Recall that a trace on an associative algebra A over the field of real or complex numbers is a linear form, tr, on A, such that

$$tr L_1 L_2 = tr L_2 L_1 \ , \tag{3.4}$$

for all L_1, L_2 in A.

Definition 3.1.— *Let (M, P, Q) be a bihamiltonian manifold. A Lax mapping compatible with (P, Q) is an A-valued function L on M, where A is an associative algebra with unit and trace, such that the functions, $J_k = \frac{1}{k} tr L^k$, $k \in \mathbb{N}^*$, satisfy the Lenard recursion relations (3.3).*

Under this definition, by proposition 2.3, the traces of the powers of a Lax mapping compatible with (P, Q) are pairwise in involution with respect to both P and Q.

3.2 Lax-Nijenhuis operators

To understand what relates L to P and Q, we shall consider the simplest case where L is a matrix, but we shall first review some facts about the geometry of associative algebras.

Let A be an associative algebra with a trace. We assume that the symmetric bilinear form on A, $(L_1, L_2) = tr L_1 L_2$, defines an isomorphism of A with its dual A^* and, by means of this isomorphism, we identify A^* with A. We equip A with the Lie-algebra structure defined by the associative product,

$$[L_1, L_2] = L_1 L_2 - L_2 L_1 \ .$$

Since $tr L_1 L_2 L_3 = tr L_3 L_1 L_2$, the symmetric bilinear form (,) is invariant, *i.e.*,

$$(L_1, [L_2, L_3]) = ([L_1, L_2], L_3) \ .$$

Thus the coadjoint action of the Lie algebra A on A^* is identified with the adjoint action of A on itself, and the tangent space at L in A to the coadjoint orbit of L is $\{[L, B] \mid B \in A\}$. (See also VIII.4 of Ref. 33 for the role of coadjoint orbits in the theory of Lax operators.)

Proposition 3.2.— *Let A be the algebra of square $n \times n$ matrices, where n is a positive integer. Let L be an A-valued Lax operator compatible with the bihamiltonian structure (P, Q). We assume that L is semi-simple. Then, at each point where the eigenvalues of L are distinct, and for each differential 1-form α on M, there exists a matrix $\widetilde{L}(\alpha)$ such that*

$$\mathcal{L}_{Q\alpha} L - L \mathcal{L}_{P\alpha} L = [L, \widetilde{L}(\alpha)] \ . \tag{3.5}$$

Proof. Let α be any differential 1-form on M. Then, by the definition of a Lax operator, L, compatible with (P, Q), and the skew-symmetry of P and Q, for all 1-forms α and for all $k \in \mathbb{N}^*$,

$$\frac{1}{k} tr \mathcal{L}_{Q\alpha} L^k = \frac{1}{k+1} tr \mathcal{L}_{P\alpha} L^{k+1} \ ,$$

whence

$$tr L^{k-1} (\mathcal{L}_{Q\alpha} L - L \mathcal{L}_{P\alpha} L) = 0 \ .$$

This condition expresses the fact that for all k, the vector field with value $\mathcal{L}_{Q\alpha}L - L\mathcal{L}_{P\alpha}L$ at L leaves trL^k invariant, which implies that it leaves all eigenvalues of L invariant. This condition is clearly satisfied if $\mathcal{L}_{Q\alpha}L - L\mathcal{L}_{P\alpha}L$ is tangent to the coadjoint orbit of L, and the converse holds if L is semi-simple with distinct eigenvalues. Thus, under the assumptions of the proposition on the spectrum of L, for each α there exists a matrix, $\widetilde{L}(\alpha)$, such that equation (3.5) is satisfied. ∎

We shall now allow L to be a section of an associative algebra bundle with trace, $\mathcal{A}$, over M. By this, we mean a vector bundle over M such that each fiber $\mathcal{A}_x$ of $\mathcal{A}$, for x in M, is an associative algebra with unit and trace, depending smoothly on x. Obviously an A-valued function L on M corresponds to the case where $\mathcal{A}$ is the trivial vector bundle, $\mathcal{A} = M \times A$. However, we formulate our definition in this more general situation in order to include the case of the Nijenhuis operators that was considered in section 2. At each point x in M, $\mathrm{End}(T_xM)$ is an associative algebra with trace, to which we can apply the preceding remarks. Equation (3.5) means that for each differential form α on M, and for each x in M, the vertical vector, $(\mathcal{L}_{Q\alpha}L - L(\mathcal{L}_{P\alpha}L))(x)$, is tangent to the coadjoint orbit of $L(x)$ in $\mathcal{A}_x$. It is easy to show that this is equivalent to the fact that the vertical vector, $(\mathcal{L}_{Q\alpha}L - \mathcal{L}_{P\alpha}(\frac{L^2}{2}))(x)$, is tangent to this orbit. We assume that M has a bihamiltonian structure, (P, Q). Motivated by the discussion in the previous subsections, we define:

Definition 3.3.— *A section L of an associative algebra bundle $\mathcal{A}$ with trace over a bihamiltonian manifold (M, P, Q) is a Lax-Nijenhuis operator if, for all differential forms α on M, $\mathcal{L}_{Q\alpha}L - \mathcal{L}_{P\alpha}(\frac{L^2}{2})$ is tangent to the coadjoint orbit of $L(x)$ in the fiber $\mathcal{A}_x$ of $\mathcal{A}$, for each x in M.*

Identifying a section of $T^*\mathcal{A}$ over L with a section of $T\mathcal{A}$ over L and using the identification of the dual of the vertical space at x, $(V(\mathcal{A}_x))^* = \mathcal{A}_x^*$, with the vertical space $V(\mathcal{A}_x) = \mathcal{A}_x$, we obtain immediately

Proposition 3.4.— *A section $L : M \to \mathcal{A}$ is a Lax-Nijenhuis operator if there exists a lifting of L into a section $\widehat{L} : T^*M \to T^*\mathcal{A}$ such that*

$$\mathcal{L}_{Q\alpha}L - \mathcal{L}_{P\alpha}(\frac{L^2}{2}) = [L\ , \widehat{L}(\alpha)]\ , \tag{3.6}$$

for each section α of T^*M.

Equation (3.6) is called the *Lax-Nijenhuis equation.*

Examples. By (2.6), the recursion operator of a bihamiltonian manifold (M, P, Q) with P invertible is a Lax-Nijenhuis operator.

Proposition 3.2 shows that any matrix-valued Lax operator with a simple spectrum, compatible with (P, Q) is a Lax-Nijenhuis operator.

3.3 Properties of Lax-Nijenhuis operators

We shall now prove that the traces of powers of any matrix-valued Lax-Nijenhuis operator on a bihamiltonian manifold, (M, P, Q), satisfy the Lenard recursion relations (3.3), and that the operator is therefore, by definition 3.1, a Lax operator compatible with (P, Q).

Proposition 3.5.— *Let L be a matrix-valued Lax-Nijenhuis operator on a bi-hamiltonian manifold (M, P, Q). Then the functions $J_k = \frac{1}{k} tr L^k, k \in \mathbb{N}^*$, satisfy the Lenard recursion relations (3.3), and L is a Lax operator compatible with (P, Q). Moreover, if L is invertible, relation (3.3) holds when k is a negative integer, and, if L admits a fractional power, $L^{\frac{1}{r}}$, relation (3.3) also holds when k is an integral multiple of $\frac{1}{r}$.*

Proof. Relation (3.6) implies that, for any $k \in \mathbb{N}^*$,

$$L^{k-1}\mathcal{L}_{Q\alpha}L - L^{k-1}\mathcal{L}_{P\alpha}(\frac{L^2}{2}) = [L, L^{k-1}\widehat{L}(\alpha)] \ ,$$

for any α. Taking traces of both sides implies that

$$\mathcal{L}_{Q\alpha}(\frac{1}{k} tr L^k) = \mathcal{L}_{P\alpha}(\frac{1}{k+1} tr L^{k+1}) \ ,$$

whence, with the notation of (3.2),

$$\langle Q\alpha, dJ_k\rangle = \langle P\alpha, dJ_{k+1}\rangle \ .$$

Since this relation holds for any differential form α, we obtain relation (3.3) by the skew-symmetry of P and Q. It follows from definition 3.1 that L is a Lax operator compatible with (P, Q).

We now show that relation (3.3) holds for negative and fractional powers of Lax-Nijenhuis operators, when they are defined. When α is a fixed differential form, we introduce the convenient notations

$$\mathcal{L}_{P\alpha}L = \frac{dL}{dt_1}, \quad \mathcal{L}_{Q\alpha}L = \frac{dL}{dt_2}, \quad \widehat{L}(\alpha) = B \ .$$

Using the following elementary formulae, valid for $L, B \in A$ and $k \in \mathbb{N}^*$,

(3.7) $$[L^k, B] = \sum_{j=0}^{k-1}[L, L^j B L^{k-1-j}] = \sum_{j=0}^{k-1} L^j [L, B] L^{k-1-j}$$

and

(3.8) $$\frac{dL^k}{dt} = \sum_{j=0}^{k-1} L^j \frac{dL}{dt} L^{k-1-j} \ ,$$

we obtain from (3.6), by induction on k,

(3.9) $$\frac{dL^k}{dt_2} - \frac{1}{2}(L\frac{dL^k}{dt_1} + \frac{dL^k}{dt_1}L) = [L^k, B],$$

for k a positive integer. Let us prove that, if L is invertible, (3.9) is also valid for k a negative integer. In fact, for $k=-1$,

$$\frac{dL^{-1}}{dt_2}-\frac{1}{2}(L\frac{dL^{-1}}{dt_1}+\frac{dL^{-1}}{dt_1}L)$$

$$=-L^{-1}\frac{dL}{dt_2}L^{-1}+\frac{1}{2}(\frac{dL}{dt_1}L^{-1}+L^{-1}\frac{dL}{dt_1})$$

$$=-\frac{1}{2}L^{-1}(L\frac{dL}{dt_1}+\frac{dL}{dt_1}L)L^{-1}+\frac{1}{2}(\frac{dL}{dt_1}L^{-1}+L^{-1}\frac{dL}{dt_1})-L^{-1}[L,B]L^{-1}$$

$$=-BL^{-1}+L^{-1}B=[L^{-1},B]\ ,$$

and, more generally, formula (3.9) for $k<-1$ is proved by recursion.

We now assume that L admits a fractional power D, namely $D^r=L$, for some positive integer r. Then using (3.7) and (3.8), we obtain

$$0=\frac{dD^r}{dt_2}-\frac{1}{2}(D^r\frac{dD^r}{dt_1}+\frac{dD^r}{dt_1}D^r)-[D^r,B]$$

$$=\sum_{j=1}^{r}(D^{r-j}\frac{dD}{dt_2}D^{j-1}-\frac{1}{2}(D^rD^{r-j}\frac{dD}{dt_1}D^{j-1}+D^{r-j}\frac{dD}{dt_1}D^{j-1}D^r)-D^{r-j}[D,B]D^{j-1}),$$

thus

$$\sum_{j=1}^{r}D^{r-j}(\frac{dD}{dt_2}-\frac{1}{2}(L\frac{dD}{dt_1}+\frac{dD}{dt_1}L)-[D,B])D^{j-1}=0\ . \tag{3.10}$$

Still more generally, if $D^r=L$, and h is a positive integer, we can prove

$$\sum_{j=1}^{r}D^{r-j}(\frac{dD^h}{dt_2}-\frac{1}{2}(L\frac{dD^h}{dt_1}+\frac{dD^h}{dt_1}L)-[D^h,B])D^{j-1}=0\ . \tag{3.11}$$

In fact, we first write (3.10), left-multiply by D^{h-1-i}, then right-multiply by D^i and sum from $i=0$ to $h-1$. In the resulting equality, we use (3.7) and (3.8), to obtain (3.11). Taking traces in (3.11) and using

$$\frac{1}{b}trD^a\frac{dD^b}{dt}=\frac{1}{a+b}tr\frac{dD^{a+b}}{dt},$$

we obtain,

$$\frac{1}{r+h-1}tr\frac{dD^{r+h-1}}{dt_2}=\frac{1}{2r+h-1}tr\frac{dD^{2r+h-1}}{dt_1}\ . \tag{3.12}$$

Setting $\frac{r+h-1}{r}=k$, we obtain relation (3.3) for any k that is an integral multiple of $\frac{1}{r}$. Combining the previous results, we see that this formula also holds for negative rational numbers, when such powers of L are defined. ∎

IV. Bihamiltonian hierarchies and Lax formulation

We have emphasized the striking analogies between the properties of a Lax operator compatible with a bihamiltonian structure and those of a Nijenhuis operator. In this section, we shall continue by considering the "hereditary properties" of both types of operators.

The simplest hereditary property of a Nijenhuis operator, N, is that it maps symmetries of N into symmetries of N. In fact, if a vector field X is a symmetry of N, *i.e.*, is such that

$$\mathcal{L}_X N = 0 \ ,$$

then, by relation (2.4), and as a consequence of the vanishing of the torsion of N,

$$\mathcal{L}_{NX} N = 0 \ .$$

Thus, if X is a symmetry of N, so are NX and, more generally, $N^k X$, for $k \in \mathbb{N}$. The property has been "inherited" by the iterated vector fields NX, N^2X,...

