How large dimension guarantees a given angle?

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Abstract

We study the following two problems:

(1) Given $n \geq 2$ and α, how large Hausdorff dimension can a compact set $A \subset \mathbb{R}^n$ have if A does not contain three points that form an angle α?

(2) Given α and δ, how large Hausdorff dimension can a compact subset A of a Euclidean space have if A does not contain three points that form an angle in the δ-neighborhood of α?

Some angles $(0, 60^\circ, 90^\circ, 120^\circ, 180^\circ)$ turn out to behave differently than other $\alpha \in [0, 180^\circ]$.

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1 Introduction

The task of guaranteeing given patterns in a sufficiently large set has been a central problem in different areas of mathematics for a long time. Perhaps the most famous example is the celebrated theorem of Szemerédi [9], which states that any sequence of positive integers with positive upper density contains arbitrarily long arithmetic progressions.

More closely related to the present paper are the results of the second [5] and the fourth [6] author, which state that for any three points in $\mathbb{R}$ or in $\mathbb{R}^2$ there exists a set of full Hausdorff dimension that contains no similar copy to the three given points. It is open whether the analogous result holds in higher dimension. In case of a negative answer it would be natural to ask what Hausdorff dimension guarantees a similar copy of three given points. Since the similar copy of a triangle has the same angles as the original one, the following question arose.

Question 1.1. *For given n and α, what is the smallest d for which any compact set $A \subset \mathbb{R}^n$ with Hausdorff dimension larger than d contains three points that form an angle α?*

We use the following terminology.

Definition 1.2. We say that the set $A \subset \mathbb{R}^n$ *contains the angle* $\alpha \in [0, 180°]$ if there exist distinct points $x, y, z \in A$ such that the angle between the vectors $y - x$ and $z - x$ is α.

Definition 1.3. If $n \geq 2$ is an integer and $\alpha \in [0, 180°]$, then let

$$C(n, \alpha) = \sup\{s : \exists A \subset \mathbb{R}^n \text{ compact such that } \dim(A) = s \text{ and } A \text{ does not contain the angle } \alpha\}.$$

Clearly, answering Question 1.1 is the same as finding $C(n, \alpha)$. Somewhat surprisingly our results highly depend on the given angle. For 90° we show (Theorem 2.4) that $C(n, 90°) \leq [(n+1)/2]$ (where $[a]$ denotes the integer part of a) while for other angles we prove (Theorem 2.2) only $C(n, \alpha) \leq n - 1$, which is sharp for $\alpha = 0$ and $\alpha = 180°$.

In the other direction for any $\alpha \in (0, 180°) \setminus \{60°, 90°, 120°\}$ we construct (Theorem 3.3) a self-similar compact set with Hausdorff dimension $c(\alpha) \log n$ that does not contain the angle α. We show (Theorem 3.4) that it is impossible to avoid 90° with the same type self-similar construction. We mention though that the first and fifth authors constructed high Hausdorff dimensional compact sets for the angles 60°, 90°, 120° as well. In particular, the fifth author constructed an $n/2$-dimensional compact set that does not contain 90°. These constructions are more complicated and they will be published separately.

It is also natural to ask what can be said if we only want to guarantee an angle near to a given angle. In Section 4 we show that the previously mentioned special angles $(0, 60°, 90°, 120°, 180°)$ are really very special. If we fix α and a sufficiently small δ (but do not fix n) then for all other angles the above-mentioned self-similar construction gives a compact set with arbitrarily large Hausdorff dimension that does not contain any angle from the δ-neighborhood of α, while for the special angles this is not the case. More precisely, we show that any compact set with Hausdorff (or even upper Minkowski) dimension larger than 1 contains angles arbitrarily close to the right angle (Theorems 4.1 and 4.2), and that any compact set with Hausdorff dimension larger than $\frac{C}{\delta} \log(\frac{1}{\delta})$ (with an absolute constant C) contains angles from the δ-neighborhoods of 60° and 120° (Corollary 4.7

and Theorem 4.12). For the angles 0 and 180° it was already known by Erdős and Füredi [1] that any infinite set contains angles arbitrarily close to 0 and angles arbitrarily close to 180°. Table 1 at the end of Section 4 summarizes the above-mentioned results.

We emphasize the difference between the tasks of finding an angle precisely and finding it approximately. For example, we can find angles arbitrarily close to 90° given that the dimension of the set is greater than 1, while if we want to find 90° precisely in the set, we need to know that its dimension is greater than $n/2$.

We mention a related result of Iosevich, Mourgoglou and Senger claiming that if the Hausdorff dimension of a set $A \subset \mathbb{R}^n$ is greater than $n/2 - 1/6$ for $n \geq 3$, then the set of angles contained by A has positive Lebesgue measure [4]. The fifth author also has similar results.

Notation 1.4. We denote the s-dimensional Hausdorff measure by $\mathcal{H}^s$.

By dim we denote the Hausdorff dimension.

Recall that compact sets having the property $0 < \mathcal{H}^s(K) < \infty$ are called *compact s-sets.*

Using the well-known fact that an analytic set A with positive $\mathcal{H}^s$ measure contains a compact s-set (see e.g. [2, 2.10.47-48]) we get that in all of the above-mentioned results instead of compactness it is enough to assume that the set is analytic (or Borel) and on the other hand, we can always suppose that the given compact or analytic set is a compact s-set. Thus $C(n, \alpha)$ can be also expressed as

$$\begin{aligned} C(n,\alpha) = \sup\{s : \exists A \subset \mathbb{R}^n &\text{ analytic such that} \\ &\dim(A) = s \text{ and } A \text{ does not contain the angle } \alpha\}, \end{aligned}$$

or

$$\begin{aligned} C(n,\alpha) = \sup\{s : \exists K \subset \mathbb{R}^n &\text{ compact such that} \\ &0 < \mathcal{H}^s(K) < \infty \text{ and } K \text{ does not contain the angle } \alpha\}. \end{aligned}$$

However, as we prove it in the Appendix (Theorem 5.4), some assumption about the set is necessary, otherwise the above function would be n for any α. In fact, for any given n and α we construct by transfinite induction a set in $\mathbb{R}^n$ with positive Lebesgue outer measure that does not contain the angle α.

The following theorem, which is the first statement of [8, Theorem 10.11], plays essential role in some of our proofs.

Notation 1.5. The set of k-dimensional subspaces of $\mathbb{R}^n$ will be denoted by $G(n, k)$ and the natural probability measure on it by $\gamma_{n,k}$ (see e.g. [8] for more details).

Theorem 1.6. *If $m < s < n$ and A is an $\mathcal{H}^s$ measurable subset of $\mathbb{R}^n$ with $0 < \mathcal{H}^s(A) < \infty$, then*

$$\dim\big(A \cap (W + x)\big) = s - m$$

for $\mathcal{H}^s \times \gamma_{n,n-m}$ almost all $(x, W) \in A \times G(n, n-m)$.

In two dimensions it says that for $\mathcal{H}^s$ almost all $x \in A$, almost all lines through x intersect A in a set of dimension $s - 1$. One would expect that this theorem also holds for half-lines instead of lines. Indeed, Marstrand proved it in [7, Lemma 17]. Although the lemma only says that it holds for lines, he actually proves it for half-lines. Therefore the following theorem is also true.

Theorem 1.7. *Let $1 < s < 2$ and let $A \subset \mathbb{R}^2$ be $\mathcal{H}^s$ measurable with $0 < \mathcal{H}^s(A) < \infty$. For any $x \in \mathbb{R}^2$ and $\vartheta \in [0, 2\pi)$ let $L_{x,\vartheta} = \{x + te^{i\vartheta} : t \geq 0\}$. Then*

$$\dim\bigl(A \cap L_{x,\vartheta}\bigr) = s - 1$$

for $\mathcal{H}^s \times \lambda$ almost all $(x, \vartheta) \in A \times [0, 2\pi)$.

2 Finding a given angle

In this section we give estimates to $C(n, \alpha)$. For $n = 2$ we get the following exact result.

Theorem 2.1. *For any $\alpha \in [0, 180°]$ we have $C(2, \alpha) = 1$.*

Proof. A line has dimension 1 and it contains only the angles 0 and 180°. A circle also has dimension 1, but does not contain the angles 0 and 180°. Therefore $C(2, \alpha) \geq 1$ for all $\alpha \in [0, 180°]$.