We now consider a sequence of differential 1-forms, α_k, $k \in \mathbb{N}$, on a bihamiltonian manifold, (M, P, Q), that satisfy the recursion relations,

$$Q\alpha_k = P\alpha_{k+1} \ . \tag{4.1}$$

Let us consider the sequence of vector fields,

$$X_k = Q\alpha_k = P\alpha_{k+1} \ . \tag{4.2}$$

If P is invertible, then, by lemma 2.5, $N = QP^{-1}$ is a Nijenhuis operator and it satisfies

$$X_{k+1} = NX_k \ . \tag{4.3}$$

If we now assume that the vector field X_0 is a symmetry of the Nijenhuis operator N, then so is X_k, for each $k \in \mathbb{N}$, by the hereditary property of N, recalled above.

Let us now examine the corresponding property for Lax-Nijenhuis operators. Let L be a matrix-valued Lax-Nijenhuis operator on a bihamiltonian manifold (M, P, Q), in the sense of definition 3.3, and let α_k and X_k be forms and vectors as above. We assume that the vector field X_0 is such that there exists a matrix-valued mapping A_0 on M satisfying

$$\frac{dL}{dt_0} = [L, A_0], \quad \text{where} \quad \frac{dL}{dt_0} = \mathcal{L}_{X_0} L \ . \tag{4.4}$$

We shall prove by recursion that, for any vector field X_k in the associated sequence, there exists a matrix-valued mapping A_k on M satisfying

$$\frac{dL}{dt_k} = [L, A_k], \quad \text{where} \quad \frac{dL}{dt_k} = \mathcal{L}_{X_k} L \ . \tag{4.5}$$

In fact, let us assume (4.5) for $0, 1, \ldots, k-1$. Then, from (3.6), we obtain

$$\begin{aligned}\mathcal{L}_{X_k} L = \mathcal{L}_{Q\alpha_k} L &= \mathcal{L}_{P\alpha_k} \frac{L^2}{2} + [L, \widehat{L}(\alpha_k)] \\ &= \frac{1}{2}(L\mathcal{L}_{X_{k-1}} L + (\mathcal{L}_{X_{k-1}} L)L) + [L, \widehat{L}(\alpha_k)] \\ &= [L, \frac{1}{2}(LA_{k-1} + A_{k-1}L) + \widehat{L}(\alpha_k)] \ .\end{aligned}$$

Setting $A_k = \frac{1}{2}(LA_{k-1} + A_{k-1}L) + \widehat{L}(\alpha_k)$, we obtain (4.5) for k.

Remark. Setting $\widehat{L}(\alpha_k) = C_k, k \in \mathbb{N}$, an explicit expression for $A_k, k \in \mathbb{N}^*$, is

$$A_k = \frac{1}{2^k} \sum_{h=0}^{k} \binom{k}{h} L^h A_0 L^{k-h} + \sum_{h=0}^{k-1} \sum_{p=0}^{h} \frac{1}{2^h} \binom{h}{p} L^p C_{k-h} L^{h-p} \ .$$

In fact, this formula is valid for $k = 1$ and is proved by recursion.

The following proposition summarizes this discussion.

Proposition 4.1.— *Let α_k , $k \in \mathbb{N}$, be a sequence of differential 1-forms on the bihamiltonian manifold (M, P, Q), with α_k satisfying recursion relations (4.1) and let $X_k = Q\alpha_k = P\alpha_{k+1} = \frac{d}{dt_k}$ be the corresponding sequence of vector fields. If the vector field $X_0 = \frac{d}{dt_0}$ admits a Lax formulation,*

$$\frac{dL}{dt_0} = [L, A_0] \ ,$$

where L is a matrix-valued Lax-Nijenhuis operator, then for each $k \in \mathbb{N}$, there exists a matrix-valued mapping A_k on M satisfying (4.5).

In particular, we shall consider the case when there exists a sequence of closed differential 1-forms α_k satisfying recursion relations (4.1). When P is invertible, we set $N = QP^{-1}$, and we denote the transpose of N by ${}^tN$. Then (4.1) is written

$$\alpha_{k+1} = ({}^tN)(\alpha_k) \quad \text{or} \quad \alpha_k = ({}^tN)^k \alpha_0 \ ,$$

and (4.3) is written

$$X_k = N^k(X_0) \ .$$

Proposition 4.2.— *Let (M, P, Q) be a bihamiltonian manifold with P invertible. Assume that the differential 1-forms α_0 and α_1 are closed. Then all α_k's are closed and the vector fields $X_k, k \in \mathbb{N}$, are (locally) bihamiltonian vector fields which commute in pairs.*

Proof. Using the fact that N has vanishing Nijenhuis torsion (lemma 2.5), we find that

$$d\alpha_k(X,Y) = d\alpha_{k-1}(NX,Y) + d\alpha_{k-1}(X,NY) - d\alpha_{k-2}(NX,NY) \ ,$$

for $k \geq 2$ and for all vector fields X, Y on M. Thus all the α_k's are closed.

Therefore each vector field X_k is (locally) bihamiltonian,

$$\mathcal{L}_{X_k} P = 0 \ , \ \mathcal{L}_{X_k} Q = 0 \ ,$$

and hence each X_k is a symmetry of N,

$$\mathcal{L}_{X_k} N = 0 \ .$$

(This fact also follows from $\mathcal{L}_{X_0} N = 0$ and the hereditary property of N.)

Thus

$$\begin{aligned}
[X_k, X_\ell] &= \mathcal{L}_{X_k}(N^\ell X_0) = (\mathcal{L}_{X_k} N^\ell) X_0 + N^\ell \mathcal{L}_{X_k} X_0 \\
&= -N^\ell \mathcal{L}_{X_0}(N^k X_0) \\
&= -N^\ell (\mathcal{L}_{X_0} N^k)(X_0) - N^{\ell+k} \mathcal{L}_{X_0} X_0 = 0 \ . \ \blacksquare
\end{aligned}$$

Remark. If X is any (locally) bihamiltonian vector field, then $\mathcal{L}_X N = 0$. It follows that if Y is a symmetry of X, so is NY. This justifies the term "recursion operator" for the Nijenhuis operator N of a bihamiltonian structure (P,Q), with P invertible.

Remark. Let k and ℓ be nonnegative integers. For any Nijenhuis operator N and vector field X, it follows from (2.4) by recursion that

$$\mathcal{L}_{NX}(N^\ell) = N \mathcal{L}_X (N^\ell)$$

and that

$$\mathcal{L}_{N^k X}(N^\ell) = N^k \mathcal{L}_X (N^\ell) \ .$$

For $k = \ell$, we recover the well known fact that any positive power of a Nijenhuis operator is a Nijenhuis operator, and that negative and fractional powers of a Nijenhuis operator, when they are defined, are also Nijenhuis operators.

A sequence of commuting bihamiltonian vector fields is called a *bihamiltonian hierarchy.* When $X_k, k \in \mathbb{N}$, is a bihamiltonian hierarchy, we obtain further properties for the sequence of Lax equations (4.5). In fact, writing that $\mathcal{L}_{X_j}\mathcal{L}_{X_k} L - \mathcal{L}_{X_k}\mathcal{L}_{X_j} L = 0$ for all j, $k \in \mathbb{N}$, we obtain

$$\begin{aligned}
0 &= \mathcal{L}_{X_j}[L, A_k] - \mathcal{L}_{X_k}[L, A_j] \\
&= [\frac{\partial L}{\partial t_j} \ , \ A_k] + [L, \frac{\partial A_k}{\partial t_j}] - [\frac{\partial L}{\partial t_k}, A_j] - [L, \frac{\partial A_j}{\partial t_k}] \\
&= [[L, A_j], A_k] - [[L, A_k], A_j] + [L, \frac{\partial A_k}{\partial t_j} - \frac{\partial A_j}{\partial t_k}] \\
&= [L, [A_j, A_k] + \frac{\partial A_k}{\partial t_j} - \frac{\partial A_j}{\partial t_k}] \ ,
\end{aligned}$$

by the Jacobi identity. Thus the operator, which can be called the curvature of the connection defined by A_k,

$$
[A_j, A_k] + \frac{\partial A_k}{\partial t_j} - \frac{\partial A_j}{\partial t_k}
$$

commutes with L. Summarizing, we obtain

Proposition 4.3.— *When X_k is a bihamiltonian hierarchy with Lax formulation (4.5), the curvature of the connection with components A_k commutes with the Lax operator L .*

V. Examples

In this section, we give four examples showing how the Lax-Nijenhuis equation can be used to compute higher-order Hamiltonian structures associated with a Hamiltonian system admitting a Lax representation. We write the Lax-Nijenhuis equation (3.6) in the form

$$
\mathcal{L}_{Q\alpha} L = \frac{1}{2}(L \mathcal{L}_{P\alpha} L + (\mathcal{L}_{P\alpha} L) L) + [L, \widehat{L}(\alpha)], \tag{5.1}
$$

and we use the information on L and the arbitrariness of α to split this equation into two parts: the first determines the unknown map $\widehat{L}$, up to some still arbitrary constants, the second part determines $\mathcal{L}_{Q\alpha} L$ and then Q. The condition of skewsymmetry on Q then determines the constants. The discussion is quite similar to a problem with constraints coming from the restrictions imposed on L, where the role of the Lagrange multipliers is played by the mapping $\widehat{L}$.

5.1 The Toda system

We shall consider the Toda system and its well-known Lax formulation. Let $M = \mathbb{R}^{2n+1}$ with coordinates $x^I = (a_j, b_\ell)$, $j = 1, \cdots, n$, $\ell = 1, \cdots, n+1$, $I = 1, \cdots, 2n+1$. We consider the Poisson bivector P_0 defined by

$$
\begin{aligned}
\{a_j, b_j\} &= -a_j \ , \ j = 1, 2, \cdots, n \\
\{a_j, b_{j+1}\} &= a_j \ , \ j = 1, 2, \cdots, n \ ,
\end{aligned} \tag{5.2}
$$

all other Poisson brackets being equal to 0. Thus the Poisson bivector P_0 has matrix

$$
(P_0^{IJ}) = \begin{pmatrix}
 & & & & a_1 & -a_1 & & & \\
 & 0 & & & & a_2 & -a_2 & & 0 \\
 & & & & & & \ddots & \ddots & \\
 & & & & 0 & & & \ddots & \ddots \\
 & & & & & & & a_n & -a_n \\
-a_1 & & & & & & & & \\
a_1 & -a_2 & & 0 & & & & & \\
 & a_2 & \ddots & & & & & & \\
 & & \ddots & \ddots & & & 0 & & \\
 & 0 & & \ddots & -a_n & & & & \\
 & & & & a_n & & & &
\end{pmatrix}
$$

When $H = \sum_{j=1}^{n} a_j^2 + \frac{1}{2}\sum_{\ell=1}^{n+1} b_\ell^2$, the Hamiltonian vector field $X = P_0 dH$ is the Toda vector field. In fact, the evolution equation $\frac{dx}{dt} = X(x)$ is the system

$$\begin{cases} \dfrac{da_j}{dt} = \{a_j, H\} \ , \ j = 1, 2, \cdots, n \ , \\ \dfrac{db_\ell}{dt} = \{b_\ell, H\} \ , \ \ell = 1, 2, \cdots, n+1 \ , \end{cases}$$

and, setting $a_0 = a_{n+1} = 0$, this system becomes

$$\begin{cases} \dfrac{da_j}{dt} = a_j(b_{j+1} - b_j) \ , \ j = 1, 2, \cdots, n \ , \\ \dfrac{db_\ell}{dt} = 2(a_\ell^2 - a_{\ell-1}^2), \ell = 1, 2, \cdots, n+1 \ , \end{cases} \tag{5.3}$$

which are the equations of the Toda system in Flaschka coordinates[15]. A Lax formulation for the Toda system is

$$\frac{dL}{dt} = [L, A] \ ,$$

where
(5.4)

$$L = \begin{pmatrix} b_1 & a_1 & & & & & \\ a_1 & b_2 & a_2 & & & 0 & \\ & a_2 & b_3 & a_3 & & & \\ & & a_3 & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & 0 & & & \ddots & b_n & a_n \\ & & & & & a_n & b_{n+1} \end{pmatrix}, A = \begin{pmatrix} 0 & a_1 & & & & & \\ a_1 & 0 & a_2 & & & 0 & \\ & a_2 & 0 & a_3 & & & \\ & & a_3 & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & 0 & & & \ddots & 0 & a_n \\ & & & & & a_n & 0 \end{pmatrix}$$

Now let P and Q be Poisson structures such that (5.1) is satisfied for L symmetric and tridiagonal, as above. We assume that P is known, and we consider Q to be an unknown, higher-order, Poisson structure. We shall assume that $\widehat{L}(\alpha)$ depends linearly on α, and we shall denote the matrix $\widehat{L}(dx^I)$ of order $n+1$ by C^I. Then (5.1) becomes

$$L(P^{IJ}\frac{\partial L}{\partial x^J} + 2C^I) + (P^{IJ}\frac{\partial L}{\partial x^J} - 2C^I)L = 2Q^{IJ}\frac{\partial L}{\partial x^J} \ , \tag{5.5}$$

where the summation over $J = 1, 2, \cdots, 2n+1$ is understood.

We shall assume that C^I is a skewsymmetric tridiagonal matrix, for each $I = 1, \cdots, 2n+1$,

$$C^I = \begin{pmatrix} 0 & c^{I1} & & & & & \\ -c^{I1} & 0 & c^{I2} & & & 0 & \\ & -c^{I2} & 0 & c^{I3} & & & \\ & & -c^{I3} & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & 0 & & & \ddots & 0 & c^{In} \\ & & & & & -c^{In} & 0 \end{pmatrix} \tag{5.6}$$

where $c_1^I, \cdots, c_n^I$ are to be determined. This choice is the simplest possible and it guarantees that $[L, C^I]$ is a symmetric penta-diagonal matrix. Any other choice of C^I would yield a matrix $[L, C^I]$ with more entries, and so with more constraints to be imposed.

For each $I = 1, \cdots, 2n+1$, $P^{IJ}\frac{\partial L}{\partial x^J}$ is the symmetric tridiagonal matrix

$$\begin{pmatrix} P^{I,n+1} & P^{I1} & & & 0 \\ P^{I1} & P^{I,n+2} & P^{I2} & & \\ & P^{I2} & \ddots & \ddots & \\ & & \ddots & P^{I,2n} & P^{In} \\ 0 & & & P^{In} & P^{I,2n+1} \end{pmatrix},$$

and therefore $(P^{IJ}\frac{\partial L}{\partial x^J} - 2C^I)L$ is the transpose of $L(P^{IJ}\frac{\partial L}{\partial x^J} + 2C^I)$. Thus condition (5.5) implies that, for fixed I, matrix C^I is such that the symmetric part of $L(P^{IJ}\frac{\partial L}{\partial x^J} + 2C^I)$ is tridiagonal. Writing this condition explicitly, one obtains

$$\sum_{k=1}^{n-1}(a_k(P^{J,k+1} + 2c^{J,k+1}) + a_{k+1}(P^{Jk} - 2c^{Jk})) = 0 \ .$$

Solving this system for c^{Jk} in terms of P^{Jk} yields the existence of multipliers λ_k^J, $k = 1, 2, \cdots, n$, such that

$$2c^{Jk} = P^{Jk} + 2\lambda_k^J a_k \ .$$

We assume that P^{Jk} is divisible by a_k. This assumption is satisfied for $P = P_0$ defined by (5.2). Setting $\lambda_1^J = \lambda^J$, we obtain

$$2c^{Jk} = P^{Jk} + 2a_k(\lambda^J - \sum_{j=2}^{k}\frac{P^{Jj}}{a_j}) \ .$$

(By convention, here and below the last sum vanishes if $k < 2$.)

From relations (5.5), we then obtain the coefficients of the higher-order Poisson structure Q,

$$Q^{Jk} = \frac{1}{2}a_k(P^{J,n+k} + P^{J,n+k+1}) + \frac{1}{2}b_k(P^{Jk} + 2c^{Jk}) + \frac{1}{2}b_{k+1}(P^{Jk} - 2c^{Jk}) \ ,$$

$$Q^{J,n+\ell} = a_\ell(P^{J\ell} - 2c^{J\ell}) + a_{\ell-1}(P^{J,\ell-1} + 2c^{J,\ell-1}) + b_\ell P^{J,n+\ell} \ .$$

We now replace the c^{Jk}'s by their values in terms of the parameters λ^J, and we impose the conditions that the diagonal terms of Q vanish. These $2n+1$ conditions imply

$$2(b_{k+1} - b_k)(\lambda^k - \sum_{j=2}^{k-1}\frac{P^{kj}}{a_j}) = P^{k,n+k} + P^{k,n+k+1},$$

$$a_1^2\lambda^{n+1} = 0 \ ,$$

$$(a_\ell^2 - a_{\ell-1}^2)\left(\lambda^{n+\ell} - \sum_{j=2}^{\ell-1}\frac{P^{n+\ell,j}}{a_j}\right) = a_{\ell-1}P^{n+\ell,\ell-1} + a_\ell P^{n+\ell,\ell},$$

for $\ell = 2, \cdots, n+1$.

Thus, we have obtained

$$Q^{Jk} = \frac{1}{2}a_k\left(P^{J,n+k} + P^{J,n+k+1}\right) + b_k P^{Jk} + a_k(b_k - b_{k+1})\left(\lambda^J - \sum_{j=2}^{k-1}\frac{P^{Jj}}{a_j}\right) \ ,$$

$$Q^{J,n+\ell} = 2\left(a_{\ell-1}P^{J,\ell-1} + a_\ell P^{J\ell}\right) + b_\ell P^{J,n+\ell} + 2(a_{\ell-1}^2 - a_\ell^2)\left(\lambda^J - \sum_{j=2}^{\ell-1}\frac{P^{Jj}}{a_j}\right) \ ,$$

where the λ^J's are given above in terms of a_k, b_ℓ, P^{IJ}.

Let us assume that all a_k's are nonvanishing and let us introduce the matrix M of order $2n+1$, depending on $a_1, \cdots, a_n, b_1, \cdots, b_{n+1}$, such that M applied to the column with entries $A_1, \cdots, A_n, B_1, \cdots, B_{n+1}$, is the column with entries

$$(5.7) \qquad \begin{cases} \overline{A}_k = \frac{1}{2}a_k(B_k + B_{k+1}) + b_k A_k + a_k(b_{k+1} - b_k)\sum_{j=2}^{k}\frac{A_j}{a_j}, \\ \overline{B}_1 = b_1 B_1, \ \overline{B}_2 = b_2 B_2 + 2(a_1 A_1 + a_2 A_2), \\ \overline{B}_\ell = b_\ell B_\ell + 2(a_{\ell-1}A_{\ell-1} + a_\ell A_\ell) + 2(a_\ell^2 - a_{\ell-1}^2)\sum_{j=2}^{\ell-1}\frac{A_j}{a_j} \ , \end{cases}$$

for $\ell = 3, \cdots, n+1$.

We see that

$$Q = MP + X \otimes \lambda \ ,$$

where X is the Toda vector field with components

$$a_1(b_2 - b_1), a_2(b_3 - b_2), \cdots, a_n(b_{n+1} - b_n), 2a_1^2, 2(a_2^2 - a_1^2), \cdots, -2a_n^2 \ ,$$

and λ is the vector with components $\lambda^1, \cdots, \lambda^n, \lambda^{n+1}, \cdots, \lambda^{2n+1}$. We observe that, although Q is skewsymmetric, this expression does not constitute a decomposition of Q into a sum of skewsymmetric 2-tensors.

Thus the Lax-Nijenhuis equation yields an explicit determination of the bivector Q in terms of P. For $P = P_0$, we see that the corresponding vector $\lambda = \lambda_{(0)}$ is the row-matrix with entries

$$(5.8) \qquad \lambda_{(0)}^J = -\frac{1}{2}\delta_{n+2}^J \ .$$

It is easy to check that

$$P_1 = MP_0 + X \otimes \lambda_{(0)}$$

coincides with the second Poisson structure of the Toda system[17,19,20]. For example, if $n = 3$, the matrix MP_0 is equal to

$$\begin{pmatrix} b_1 & 0 & 0 & \frac{a_1}{2} & \frac{a_1}{2} & 0 & 0 \\ 0 & b_3 & 0 & 0 & \frac{a_2}{2} & \frac{a_2}{2} & 0 \\ 0 & U & b_4 & 0 & 0 & \frac{a_3}{2} & \frac{a_3}{2} \\ 0 & 0 & 0 & b_1 & 0 & 0 & 0 \\ 2a_1 & 2a_2 & 0 & 0 & b_2 & 0 & 0 \\ 0 & 2\frac{a_3^2}{a_2} & 2a_3 & 0 & 0 & b_3 & 0 \\ 0 & -2\frac{a_3^2}{a_2} & 0 & 0 & 0 & 0 & b_4 \end{pmatrix} \begin{pmatrix} & & & a_1 & -a_1 & 0 & 0 \\ & 0 & & 0 & a_2 & -a_2 & 0 \\ & & & 0 & 0 & a_3 & -a_3 \\ -a_1 & 0 & 0 & & & & \\ a_1 & -a_2 & 0 & & & & \\ 0 & a_2 & -a_3 & & & 0 & \\ 0 & 0 & a_3 & & & & \end{pmatrix}$$

where $U = \frac{a_3}{a_2}(b_4 - b_3)$, and

$$X \otimes \lambda_{(0)} = \begin{pmatrix} a_1(b_2 - b_1) \\ a_2(b_3 - b_2) \\ a_3(b_4 - b_3) \\ 2a_1^2 \\ 2(a_2^2 - a_1^2) \\ 2(a_3^2 - a_2^2) \\ -2a_3^2 \end{pmatrix} \otimes {}^t \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix} ,$$

so that

$$P_1 = MP_0 + X \otimes \lambda_{(0)} = \begin{pmatrix} 0 & -\frac{1}{2}a_1a_2 & 0 & a_1b_1 & -a_1b_2 & 0 & 0 \\ \frac{1}{2}a_1a_2 & 0 & -\frac{1}{2}a_2a_3 & 0 & a_2b_2 & -a_2b_3 & 0 \\ 0 & \frac{1}{2}a_2a_3 & 0 & 0 & 0 & a_3b_3 & -a_3b_4 \\ -a_1b_1 & 0 & 0 & 0 & -2a_1^2 & 0 & 0 \\ a_1b_2 & -a_2b_2 & 0 & 2a_1^2 & 0 & -2a_2^2 & 0 \\ 0 & a_2b_3 & -a_3b_3 & 0 & 2a_2^2 & 0 & -2a_3^2 \\ 0 & 0 & a_3b_4 & 0 & 0 & 2a_3^2 & 0 \end{pmatrix} .$$

Repeating the process, we have to compute the bivector

$$P_2 = MP_1 + X \otimes \lambda_{(1)}$$

where $\lambda_{(1)}$ is the vector corresponding to P_1. We can show that

$$\lambda_{(1)} = M\lambda_{(0)} \ .$$

For example, if $n = 3$,

$$\lambda_{(1)} = (-\frac{1}{4}a_1, -\frac{1}{4}a_2, 0, 0, -\frac{1}{2}b_2, 0, 0)$$

and P_2 is the skewsymmetric matrix

$$P_2 = MP_1 + X \otimes \lambda_{(1)} =$$

$$\begin{pmatrix} 0 & -a_1a_2b_2 & 0 & a_1b_1^2 + a_1^3 & -a_1b_2^2 - a_1^3 & -a_1a_2^2 & 0 \\ & 0 & -a_2a_3b_3 & a_1^2a_2 & a_2b_2^2 + a_2^3 & -a_2b_3^2 - a_2^3 & -a_2a_3^2 \\ & & 0 & 0 & a_2^2a_3 & a_3b_3 + a_3^2 & -a_3b_4^2 - a_3^3 \\ & & & 0 & -2a_1^2(b_1+b_2) & 0 & 0 \\ & & & & 0 & -2a_2^2(b_2+b_3) & 0 \\ & & & & & 0 & -2a_3^2(b_3+b_4) \\ & & & & & & 0 \end{pmatrix}$$

In this case, P_2 coincides with the opposite of the third Poisson structure of the Toda system described in Refs. 17 and 20.

Thus if M is the matrix of order $2n+1$ defined by (5.7), the bivectors P_i obtained from the Lax-Nijenhuis equation satisfy

$$P_{i+1} = MP_i + X \otimes \lambda_{(i)} \ ,$$

where X is the Toda vector field and $\lambda_{(i)}$ is the vector corresponding to P_i.

In particular, it follows that each coefficient P_i^{Jk} of the bivector P_i is divisible by a_k, so the iteration can be carried out.

Let us show that in fact the skewsymmetry of P_i, P_{i+1} implies that

$$\lambda_{(i+1)} = M\lambda_{(i)} \ .$$

Thus we consider

$$P_i = MP_{i-1} + X \otimes \lambda_{(i-1)} \ ,$$

$$P_{i+1} = MP_i + X \otimes \lambda_{(i)} \ .$$

Since P_{i+1} is assumed to be skewsymmetric, $\lambda_{(i)}$ must satisfy

$$P_i \, {}^tM - MP_i = X \otimes \lambda_{(i)} + \lambda_{(i)} \otimes X \ .$$

Moreover, from the skewsymmetry of P_i , we obtain

$$P_i = MP_{i-1} + X \otimes \lambda_{(i-1)} = P_{i-1}{}^tM - \lambda_{(i-1)} \otimes X \ ,$$

whence

$$P_i \, {}^tM - MP_i = X \otimes M\lambda_{(i-1)} + M\lambda_{(i-1)} \otimes X \ .$$

We thus obtain $\lambda_{(i)} = M\lambda_{(i-1)}$. The following proposition summarizes this discussion.

Proposition 5.1.— *Let M be the matrix of order $2n+1$ defined by (5.7). The bivectors obtained from the Lax-Nijenhuis equation satisfy*

$$P_{i+1} = MP_i + X \otimes M^i \lambda_{(0)} \ ,$$

where $\lambda_{(0)}$ is the vector given by (5.8).

5.2 The n-dimensional rigid body

The Euler equations for the n-dimensional rigid body rotating about a fixed point can be written

$$\dot{M} = [M, \Omega] \ ,$$

where M is the angular momentum, a time-dependent element of the Lie algebra $so(n)$, and

$$M = J\Omega + \Omega J \ .$$

Here J is a diagonal matrix with positive entries $J_1, J_2, \cdots, J_n$ defined in terms of the principal moments of inertia, and Ω is the angular velocity. These equations admit a Lax formulation, with a spectral parameter,

$$\dot{L} = [L, B] \ ,$$

where $L = \frac{M}{\lambda} + J^2$ and $B = \Omega + \lambda J$.

Moreover, it is well-known (see *e.g.*, Refs. 10, 34) that these equations can be written in Hamiltonian form, with respect to the linear Poisson structure of $so(n)$ induced by the identification of the Lie algebra $so(n)$ with its dual by means of the trace of the product of matrices. Making use of this identification, this Poisson structure P is defined by $P_M : so(n) \to so(n)$, for each M in $so(n)$, where

$$P_M = ad_M \ .$$

If $K(M) = \frac{1}{2} tr(M\Omega)$, then the gradient of K (the differential of K identified with a matrix in $so(n)$) is the constant matrix Ω, and therefore the Euler equations can be written as the Hamiltonian equation

$$\dot{M} = P(dK).$$

Let us use the Lax-Nijenhuis equation in order to find a possible form of a second Hamiltonian structure that will make this equation a bihamiltonian system. Setting $\dot{L} = \frac{dL}{dt_1}$, it follows from the definitions and from the Euler equation that

$$\frac{d(L^2)}{dt_1} = \frac{1}{\lambda}(L[M, \Omega] + [M, \Omega]L) \ ,$$

and therefore, by a simple computation,

$$\frac{1}{2}\frac{d(L^2)}{dt_1} = \frac{1}{2\lambda}[L, M\Omega + \Omega M] + \frac{1}{\lambda}(M\Omega L - L\Omega M) \ .$$

By the definition of L, $M\Omega L - L\Omega M = M\Omega J^2 - J^2\Omega M$. This suggests that we should set

$$\frac{dM}{dt_2} = M\Omega J^2 - J^2\Omega M.$$

We observe that if M and Ω are skewsymmetric, so is $\frac{dM}{dt_2}$. In fact, Q defined by

$$Q_M(V) = MVJ^2 - J^2VM,$$

for $M, V \in so(n)$, is the second, compatible Poisson structure on $so(n)$ that was recently found by Morosi and Pizzocchero[34]. This second Poisson structure Q is actually a deformation of the first, linear one, P, under the linear map $M \to JMJ$. The Euler equation can be written

$$\dot{M} = Q(dH) \ ,$$

where $H(M) = -\frac{1}{2}tr(J^{-1}MJ^{-1}\Omega)$, since the gradient of H is $-J^{-1}\Omega J^{-1}$.

We now show how to extend this procedure to the determination of Poisson structures compatible with the linear Poisson structures on the sum of several copies of a simple Lie algebra, considered in Refs. 30 and 35. Let us consider, for instance, the Hamiltonian system

$$\frac{dM_0}{dt_1} = [M_0, V_1]$$

$$\frac{dM_1}{dt_1} = [M_0, V_0] + [M_1, V_1],$$

where V_0, V_1 are the components of the gradient of a Hamiltonian function K . We introduce the Lax matrix depending on the spectral parameter λ,

$$L = \frac{M_0}{\lambda^2} + \frac{M_1}{\lambda} + A.$$

Computing the derivative of the square of this matrix, we find that

$$\frac{1}{2}\frac{d(L^2)}{dt_1} = [L, \frac{(M_0V_1 + V_1M_0)}{2\lambda^2} + \frac{(M_0V_0 + V_0M_0) + (M_1V_1 + V_1M_1)}{2\lambda}]$$

$$+\frac{1}{\lambda^2}(M_0V_0M_1 - M_1V_0M_0 + M_0V_1A - AV_1M_0) + \frac{1}{\lambda}(M_0V_0A - AV_0M_0 + M_1V_1A - AV_1M_1).$$

Therefore the Lax-Nijenhuis equation suggests that we should set

$$\frac{dM_0}{dt_2} = (M_0V_0M_1 - M_1V_0M_0) + (M_0V_1A - AV_1M_0)$$

$$\frac{dM_1}{dt_2} = (M_0V_0A - AV_0M_0) + (M_1V_1A - AV_1M_1).$$

A computation shows that this is actually a Poisson structure on the direct sum of two copies of $so(n)$. It is clearly compatible with the first, because it can be otained by deforming A into $A + \lambda I$.

5.3 The KdV equation

We now enter the field of nonlinear partial differential equations by considering the Korteweg-de Vries equation

$$\frac{du}{dt} = u_{xxx} - 6uu_x \ .$$

We use the notations of the formal calculus of variations[9,10,8]. As is well-known, the KdV equation is Hamiltonian since if can be written in the form

$$\frac{\partial u}{\partial t} = \partial \frac{\delta H}{\delta u} \ ,$$

where

$$H(u) = -\int (\frac{1}{2}u_x^2 + u^3)dx \ ,$$

and $\partial = \frac{d}{dx}$ is the Gardner Hamiltonian structure[13]. It admits a Lax representation, $\frac{du}{dt} = [L, B]$, with

$$L = \partial^2 - u, \quad B = 4\partial^3 - 3(u\partial + \partial u) \ .$$

We now want to use the Lax-Nijenhuis equation (5.1), where $P = \partial$, to find the second Hamiltonian structure of the KdV equation. Since L is a second-order differential operator, we assume that $\widehat{L}(\alpha)$ is a first-order differential operator,

$$\widehat{L}(\alpha) = \lambda + \mu\partial \ ,$$

where λ and μ depend linearly on the 1-form α. A simple computation yields:

$$\frac{1}{2}\mathcal{L}_{P\alpha}(L^2) = -\alpha_x\partial^2 - \alpha_{xx}\partial + (u\alpha_x - \frac{1}{2}\alpha_{xxx}) \ ,$$

$$[L, \widehat{L}(\alpha)] = 2\mu_x\partial^2 + (2\lambda_x + \mu_{xx})\partial + (\lambda_{xx} + \mu u_x) \ .$$

By inserting these formulas into the Lax-Nijenhuis equation and by equating the coefficients of ∂^2, ∂ and ∂^0, we get:

$$\begin{aligned} &(2\mu - \alpha)_x = 0 \ , \\ &(2\lambda + \mu_x - \alpha_x)_x = 0 \ , \\ &-\mathcal{L}_{Q\alpha}(u) = u\alpha_x - \frac{1}{2}\alpha_{xxx} + \lambda_{xx} + \mu u_x \ . \end{aligned}$$

The first two equations yield the solution $\mu = \frac{1}{2}\alpha, \ \lambda = \frac{1}{4}\alpha_x$, while the third one yields the second Hamiltonian structure of the KdV equation,

$$Q_u(\alpha) = \frac{1}{4}\alpha_{xxx} - u\alpha_x - \frac{1}{2}u_x\alpha \ .$$

The recursion operator is the nonlocal operator, $R_u = \frac{1}{4}\partial^2 - u - \frac{1}{2}u_x\partial^{-1}$.

5.4 The second Adler-Gelfand-Dickey bracket

We now generalize the previous example to the first equation of the n-th KdV hierarchy. The unknowns are functions $u_0, u_1, \ldots, u_{n-1}$ on the circle, whose time evolution is being studied. It admits a Lax formulation,

$$\frac{dL}{dt} = [L, B] ,$$

where

$$L = \partial^n + u_{n-1}\partial^{n-1} + \cdots + u_0 ,$$

and B is a suitable differential operator of order $n+1$. Here L takes values in a manifold $\mathcal{L}_n$ of invertible elements in the algebra A_n of formal pseudodifferential operators of order $\leq n$ on the circle[36]. This equation is Hamiltonian with respect to the Poisson structure P on $\mathcal{L}_n$ which, in the operator formalism, is defined by

$$P_L(\alpha) = [\alpha, L]_+ , \tag{5.9}$$

where α is the pseudodifferential operator,

$$\alpha = \partial^{-1}\alpha_0 + \partial^{-2}\alpha_1 + \cdots + \partial^{-n}\alpha_{n-1} ,$$

which is considered as a 1-form on $\mathcal{L}_n$. The value of α on any tangent vector $U = U_{n-1}\partial^{n-1} + \cdots + U_1\partial + U_0$ is, by definition,

$$\langle \alpha, U \rangle = \int \mathrm{res}_{\partial^{-1}}(\alpha \circ U) = \int (\alpha_0 U_0 + \cdots + \alpha_{n-1}U_{n-1})dx .$$

In equation (5.9) the symbol $[L, \alpha]_+$ means that we consider the differential part of the pseudodifferential operator obtained by computing the commutator of the operators L and α by the usual (formal) rules of the algebra of pseudodifferential operators. See, *e.g.*, Refs. 36, 37.

The Lax-Nijenhuis equation (5.1) then takes the form

$$\mathcal{L}_{Q\alpha}(L) = \frac{1}{2}(L[\alpha, L]_+ + [\alpha\ , L]_+ L) + [L, \widehat{L}(\alpha)] .$$

Since L is a monic differential operator of order n and

$$\mathcal{L}_{Q\alpha}(L) = \mathcal{L}_{Q\alpha}(u_0) + \mathcal{L}_{Q\alpha}(u_1)\partial + \cdots + \mathcal{L}_{Q\alpha}(u_{n-1})\partial^{n-1},$$

we can solve this equation by looking for operators

$$\widehat{L}(\alpha) = \lambda_0 + \lambda_1\partial + \cdots + \lambda_{n-1}\partial^{n-1} .$$

(The reasons for this choice and that made in the case of the Toda system are similar.) Then we observe that the Lax-Nijenhuis equation can also be written in the form

$$\mathcal{L}_{Q\alpha}(L) = [\alpha, L]_+ L + [L, M(\alpha)] , \tag{5.10}$$

if we set

$$M(\alpha) = \widehat{L}(\alpha) + \frac{1}{2}[\alpha, L]_+ \ .$$

To split equation (5.10) in two parts, one determining $M(\alpha)$ and the other determining $\mathcal{L}_{Q\alpha}(L)$, we observe that the constraints on L imply that

$$(\mathcal{L}_{Q\alpha}(L)L^{-1})_+ = 0 \ .$$

Then we get

$$(\mathcal{L}_{Q\alpha}(L) \cdot L^{-1})_+ = ([\alpha, L]_+ + [L, M(\alpha)L^{-1}])_+ = 0$$

$$(\mathcal{L}_{Q\alpha}(L) \cdot L^{-1})_- = ([\alpha, L]_+ + [L, M(\alpha)L^{-1}])_-$$

or

(5.11) $$[L, (M(\alpha) - \alpha L)L^{-1}]_+ = 0$$

(5.12) $$\mathcal{L}_{Q\alpha}(L) = [L \ , \ M(\alpha)L^{-1}]_- \cdot L \ .$$

Now $[L, (\alpha L)_- L^{-1}]_+ = 0$, since

$$(L(\alpha L)_- L^{-1} - (\alpha L)_-)_+ = (L(\alpha L)_- L^{-1})_+ = 0 \ .$$

In fact, we know that for any strictly pseudodifferential operator X, such that $X_+ = 0$,

$$(LXL^{-1})_+ = 0 \ .$$

The constraint equation (5.11) can therefore be written in the form

$$[L, (M(\alpha) - (\alpha L)_+)L^{-1}]_+ = 0 \ ,$$

and the simplest solution of (5.11) is thus

$$M(\alpha) = (\alpha L)_+ \ .$$

If we now insert this solution into equation (5.12), we get

$$\mathcal{L}_{Q\alpha}(L) = [L, (\alpha L)_+ L^{-1}]_- \cdot L \ ,$$

or

$$\begin{aligned} Q_L(\alpha) &= (L(\alpha L)_+ L^{-1})_- L \\ &= L(\alpha L)_+ L^{-1} L - (L(\alpha L)_+ L^{-1})_+ L \\ &= L(\alpha L)_+ - (L\alpha L L^{-1} - L(\alpha L)_- L^{-1})_+ L \\ &= L(\alpha L)_+ - (L\alpha)_+ L \ . \end{aligned}$$

This is the second Adler-Gelfand-Dickey bracket[21,16,10,37].

5.5 The R-matrix bracket

It is well-known that the Poisson structure (5.9) on $\mathcal{L}_n$ is a particular case of the Poisson structure P defined by

$$P_L(\alpha) = R([L,\alpha]) - [L, R\alpha] \tag{5.13}$$

associated with any skewsymmetric R-matrix satisfying the modified classical Yang-Baxter equation,

$$[RX, RY] - R([RX, Y] + [X, RY]) = -[X, Y] \ .$$

Indeed to obtain (5.9) from (5.13) it is enough to choose as an R-matrix on the algebra of formal pseudodifferential operators half the difference,

$$R = \frac{1}{2}(\pi_+ - \pi_-),$$

between the projections π_+ and π_- onto the positive and negative parts into which the algebra of formal pseudodifferential operators naturally splits. In fact

$$R([L,\alpha]) - [L, R\alpha] = \frac{1}{2}[L,\alpha]_+ - \frac{1}{2}[L,\alpha]_- + \frac{1}{2}[L,\alpha] = [L,\alpha]_+ \ .$$

Therefore, it is natural to try to generalize the previous example by solving the Lax-Nijenhuis equation corresponding to

$$\mathcal{L}_{P\alpha}(L) = R([L,\alpha]) - [L, R(\alpha)] \ .$$

To this end we remark that

$$\mathcal{L}_{P\alpha}(L^2) = LR([L,\alpha]) + R([L,\alpha])L - [L, LR(\alpha) + R(\alpha)L]$$

so that the Lax-Nijenhuis condition takes the form

$$\mathcal{L}_{Q\alpha}(L) = \frac{1}{2}(LR([L,\alpha]) + R([L,\alpha])L) + [L, \widehat{L}(\alpha) - \frac{1}{2}(LR(\alpha) + R(\alpha)L)] \ .$$

In this case we have no obvious supplementary conditions on L to be used to determine $\widehat{L}(\alpha)$. However to do this we can use the skewsymmetry of Q (as in the Toda example). The idea is to split the linear operator

$$M_L(\alpha) = LR([L,\alpha]) + R([L,\alpha])L$$

into its symmetric and skewsymmetric parts. Since

$${}^tM_L(\alpha) = [L, R(\alpha L + L\alpha)] \ ,$$

we can write

$$\mathcal{L}_{Q\alpha}(L) = \frac{1}{2}(M_L(\alpha) - {}^tM_L(\alpha)) + [L, \widehat{L}(\alpha) + \frac{1}{2}(R(\alpha L + L\alpha)) - LR(\alpha) - R(\alpha)L] \ .$$

Now we can choose

$$\widehat{L}(\alpha) = \frac{1}{2}(LR(\alpha) + R(\alpha)L) - \frac{1}{2}R(\alpha L + L\alpha)$$

so as to annihilate the commutator and to get $\mathcal{L}_{Q\alpha}(L) = \frac{1}{2}(M_L(\alpha) - {}^tM_L(\alpha))$, a manifestly skewsymmetric mapping. The explicit result that we finally get is

$$\mathcal{L}_{Q\alpha}(L) = \frac{1}{2}(LR([L,\alpha]) + R([L,\alpha])L - [L, R(\alpha L + L\alpha)]),$$

and thus

$$Q_L(\alpha) = R(L\alpha)L - LR(\alpha L) \ .$$

This is the well-known form[23] of the second (quadratic) Poisson bivector associated with the R-bracket (5.13).

Conclusion. These examples may help to explain the role of the Lax-Nijenhuis equation and its limits. This equation does not define the second ("quadratic") Poisson bracket, Q, associated with a Lax operator, but it provides a systematic way of deriving this bracket. The previous examples show that, in many cases, the form of L and the form of the first, given Poisson tensor suggest natural choices for the form of $\widehat{L}(\alpha)$ which make Q uniquely defined. This is the value of the method. Its limits are that it does not provide a proof of the fact that we indeed obtain a second Poisson tensor compatible with the given one.

In Ref. 38, there appears a Lax formulation for the evolution of the recursion operator of the KdV hierarchy, whose geometric interpretation along the lines of the present exposition remains to be clarified.

References

[1] P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. **21**, 467-490 (1968).

[2] F. Magri, *A simple model of the integrable Hamiltonian equation*, J. Math. Phys. **19**, 1156-1162 (1978).

[3] I. M. Gelfand and I. Ya. Dorfman, *Hamiltonian operators and algebraic structures related to them*, Funct. Anal. Appl. **13**, 248-262 (1979), *The Schouten bracket and Hamiltonian operators*, Funct. Anal. Appl. **14**, 223-226 (1980).

[4] A. G. Reiman and M. A. Semenov-Tian-Shansky, *A family of Hamiltonian structures, hierarchy of Hamiltonians, and reduction for first-order matrix differential operators*, Funct. Anal. Appl. **14**, 146-148 (1980).

[5] H. P. McKean, *Compatible brackets in Hamiltonian mechanics* in *Important Developments in Soliton Theory*, A. S. Fokas and V. E. Zakharov eds. (Springer-Verlag, 1993), pp. 344-354.

[6] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer-Verlag, 1993), 2nd edition.

[7] Y. Kosmann-Schwarzbach, *Géométrie des sytèmes bihamiltoniens* in *Systèmes dynamiques non linéaires : intégrabilité et comportement qualitatif*, P. Winternitz ed., Séminaire de Mathématiques Supérieures **102** (Presses de l'Université de Montréal, 1986), pp. 185-216.

[8] I. Ya. Dorfman, *Dirac Structures and Integrability of Nonlinear Evolution Equations* (Wiley and Sons, 1993).

[9] I. M. Gelfand and L. A. Dickey, *Asymptotic behaviour of the resolvent of Sturm-Liouville equations and the algebra of the Korteweg-de Vries equation*, Russian Math. Surveys **30**, 77-113 (1975), *A Lie algebra structure in the formal variational calculus*, Funct. Anal. Appl. **10**, 16-22 (1976).

[10] L. A. Dickey, *Soliton Equations and Hamiltonian Systems* (World Scientific, 1991).

[11] P. D. Lax, *Almost periodic solutions of the KdV equation*, S.I.A.M. Review **18**, 351-375 (1976).

[12] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, *Korteweg-de Vries equation and generalizations. VI Methods for exact solution*, Comm. Pure Appl. Math. **27**, 97-133 (1974).

[13] C. S. Gardner, *Korteweg-de Vries equation and generalizations. IV The Korteweg-de Vries equation as a Hamiltonian system*, J. Math. Phys. **12**, 1548-1551 (1971).

[14] A. S. Fokas and B. Fuchssteiner, *On the structure of symplectic operators and hereditary symmetries*, Lett. Nuovo Cimento **28**(8), 299-303 (1980).

[15] H. Flaschka, *The Toda lattice. I Existence of integrals*, Phys. Rev. B **9**, 1924-1925 (1974).

[16] M. Adler, *On a trace functional for formal pseudo-differential operators and the symplectic structure of the Korteweg-de Vries type equations*, Invent. Math. **50**, 403-432 (1979).

[17] B. Kupershmidt, *Discrete Lax Equations and Differential Difference Calculus*, Astérisque **123** (1985).

[18] W. Oevel and O. Ragnisco, *R-matrices and higher-order Poisson brackets for integrable systems*, Physica A **161**, 181-220 (1989).

[19] A. Das and S. Okubo, *A systematic study of the Toda lattice*, Ann. Phys. **190**, 215-232 (1989).

[20] P. A. Damianou, *Master symmetries and R-matrices for the Toda lattice*, Lett. Math. Phys. **20**, 101-112 (1990).

[21] I. M. Gelfand and L. A. Dickey, *A family of Hamiltonian structures connected with integrable nonlinear differential equations*, preprint **136** (Inst. Appl. Math. Moscow, 1978), published in English translation in *Collected Papers of I. M. Gelfand*, vol. 1 (Springer-Verlag, 1987), pp. 625-646.

[22] B. A. Kupershmidt and G. Wilson, *Modifying Lax equations and the second Hamiltonian structure*, Invent. Math. **62**, 403-436 (1981).

[23] M. A. Semenov-Tian-Shansky, *What is a classical r-matrix?*, Funct. Anal. Appl. **17**, 259-272 (1983).

[24] F. Magri, *A geometrical approach to the nonlinear solvable equations*, Lect. Notes Phys. **120** (Springer-Verlag, 1980), pp. 233-263.

[25] B. Fuchssteiner, *The Lie algebra structure of nonlinear evolution equations admitting infinite-dimensional Abelian symmetry groups*, Prog. Theor. Physics, **65** (3), 861-876 (1981).

[26] F. Magri, C. Morosi and O. Ragnisco, *Reduction techniques for infinite- dimensional Hamiltonian systems: some ideas and applications*, Comm. Math. Phys. **99**, 115-140 (1985).

[27] Y. Kosmann-Schwarzbach and F. Magri, *Poisson-Nijenhuis structures*, Ann. Inst. Henri Poincaré, A (Phys. Théor.) **53**, 35-81 (1990).

[28] A. Nijenhuis, *X_{n-1}-forming sets of eigenvectors*, Proc. Kon. Ned. Akad. Wet. Amsterdam **54**, 200-212 (1951).

[29] Yu. I. Manin, *Algebraic aspects of nonlinear differential equations*, J. Soviet Math. **11**, 1-122 (1979), in Russian, Itogi **11** , 5-152 (1978). See IV 5.3 in particular.

[30] A. G. Reiman and M. A. Semenov-Tian-Shansky, *Group-theoretical methods in the theory of finite-dimensional integrable systems* in *Dynamical Systems VII*, V. I. Arnold and S. P. Novikov, eds., Encycl. Math. Sci. **16** (Springer-Verlag, 1994).

[31] C. Morosi and L. Pizzocchero, *On the bihamiltonian interpretation of the Lax formalism*, Rep. Math. Phys. **7**(3), 389-430 (1995).

[32] V. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, 1978).

[33] I. S. Krasilshchik, V. V. Lychagin and A. M. Vinogradov, *Geometry of Jet Spaces and Nonlinear Partial Differential Equations* (Gordon and Breach, 1986).

[34] C. Morosi and L. Pizzocchero, *On the Euler equation: bi-Hamiltonian structure and integrals in involution*, Lett. Math. Phys. **37**, 117-135 (1996).

[35] G. Magnano and F. Magri, *Poisson-Nijenhuis structures and Sato hierarchy*, Rev. Math. Phys. **3**, 403-466 (1991).

[36] A. Radul, *Lie algebras of differential operators, their central extensions and W-algebras*, Funct. Anal. Appl. **25** (1), 25-39 (1991).

[37] B. Enriquez, S. Khoroshkin, A. Radul, A. Rosly and V. Rubtsov, *Poisson-Lie group aspects of the classical W-algebras*, Adv. in Math. Sci., Amer. Math. Soc. Transl., ser. 2, **167**, V. Lychagin, ed. (1995).

[38] B. Enriquez, A. Orlov and V. Rubtsov, *Higher Hamiltonian structures on the KdV phase space*, JETP Letters **58** (8), 677-683 (1993).

V. Examples

In this section, we give four examples showing how the Lax-Nijenhuis equation can be used to compute higher-order Hamiltonian structures associated with a Hamiltonian system admitting a Lax representation. We write the Lax-Nijenhuis equation (3.6) in the form

$$\mathcal{L}_{Q\alpha}L = \frac{1}{2}(L\mathcal{L}_{P\alpha}L + (\mathcal{L}_{P\alpha}L)L) + [L, \widehat{L}(\alpha)], \tag{5.1}$$

and we use the information on L and the arbitrariness of α to split this equation into two parts: the first determines the unknown map $\widehat{L}$, up to some still arbitrary constants, the second part determines $\mathcal{L}_{Q\alpha}L$ and then Q. The condition of skewsymmetry on Q then determines the constants. The discussion is quite similar to a problem with constraints coming from the restrictions imposed on L, where the role of the Lagrange multipliers is played by the mapping $\widehat{L}$.

5.1 The Toda system

We shall consider the Toda system and its well-known Lax formulation. Let $M = \mathbb{R}^{2n+1}$ with coordinates $x^I = (a_j, b_\ell)$, $j = 1, \cdots, n, \ell = 1, \cdots, n+1$, $I = 1, \cdots, 2n+1$. We consider the Poisson bivector P_0 defined by

$$\begin{aligned} \{a_j, b_j\} &= -a_j \ , \ j = 1, 2, \cdots, n \\ \{a_j, b_{j+1}\} &= a_j \ , \ j = 1, 2, \cdots, n \ , \end{aligned} \tag{5.2}$$

all other Poisson brackets being equal to 0. Thus the Poisson bivector P_0 has matrix

$$(P_0^{IJ}) = \begin{pmatrix} 0 & \begin{matrix} a_1 & -a_1 & & & & 0 \\ & a_2 & -a_2 & & & \\ & & \ddots & \ddots & & \\ & 0 & & \ddots & \ddots & \\ & & & & a_n & -a_n \end{matrix} \\ \begin{matrix} -a_1 & & & & \\ a_1 & -a_2 & & 0 & \\ & a_2 & \ddots & & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & -a_n \\ & & & & a_n \end{matrix} & 0 \end{pmatrix}$$

When $H = \sum_{j=1}^{n} a_j^2 + \frac{1}{2}\sum_{\ell=1}^{n+1} b_\ell^2$, the Hamiltonian vector field $X = P_0 dH$ is the Toda vector field. In fact, the evolution equation $\frac{dx}{dt} = X(x)$ is the system

$$\begin{cases} \dfrac{da_j}{dt} = \{a_j, H\} \ , \ j = 1, 2, \cdots, n \ , \\ \dfrac{db_\ell}{dt} = \{b_\ell, H\} \ , \ \ell = 1, 2, \cdots, n+1 \ , \end{cases}$$

and, setting $a_0 = a_{n+1} = 0$, this system becomes

$$\begin{cases} \dfrac{da_j}{dt} = a_j(b_{j+1} - b_j) \ , \ j = 1, 2, \cdots, n \ , \\ \dfrac{db_\ell}{dt} = 2(a_\ell^2 - a_{\ell-1}^2), \ell = 1, 2, \cdots, n+1 \ , \end{cases} \tag{5.3}$$

which are the equations of the Toda system in Flaschka coordinates[15]. A Lax formulation for the Toda system is

$$\frac{dL}{dt} = [L, A] \ ,$$

where
(5.4)

$$L = \begin{pmatrix} b_1 & a_1 & & & & \\ a_1 & b_2 & a_2 & & 0 & \\ & a_2 & b_3 & a_3 & & \\ & & a_3 & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & 0 & & & \ddots & b_n & a_n \\ & & & & & a_n & b_{n+1} \end{pmatrix}, A = \begin{pmatrix} 0 & a_1 & & & & \\ a_1 & 0 & a_2 & & 0 & \\ & a_2 & 0 & a_3 & & \\ & & a_3 & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & 0 & & & \ddots & 0 & a_n \\ & & & & & a_n & 0 \end{pmatrix}$$

Now let P and Q be Poisson structures such that (5.1) is satisfied for L symmetric and tridiagonal, as above. We assume that P is known, and we consider Q to be an unknown, higher-order, Poisson structure. We shall assume that $\widehat{L}(\alpha)$ depends linearly on α, and we shall denote the matrix $\widehat{L}(dx^I)$ of order $n+1$ by C^I. Then (5.1) becomes

$$L(P^{IJ}\frac{\partial L}{\partial x^J} + 2C^I) + (P^{IJ}\frac{\partial L}{\partial x^J} - 2C^I)L = 2Q^{IJ}\frac{\partial L}{\partial x^J} \ , \tag{5.5}$$

where the summation over $J = 1, 2, \cdots, 2n+1$ is understood.

We shall assume that C^I is a skewsymmetric tridiagonal matrix, for each $I = 1, \cdots, 2n+1$,

$$C^I = \begin{pmatrix} 0 & c^{I1} & & & & & \\ -c^{I1} & 0 & c^{I2} & & & 0 & \\ & -c^{I2} & 0 & c^{I3} & & & \\ & & -c^{I3} & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & 0 & & & \ddots & 0 & c^{In} \\ & & & & & -c^{In} & 0 \end{pmatrix} \tag{5.6}$$

where $c_1^I, \cdots, c_n^I$ are to be determined. This choice is the simplest possible and it guarantees that $[L, C^I]$ is a symmetric penta-diagonal matrix. Any other choice of C^I would yield a matrix $[L, C^I]$ with more entries, and so with more constraints to be imposed.

For each $I = 1, \cdots, 2n+1$, $P^{IJ} \frac{\partial L}{\partial x^J}$ is the symmetric tridiagonal matrix

$$\begin{pmatrix} P^{I,n+1} & P^{I1} & & & 0 \\ P^{I1} & P^{I,n+2} & P^{I2} & & \\ & P^{I2} & \ddots & \ddots & \\ & & \ddots & P^{I,2n} & P^{In} \\ 0 & & & P^{In} & P^{I,2n+1} \end{pmatrix},$$

and therefore $(P^{IJ} \frac{\partial L}{\partial x^J} - 2C^I)L$ is the transpose of $L(P^{IJ} \frac{\partial L}{\partial x^J} + 2C^I)$. Thus condition (5.5) implies that, for fixed I, matrix C^I is such that the symmetric part of $L(P^{IJ} \frac{\partial L}{\partial x^J} + 2C^I)$ is tridiagonal. Writing this condition explicitly, one obtains

$$\sum_{k=1}^{n-1} (a_k (P^{J,k+1} + 2c^{J,k+1}) + a_{k+1}(P^{Jk} - 2c^{Jk})) = 0 \ .$$

Solving this system for c^{Jk} in terms of P^{Jk} yields the existence of multipliers $\lambda_k^J, k = 1, 2, \cdots, n$, such that

$$2c^{Jk} = P^{Jk} + 2\lambda_k^J a_k \ .$$

We assume that P^{Jk} is divisible by a_k. This assumption is satisfied for $P = P_0$ defined by (5.2). Setting $\lambda_1^J = \lambda^J$, we obtain

$$2c^{Jk} = P^{Jk} + 2a_k(\lambda^J - \sum_{j=2}^{k} \frac{P^{Jj}}{a_j}) \ .$$

(By convention, here and below the last sum vanishes if $k < 2$.)

From relations (5.5), we then obtain the coefficients of the higher-order Poisson structure Q,

$$Q^{Jk} = \frac{1}{2} a_k (P^{J,n+k} + P^{J,n+k+1}) + \frac{1}{2} b_k (P^{Jk} + 2c^{Jk}) + \frac{1}{2} b_{k+1} (P^{Jk} - 2c^{Jk}) \ ,$$

$$Q^{J,n+\ell} = a_\ell (P^{J\ell} - 2c^{J\ell}) + a_{\ell-1}(P^{J,\ell-1} + 2c^{J,\ell-1}) + b_\ell P^{J,n+\ell} \ .$$

We now replace the c^{Jk}'s by their values in terms of the parameters λ^J, and we impose the conditions that the diagonal terms of Q vanish. These $2n+1$ conditions imply

$$2(b_{k+1} - b_k)(\lambda^k - \sum_{j=2}^{k-1} \frac{P^{kj}}{a_j}) = P^{k,n+k} + P^{k,n+k+1},$$

$$a_1^2\lambda^{n+1} = 0 \ ,$$

$$(a_\ell^2 - a_{\ell-1}^2)\left(\lambda^{n+\ell} - \sum_{j=2}^{\ell-1}\frac{P^{n+\ell,j}}{a_j}\right) = a_{\ell-1}P^{n+\ell,\ell-1} + a_\ell P^{n+\ell,\ell},$$

for $\ell = 2, \cdots, n+1$.

Thus, we have obtained

$$Q^{Jk} = \frac{1}{2}a_k\left(P^{J,n+k} + P^{J,n+k+1}\right) + b_k P^{Jk} + a_k(b_k - b_{k+1})\left(\lambda^J - \sum_{j=2}^{k-1}\frac{P^{Jj}}{a_j}\right) \ ,$$

$$Q^{J,n+\ell} = 2\left(a_{\ell-1}P^{J,\ell-1} + a_\ell P^{J\ell}\right) + b_\ell P^{J,n+\ell} + 2(a_{\ell-1}^2 - a_\ell^2)\left(\lambda^J - \sum_{j=2}^{\ell-1}\frac{P^{Jj}}{a_j}\right) \ ,$$

where the λ^J's are given above in terms of a_k, b_ℓ, P^{IJ}.

Let us assume that all a_k's are nonvanishing and let us introduce the matrix M of order $2n+1$, depending on $a_1, \cdots, a_n, b_1, \cdots, b_{n+1}$, such that M applied to the column with entries $A_1, \cdots, A_n, B_1, \cdots, B_{n+1}$, is the column with entries

$$\text{(5.7)} \qquad \begin{cases} \overline{A}_k = \dfrac{1}{2}a_k(B_k + B_{k+1}) + b_k A_k + a_k(b_{k+1} - b_k)\displaystyle\sum_{j=2}^{k}\frac{A_j}{a_j}, \\ \overline{B}_1 = b_1 B_1, \ \overline{B}_2 = b_2 B_2 + 2(a_1A_1 + a_2A_2), \\ \overline{B}_\ell = b_\ell B_\ell + 2(a_{\ell-1}A_{\ell-1} + a_\ell A_\ell) + 2(a_\ell^2 - a_{\ell-1}^2)\displaystyle\sum_{j=2}^{\ell-1}\frac{A_j}{a_j} \ , \end{cases}$$

for $\ell = 3, \cdots, n+1$.

We see that

$$Q = MP + X \otimes \lambda \ ,$$

where X is the Toda vector field with components

$$a_1(b_2 - b_1), a_2(b_3 - b_2), \cdots, a_n(b_{n+1} - b_n), 2a_1^2, 2(a_2^2 - a_1^2), \cdots, -2a_n^2 \ ,$$

and λ is the vector with components $\lambda^1, \cdots, \lambda^n, \lambda^{n+1}, \cdots, \lambda^{2n+1}$. We observe that, although Q is skewsymmetric, this expression does not constitute a decomposition of Q into a sum of skewsymmetric 2-tensors.

Thus the Lax-Nijenhuis equation yields an explicit determination of the bivector Q in terms of P. For $P = P_0$, we see that the corresponding vector $\lambda = \lambda_{(0)}$ is the row-matrix with entries

$$\text{(5.8)} \qquad \lambda_{(0)}^J = -\frac{1}{2}\delta_{n+2}^J \ .$$

It is easy to check that

$$P_1 = MP_0 + X \otimes \lambda_{(0)}$$

coincides with the second Poisson structure of the Toda system[17,19,20]. For example, if $n = 3$, the matrix MP_0 is equal to

$$\begin{pmatrix} b_1 & 0 & 0 & \frac{a_1}{2} & \frac{a_1}{2} & 0 & 0 \\ 0 & b_3 & 0 & 0 & \frac{a_2}{2} & \frac{a_2}{2} & 0 \\ 0 & U & b_4 & 0 & 0 & \frac{a_3}{2} & \frac{a_3}{2} \\ 0 & 0 & 0 & b_1 & 0 & 0 & 0 \\ 2a_1 & 2a_2 & 0 & 0 & b_2 & 0 & 0 \\ 0 & 2\frac{a_3^2}{a_2} & 2a_3 & 0 & 0 & b_3 & 0 \\ 0 & -2\frac{a_3^2}{a_2} & 0 & 0 & 0 & 0 & b_4 \end{pmatrix} \begin{pmatrix} & & & a_1 & -a_1 & 0 & 0 \\ & 0 & & 0 & a_2 & -a_2 & 0 \\ & & & 0 & 0 & a_3 & -a_3 \\ -a_1 & 0 & 0 & & & & \\ a_1 & -a_2 & 0 & & & & \\ 0 & a_2 & -a_3 & & & 0 & \\ 0 & 0 & a_3 & & & & \end{pmatrix}$$

where $U = \frac{a_3}{a_2}(b_4 - b_3)$, and

$$X \otimes \lambda_{(0)} = \begin{pmatrix} a_1(b_2 - b_1) \\ a_2(b_3 - b_2) \\ a_3(b_4 - b_3) \\ 2a_1^2 \\ 2(a_2^2 - a_1^2) \\ 2(a_3^2 - a_2^2) \\ -2a_3^2 \end{pmatrix} \otimes {}^t \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix} ,$$

so that

$$P_1 = MP_0 + X \otimes \lambda_{(0)} = \begin{pmatrix} 0 & -\frac{1}{2}a_1a_2 & 0 & a_1b_1 & -a_1b_2 & 0 & 0 \\ \frac{1}{2}a_1a_2 & 0 & -\frac{1}{2}a_2a_3 & 0 & a_2b_2 & -a_2b_3 & 0 \\ 0 & \frac{1}{2}a_2a_3 & 0 & 0 & 0 & a_3b_3 & -a_3b_4 \\ -a_1b_1 & 0 & 0 & 0 & -2a_1^2 & 0 & 0 \\ a_1b_2 & -a_2b_2 & 0 & 2a_1^2 & 0 & -2a_2^2 & 0 \\ 0 & a_2b_3 & -a_3b_3 & 0 & 2a_2^2 & 0 & -2a_3^2 \\ 0 & 0 & a_3b_4 & 0 & 0 & 2a_3^2 & 0 \end{pmatrix} .$$

Repeating the process, we have to compute the bivector

$$P_2 = MP_1 + X \otimes \lambda_{(1)}$$

where $\lambda_{(1)}$ is the vector corresponding to P_1. We can show that

$$\lambda_{(1)} = M\lambda_{(0)} .$$

For example, if $n = 3$,

$$\lambda_{(1)} = (-\frac{1}{4}a_1, -\frac{1}{4}a_2, 0, 0, -\frac{1}{2}b_2, 0, 0)$$

and P_2 is the skewsymmetric matrix

$$P_2 = MP_1 + X \otimes \lambda_{(1)} =$$

$$\begin{pmatrix} 0 & -a_1a_2b_2 & 0 & a_1b_1^2 + a_1^3 & -a_1b_2^2 - a_1^3 & -a_1a_2^2 & 0 \\ & 0 & -a_2a_3b_3 & a_1^2a_2 & a_2b_2^2 + a_2^3 & -a_2b_3^2 - a_2^3 & -a_2a_3^2 \\ & & 0 & 0 & a_2^2a_3 & a_3b_3 + a_3^2 & -a_3b_4^2 - a_3^2 \\ & & & 0 & -2a_1^2(b_1 + b_2) & 0 & 0 \\ & & & & 0 & -2a_2^2(b_2 + b_3) & 0 \\ & & & & & 0 & -2a_3^2(b_3 + b_4) \\ & & & & & & 0 \end{pmatrix}$$

In this case, P_2 coincides with the opposite of the third Poisson structure of the Toda system described in Refs. 17 and 20.

Thus if M is the matrix of order $2n+1$ defined by (5.7), the bivectors P_i obtained from the Lax-Nijenhuis equation satisfy

$$P_{i+1} = MP_i + X \otimes \lambda_{(i)} \ ,$$

where X is the Toda vector field and $\lambda_{(i)}$ is the vector corresponding to P_i.

In particular, it follows that each coefficient P_i^{Jk} of the bivector P_i is divisible by a_k, so the iteration can be carried out.

Let us show that in fact the skewsymmetry of P_i, P_{i+1} implies that

$$\lambda_{(i+1)} = M\lambda_{(i)} \ .$$

Thus we consider

$$P_i = MP_{i-1} + X \otimes \lambda_{(i-1)} \ ,$$

$$P_{i+1} = MP_i + X \otimes \lambda_{(i)} \ .$$

Since P_{i+1} is assumed to be skewsymmetric, $\lambda_{(i)}$ must satisfy

$$P_i\,{}^tM - MP_i = X \otimes \lambda_{(i)} + \lambda_{(i)} \otimes X \ .$$

Moreover, from the skewsymmetry of P_i , we obtain

$$P_i = MP_{i-1} + X \otimes \lambda_{(i-1)} = P_{i-1}{}^tM - \lambda_{(i-1)} \otimes X \ ,$$

whence

$$P_i\,{}^tM - MP_i = X \otimes M\lambda_{(i-1)} + M\lambda_{(i-1)} \otimes X \ .$$

We thus obtain $\lambda_{(i)} = M\lambda_{(i-1)}$. The following proposition summarizes this discussion.

Proposition 5.1.— *Let M be the matrix of order $2n+1$ defined by (5.7). The bivectors obtained from the Lax-Nijenhuis equation satisfy*

$$P_{i+1} = MP_i + X \otimes M^i \lambda_{(0)} \ ,$$

where $\lambda_{(0)}$ is the vector given by (5.8).

5.2 The n-dimensional rigid body

The Euler equations for the n-dimensional rigid body rotating about a fixed point can be written

$$\dot{M} = [M, \Omega] \ ,$$

where M is the angular momentum, a time-dependent element of the Lie algebra $so(n)$, and

$$M = J\Omega + \Omega J \ .$$

Here J is a diagonal matrix with positive entries $J_1, J_2, \cdots, J_n$ defined in terms of the principal moments of inertia, and Ω is the angular velocity. These equations admit a Lax formulation, with a spectral parameter,

$$\dot{L} = [L, B] \ ,$$

where $L = \frac{M}{\lambda} + J^2$ and $B = \Omega + \lambda J$.

Moreover, it is well-known (see *e.g.*, Refs. 10, 34) that these equations can be written in Hamiltonian form, with respect to the linear Poisson structure of $so(n)$ induced by the identification of the Lie algebra $so(n)$ with its dual by means of the trace of the product of matrices. Making use of this identification, this Poisson structure P is defined by $P_M : so(n) \to so(n)$, for each M in $so(n)$, where

$$P_M = ad_M \ .$$

If $K(M) = \frac{1}{2} tr(M\Omega)$, then the gradient of K (the differential of K identified with a matrix in $so(n)$) is the constant matrix Ω, and therefore the Euler equations can be written as the Hamiltonian equation

$$\dot{M} = P(dK).$$

Let us use the Lax-Nijenhuis equation in order to find a possible form of a second Hamiltonian structure that will make this equation a bihamiltonian system. Setting $\dot{L} = \frac{dL}{dt_1}$, it follows from the definitions and from the Euler equation that

$$\frac{d(L^2)}{dt_1} = \frac{1}{\lambda}(L[M,\Omega] + [M,\Omega]L) \ ,$$

and therefore, by a simple computation,

$$\frac{1}{2}\frac{d(L^2)}{dt_1} = \frac{1}{2\lambda}[L, M\Omega + \Omega M] + \frac{1}{\lambda}(M\Omega L - L\Omega M) \ .$$

By the definition of L, $M\Omega L - L\Omega M = M\Omega J^2 - J^2\Omega M$. This suggests that we should set

$$\frac{dM}{dt_2} = M\Omega J^2 - J^2\Omega M.$$

We observe that if M and Ω are skewsymmetric, so is $\frac{dM}{dt_2}$. In fact, Q defined by

$$Q_M(V) = MVJ^2 - J^2VM,$$

for $M, V \in so(n)$, is the second, compatible Poisson structure on $so(n)$ that was recently found by Morosi and Pizzocchero[34]. This second Poisson structure Q is actually a deformation of the first, linear one, P, under the linear map $M \to JMJ$. The Euler equation can be written

$$\dot{M} = Q(dH) \ ,$$

where $H(M) = -\frac{1}{2}tr(J^{-1}MJ^{-1}\Omega)$, since the gradient of H is $-J^{-1}\Omega J^{-1}$.

We now show how to extend this procedure to the determination of Poisson structures compatible with the linear Poisson structures on the sum of several copies of a simple Lie algebra, considered in Refs. 30 and 35. Let us consider, for instance, the Hamiltonian system

$$\frac{dM_0}{dt_1} = [M_0, V_1]$$
$$\frac{dM_1}{dt_1} = [M_0, V_0] + [M_1, V_1],$$

where V_0, V_1 are the components of the gradient of a Hamiltonian function K . We introduce the Lax matrix depending on the spectral parameter λ,

$$L = \frac{M_0}{\lambda^2} + \frac{M_1}{\lambda} + A.$$

Computing the derivative of the square of this matrix, we find that

$$\frac{1}{2}\frac{d(L^2)}{dt_1} = [L, \frac{(M_0V_1 + V_1M_0)}{2\lambda^2} + \frac{(M_0V_0 + V_0M_0) + (M_1V_1 + V_1M_1)}{2\lambda}]$$

$$+\frac{1}{\lambda^2}(M_0V_0M_1 - M_1V_0M_0 + M_0V_1A - AV_1M_0) + \frac{1}{\lambda}(M_0V_0A - AV_0M_0 + M_1V_1A - AV_1M_1).$$

Therefore the Lax-Nijenhuis equation suggests that we should set

$$\frac{dM_0}{dt_2} = (M_0V_0M_1 - M_1V_0M_0) + (M_0V_1A - AV_1M_0)$$
$$\frac{dM_1}{dt_2} = (M_0V_0A - AV_0M_0) + (M_1V_1A - AV_1M_1).$$

A computation shows that this is actually a Poisson structure on the direct sum of two copies of $so(n)$. It is clearly compatible with the first, because it can be otained by deforming A into $A + \lambda I$.

5.3 The KdV equation

We now enter the field of nonlinear partial differential equations by considering the Korteweg-de Vries equation

$$\frac{du}{dt} = u_{xxx} - 6uu_x \ .$$

We use the notations of the formal calculus of variations[9,10,8]. As is well-known, the KdV equation is Hamiltonian since if can be written in the form

$$\frac{\partial u}{\partial t} = \partial \frac{\delta H}{\delta u} \ ,$$

where

$$H(u) = -\int (\frac{1}{2}u_x^2 + u^3)dx \ ,$$

and $\partial = \frac{d}{dx}$ is the Gardner Hamiltonian structure[13]. It admits a Lax representation, $\frac{du}{dt} = [L, B]$, with

$$L = \partial^2 - u, \quad B = 4\partial^3 - 3(u\partial + \partial u) \ .$$

We now want to use the Lax-Nijenhuis equation (5.1), where $P = \partial$, to find the second Hamiltonian structure of the KdV equation. Since L is a second-order differential operator, we assume that $\widehat{L}(\alpha)$ is a first-order differential operator,

$$\widehat{L}(\alpha) = \lambda + \mu\partial \ ,$$

where λ and μ depend linearly on the 1-form α. A simple computation yields:

$$\frac{1}{2}\mathcal{L}_{P\alpha}(L^2) = -\alpha_x\partial^2 - \alpha_{xx}\partial + (u\alpha_x - \frac{1}{2}\alpha_{xxx}) \ ,$$

$$[L, \widehat{L}(\alpha)] = 2\mu_x\partial^2 + (2\lambda_x + \mu_{xx})\partial + (\lambda_{xx} + \mu u_x) \ .$$

By inserting these formulas into the Lax-Nijenhuis equation and by equating the coefficients of ∂^2, ∂ and ∂^0, we get:

$$\begin{aligned}
&(2\mu - \alpha)_x = 0 \ , \\
&(2\lambda + \mu_x - \alpha_x)_x = 0 \ , \\
&-\mathcal{L}_{Q\alpha}(u) = u\alpha_x - \frac{1}{2}\alpha_{xxx} + \lambda_{xx} + \mu u_x \ .
\end{aligned}$$

The first two equations yield the solution $\mu = \frac{1}{2}\alpha, \ \lambda = \frac{1}{4}\alpha_x$, while the third one yields the second Hamiltonian structure of the KdV equation,

$$Q_u(\alpha) = \frac{1}{4}\alpha_{xxx} - u\alpha_x - \frac{1}{2}u_x\alpha \ .$$

The recursion operator is the nonlocal operator, $R_u = \frac{1}{4}\partial^2 - u - \frac{1}{2}u_x\partial^{-1}$.

5.4 The second Adler-Gelfand-Dickey bracket

We now generalize the previous example to the first equation of the n-th KdV hierarchy. The unknowns are functions $u_0, u_1, \ldots, u_{n-1}$ on the circle, whose time evolution is being studied. It admits a Lax formulation,

$$\frac{dL}{dt} = [L, B] \ ,$$

where

$$L = \partial^n + u_{n-1}\partial^{n-1} + \cdots + u_0 \ ,$$

and B is a suitable differential operator of order $n+1$. Here L takes values in a manifold $\mathcal{L}_n$ of invertible elements in the algebra A_n of formal pseudodifferential operators of order $\leq n$ on the circle[36]. This equation is Hamiltonian with respect to the Poisson structure P on $\mathcal{L}_n$ which, in the operator formalism, is defined by

$$P_L(\alpha) = [\alpha, L]_+ \ , \tag{5.9}$$

where α is the pseudodifferential operator,

$$\alpha = \partial^{-1}\alpha_0 + \partial^{-2}\alpha_1 + \cdots + \partial^{-n}\alpha_{n-1} \ ,$$

which is considered as a 1-form on $\mathcal{L}_n$. The value of α on any tangent vector $U = U_{n-1}\partial^{n-1} + \cdots + U_1\partial + U_0$ is, by definition,

$$\langle \alpha, U \rangle = \int \mathrm{res}_{\partial^{-1}}(\alpha \circ U) = \int (\alpha_0 U_0 + \cdots + \alpha_{n-1}U_{n-1})dx \ .$$

In equation (5.9) the symbol $[L, \alpha]_+$ means that we consider the differential part of the pseudodifferential operator obtained by computing the commutator of the operators L and α by the usual (formal) rules of the algebra of pseudodifferential operators. See, *e.g.*, Refs. 36, 37.

The Lax-Nijenhuis equation (5.1) then takes the form

$$\mathcal{L}_{Q\alpha}(L) = \frac{1}{2}(L[\alpha, L]_+ + [\alpha \ , L]_+ L) + [L, \widehat{L}(\alpha)] \ .$$

Since L is a monic differential operator of order n and

$$\mathcal{L}_{Q\alpha}(L) = \mathcal{L}_{Q\alpha}(u_0) + \mathcal{L}_{Q\alpha}(u_1)\partial + \cdots + \mathcal{L}_{Q\alpha}(u_{n-1})\partial^{n-1},$$

we can solve this equation by looking for operators

$$\widehat{L}(\alpha) = \lambda_0 + \lambda_1\partial + \cdots + \lambda_{n-1}\partial^{n-1} \ .$$

(The reasons for this choice and that made in the case of the Toda system are similar.) Then we observe that the Lax-Nijenhuis equation can also be written in the form

$$\mathcal{L}_{Q\alpha}(L) = [\alpha, L]_+ L + [L, M(\alpha)] \ , \tag{5.10}$$

if we set

$$M(\alpha) = \widehat{L}(\alpha) + \frac{1}{2}[\alpha, L]_+ \ .$$

To split equation (5.10) in two parts, one determining $M(\alpha)$ and the other determining $\mathcal{L}_{Q\alpha}(L)$, we observe that the constraints on L imply that

$$(\mathcal{L}_{Q\alpha}(L)L^{-1})_+ = 0 \ .$$

Then we get

$$(\mathcal{L}_{Q\alpha}(L) \cdot L^{-1})_+ = ([\alpha, L]_+ + [L, M(\alpha)L^{-1}])_+ = 0$$

$$(\mathcal{L}_{Q\alpha}(L) \cdot L^{-1})_- = ([\alpha, L]_+ + [L, M(\alpha)L^{-1}])_-$$

or

$$[L, (M(\alpha) - \alpha L)L^{-1}]_+ = 0 \tag{5.11}$$

$$\mathcal{L}_{Q\alpha}(L) = [L \ , \ M(\alpha)L^{-1}]_- \cdot L \ . \tag{5.12}$$

Now $[L, (\alpha L)_- L^{-1}]_+ = 0$, since

$$(L(\alpha L)_- L^{-1} - (\alpha L)_-)_+ = (L(\alpha L)_- L^{-1})_+ = 0 \ .$$

In fact, we know that for any strictly pseudodifferential operator X, such that $X_+ = 0$,

$$(LXL^{-1})_+ = 0 \ .$$

The constraint equation (5.11) can therefore be written in the form

$$[L, (M(\alpha) - (\alpha L)_+)L^{-1}]_+ = 0 \ ,$$

and the simplest solution of (5.11) is thus

$$M(\alpha) = (\alpha L)_+ \ .$$

If we now insert this solution into equation (5.12), we get

$$\mathcal{L}_{Q\alpha}(L) = [L, (\alpha L)_+ L^{-1}]_- \cdot L \ ,$$

or

$$\begin{aligned} Q_L(\alpha) &= (L(\alpha L)_+ L^{-1})_- L \\ &= L(\alpha L)_+ L^{-1} L - (L(\alpha L)_+ L^{-1})_+ L \\ &= L(\alpha L)_+ - (L\alpha L L^{-1} - L(\alpha L)_- L^{-1})_+ L \\ &= L(\alpha L)_+ - (L\alpha)_+ L \ . \end{aligned}$$

This is the second Adler-Gelfand-Dickey bracket[21,16,10,37].

5.5 The R-matrix bracket

It is well-known that the Poisson structure (5.9) on $\mathcal{L}_n$ is a particular case of the Poisson structure P defined by

$$P_L(\alpha) = R([L,\alpha]) - [L, R\alpha] \tag{5.13}$$

associated with any skewsymmetric R-matrix satisfying the modified classical Yang-Baxter equation,

$$[RX, RY] - R([RX, Y] + [X, RY]) = -[X, Y] \ .$$

Indeed to obtain (5.9) from (5.13) it is enough to choose as an R-matrix on the algebra of formal pseudodifferential operators half the difference,

$$R = \frac{1}{2}(\pi_+ - \pi_-),$$

between the projections π_+ and π_- onto the positive and negative parts into which the algebra of formal pseudodifferential operators naturally splits. In fact

$$R([L,\alpha]) - [L, R\alpha] = \frac{1}{2}[L,\alpha]_+ - \frac{1}{2}[L,\alpha]_- + \frac{1}{2}[L,\alpha] = [L,\alpha]_+ \ .$$

Therefore, it is natural to try to generalize the previous example by solving the Lax-Nijenhuis equation corresponding to

$$\mathcal{L}_{P\alpha}(L) = R([L,\alpha]) - [L, R(\alpha)] \ .$$

To this end we remark that

$$\mathcal{L}_{P\alpha}(L^2) = LR([L,\alpha]) + R([L,\alpha])L - [L, LR(\alpha) + R(\alpha)L]$$

so that the Lax-Nijenhuis condition takes the form

$$\mathcal{L}_{Q\alpha}(L) = \frac{1}{2}(LR([L,\alpha]) + R([L,\alpha])L) + [L, \widehat{L}(\alpha) - \frac{1}{2}(LR(\alpha) + R(\alpha)L)] \ .$$

In this case we have no obvious supplementary conditions on L to be used to determine $\widehat{L}(\alpha)$. However to do this we can use the skewsymmetry of Q (as in the Toda example). The idea is to split the linear operator

$$M_L(\alpha) = LR([L,\alpha]) + R([L,\alpha])L$$

into its symmetric and skewsymmetric parts. Since

$${}^tM_L(\alpha) = [L, R(\alpha L + L\alpha)] \ ,$$

we can write

$$\mathcal{L}_{Q\alpha}(L) = \frac{1}{2}(M_L(\alpha) - {}^tM_L(\alpha)) + [L, \widehat{L}(\alpha) + \frac{1}{2}(R(\alpha L + L\alpha)) - LR(\alpha) - R(\alpha)L] \ .$$

Now we can choose

$$\widehat{L}(\alpha) = \frac{1}{2}(LR(\alpha) + R(\alpha)L) - \frac{1}{2}R(\alpha L + L\alpha)$$

so as to annihilate the commutator and to get $\mathcal{L}_{Q\alpha}(L) = \frac{1}{2}(M_L(\alpha) - {}^tM_L(\alpha))$, a manifestly skewsymmetric mapping. The explicit result that we finally get is

$$\mathcal{L}_{Q\alpha}(L) = \frac{1}{2}(LR([L,\alpha]) + R([L,\alpha])L - [L, R(\alpha L + L\alpha)]),$$

and thus

$$Q_L(\alpha) = R(L\alpha)L - LR(\alpha L) \ .$$

This is the well-known form[23] of the second (quadratic) Poisson bivector associated with the R-bracket (5.13).

Conclusion. These examples may help to explain the role of the Lax-Nijenhuis equation and its limits. This equation does not define the second ("quadratic") Poisson bracket, Q, associated with a Lax operator, but it provides a systematic way of deriving this bracket. The previous examples show that, in many cases, the form of L and the form of the first, given Poisson tensor suggest natural choices for the form of $\widehat{L}(\alpha)$ which make Q uniquely defined. This is the value of the method. Its limits are that it does not provide a proof of the fact that we indeed obtain a second Poisson tensor compatible with the given one.

In Ref. 38, there appears a Lax formulation for the evolution of the recursion operator of the KdV hierarchy, whose geometric interpretation along the lines of the present exposition remains to be clarified.

References

[1] P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. **21**, 467-490 (1968).

[2] F. Magri, *A simple model of the integrable Hamiltonian equation*, J. Math. Phys. **19**, 1156-1162 (1978).

[3] I. M. Gelfand and I. Ya. Dorfman, *Hamiltonian operators and algebraic structures related to them*, Funct. Anal. Appl. **13**, 248-262 (1979), *The Schouten bracket and Hamiltonian operators*, Funct. Anal. Appl. **14**, 223-226 (1980).

[4] A. G. Reiman and M. A. Semenov-Tian-Shansky, *A family of Hamiltonian structures, hierarchy of Hamiltonians, and reduction for first-order matrix differential operators*, Funct. Anal. Appl. **14**, 146-148 (1980).

[5] H. P. McKean, *Compatible brackets in Hamiltonian mechanics* in *Important Developments in Soliton Theory*, A. S. Fokas and V. E. Zakharov eds. (Springer-Verlag, 1993), pp. 344-354.

[6] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer-Verlag, 1993), 2nd edition.

[7] Y. Kosmann-Schwarzbach, *Géométrie des sytèmes bihamiltoniens* in *Systèmes dynamiques non linéaires : intégrabilité et comportement qualitatif*, P. Winternitz ed., Séminaire de Mathématiques Supérieures **102** (Presses de l'Université de Montréal, 1986), pp. 185-216.

[8] I. Ya. Dorfman, *Dirac Structures and Integrability of Nonlinear Evolution Equations* (Wiley and Sons, 1993).

[9] I. M. Gelfand and L. A. Dickey, *Asymptotic behaviour of the resolvent of Sturm-Liouville equations and the algebra of the Korteweg-de Vries equation*, Russian Math. Surveys **30**, 77-113 (1975), *A Lie algebra structure in the formal variational calculus*, Funct. Anal. Appl. **10**, 16-22 (1976).

[10] L. A. Dickey, *Soliton Equations and Hamiltonian Systems* (World Scientific, 1991).

[11] P. D. Lax, *Almost periodic solutions of the KdV equation*, S.I.A.M. Review **18**, 351-375 (1976).

[12] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, *Korteweg-de Vries equation and generalizations. VI Methods for exact solution*, Comm. Pure Appl. Math. **27**, 97-133 (1974).

[13] C. S. Gardner, *Korteweg-de Vries equation and generalizations. IV The Korteweg-de Vries equation as a Hamiltonian system*, J. Math. Phys. **12**, 1548-1551 (1971).

[14] A. S. Fokas and B. Fuchssteiner, *On the structure of symplectic operators and hereditary symmetries*, Lett. Nuovo Cimento **28**(8), 299-303 (1980).

[15] H. Flaschka, *The Toda lattice. I Existence of integrals*, Phys. Rev. B **9**, 1924-1925 (1974).

[16] M. Adler, *On a trace functional for formal pseudo-differential operators and the symplectic structure of the Korteweg-de Vries type equations*, Invent. Math. **50**, 403-432 (1979).

[17] B. Kupershmidt, *Discrete Lax Equations and Differential Difference Calculus*, Astérisque **123** (1985).

[18] W. Oevel and O. Ragnisco, *R-matrices and higher-order Poisson brackets for integrable systems*, Physica A **161**, 181-220 (1989).

[19] A. Das and S. Okubo, *A systematic study of the Toda lattice*, Ann. Phys. **190**, 215-232 (1989).

[20] P. A. Damianou, *Master symmetries and R-matrices for the Toda lattice*, Lett. Math. Phys. **20**, 101-112 (1990).

[21] I. M. Gelfand and L. A. Dickey, *A family of Hamiltonian structures connected with integrable nonlinear differential equations*, preprint **136** (Inst. Appl. Math. Moscow, 1978), published in English translation in *Collected Papers of I. M. Gelfand*, vol. 1 (Springer-Verlag, 1987), pp. 625-646.

[22] B. A. Kupershmidt and G. Wilson, *Modifying Lax equations and the second Hamiltonian structure*, Invent. Math. **62**, 403-436 (1981).

[23] M. A. Semenov-Tian-Shansky, *What is a classical r-matrix?*, Funct. Anal. Appl. **17**, 259-272 (1983).

[24] F. Magri, *A geometrical approach to the nonlinear solvable equations*, Lect. Notes Phys. **120** (Springer-Verlag, 1980), pp. 233-263.

[25] B. Fuchssteiner, *The Lie algebra structure of nonlinear evolution equations admitting infinite-dimensional Abelian symmetry groups*, Prog. Theor. Physics, **65** (3), 861-876 (1981).

[26] F. Magri, C. Morosi and O. Ragnisco, *Reduction techniques for infinite- dimensional Hamiltonian systems: some ideas and applications*, Comm. Math. Phys. **99**, 115-140 (1985).

[27] Y. Kosmann-Schwarzbach and F. Magri, *Poisson-Nijenhuis structures*, Ann. Inst. Henri Poincaré, A (Phys. Théor.) **53**, 35-81 (1990).

[28] A. Nijenhuis, *X_{n-1}-forming sets of eigenvectors*, Proc. Kon. Ned. Akad. Wet. Amsterdam **54**, 200-212 (1951).

[29] Yu. I. Manin, *Algebraic aspects of nonlinear differential equations*, J. Soviet Math. **11**, 1-122 (1979), in Russian, Itogi **11** , 5-152 (1978). See IV 5.3 in particular.

[30] A. G. Reiman and M. A. Semenov-Tian-Shansky, *Group-theoretical methods in the theory of finite-dimensional integrable systems* in *Dynamical Systems VII*, V. I. Arnold and S. P. Novikov, eds., Encycl. Math. Sci. **16** (Springer-Verlag, 1994).

[31] C. Morosi and L. Pizzocchero, *On the bihamiltonian interpretation of the Lax formalism*, Rep. Math. Phys. **7**(3), 389-430 (1995).

[32] V. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, 1978).

[33] I. S. Krasilshchik, V. V. Lychagin and A. M. Vinogradov, *Geometry of Jet Spaces and Nonlinear Partial Differential Equations* (Gordon and Breach, 1986).

[34] C. Morosi and L. Pizzocchero, *On the Euler equation: bi-Hamiltonian structure and integrals in involution*, Lett. Math. Phys. **37**, 117-135 (1996).

[35] G. Magnano and F. Magri, *Poisson-Nijenhuis structures and Sato hierarchy*, Rev. Math. Phys. **3**, 403-466 (1991).

[36] A. Radul, *Lie algebras of differential operators, their central extensions and W-algebras*, Funct. Anal. Appl. **25** (1), 25-39 (1991).

[37] B. Enriquez, S. Khoroshkin, A. Radul, A. Rosly and V. Rubtsov, *Poisson-Lie group aspects of the classical W-algebras*, Adv. in Math. Sci., Amer. Math. Soc. Transl., ser. 2, **167**, V. Lychagin, ed. (1995).

[38] B. Enriquez, A. Orlov and V. Rubtsov, *Higher Hamiltonian structures on the KdV phase space*, JETP Letters **58** (8), 677-683 (1993).