For the other direction let $\alpha \in [0, \pi]$ and $s > 1$ fixed. We have to prove that any compact s-set contains the angle α. By Theorem 1.7, there exists $x \in K$ such that $\dim(K \cap L_{x,\vartheta}) = s - 1$ for almost all $\vartheta \in [0, 2\pi)$, where $L_{x,\vartheta} = \{x + te^{i\vartheta} : t \geq 0\}$. Hence we can take $\vartheta_1, \vartheta_2 \in [0, 2\pi)$ such that $|\vartheta_1 - \vartheta_2| = \alpha$, and $\dim(K \cap L_{x,\vartheta_i}) = s - 1$ for $i = 1, 2$. If $x_i \in L_{x,\vartheta_i} \setminus \{x\}$ then the angle between the vectors $x_1 - x$ and $x_2 - x$ is α, so indeed, K contains the angle α. □

An analogous theorem holds for higher dimensions.

Theorem 2.2. *If $n \geq 2$ and $\alpha \in [0, 180°]$ then $C(n, \alpha) \leq n - 1$.*

Proof. We have already seen the case $n = 2$, so we may assume that $n \geq 3$. It is enough to show that if $s > n - 1$ and K is a compact s-set, then K contains the angle α. By Theorem 1.6, there exists $x \in K$ such that there exists a $W \in G(n, 2)$ with $\dim(B) = s - n + 2 > 1$ for $B \stackrel{\text{def}}{=} A \cap (W + a)$. The set B lies in a two-dimensional plane, so we can think about B as a subset of $\mathbb{R}^2$. Applying Theorem 2.1 completes the proof. □

Now we are able to give the exact value of $C(n, 0)$ and $C(n, 180°)$.

Theorem 2.3. $C(n, 0) = C(n, 180°) = n - 1$ *for all* $n \geq 2$.

Proof. One of the inequalities was proven in the previous theorem, while the other one is shown by the $(n - 1)$-dimensional sphere. □

We prove a better upper bound for $C(n, 90°)$.

Theorem 2.4. *If n is even then $C(n, 90°) \leq n/2$. If n is odd then $C(n, 90°) \leq (n + 1)/2$.*

Proof. First suppose that n is even. Let $s > n/2$ and let K be a compact s-set. From Theorem 1.6 we know that there exists a point $x \in K$ such that

$$\dim\bigl(K \cap (x + W)\bigr) = s - n/2 > 0 \tag{1}$$

for $\gamma_{n,n/2}$ almost all $W \in G(n, n/2)$. There exists a $W \in G(n, n/2)$ such that (1) holds both for W and $W^\perp$. As $(x + W) \cap (x + W^\perp) = \{x\}$, by choosing a $y \in K \cap (x + W)$ and $z \in K \cap (x + W^\perp)$ such that $x \neq y$ and $x \neq z$, we find a right angle at x in the triangle xyz.

Now suppose that n is odd, $s > (n + 1)/2$ and K is a compact s-set. With a similar argument we can conclude that $\exists x \in K$ and $W \in$

$G(n,(n+1)/2)$ such that $\dim\bigl(K\cap(x+W)\bigr)=s-(n+1)/2>0$ and $\dim\bigl(K\cap(x+W^\perp)\bigr)=s-(n-1)/2>1$. If $y\in K\cap(x+W)\setminus\{x\}$ and $z\in K\cap(x+W^\perp)\setminus\{x\}$, then there is again a right angle at x in the triangle xyz. $\square$

Remark 2.5. By the following very recent result of the fifth author the above estimate is sharp if n is even: for any n there exists a compact set of Hausdorff dimension $n/2$ in $\mathbb{R}^n$ that does not contain 90°. Therefore if n is even, we have $C(n,90^\circ)=n/2$.

The construction uses number theoretic ideas and even though the set contains angles arbitrarily close to 90°, it succeeds to avoid the right angle. In the next section we will present a different approach where the constructed sets avoid not only a certain angle α but also a whole neighborhood of α.

3 A self-similar construction

In this section we construct a self-similar set in $\mathbb{R}^n$ with large dimension such that it does not contain a certain angle $\alpha\in(0,180^\circ)$. On the negative side, our method does not work for the angles 60°, 90° and 120°. On the positive side, the presented sets will avoid a whole neighborhood of α, not only α.

We start with the following very simple lemma.

Lemma 3.1. *Let $P_0,\dots,P_n$ be the vertices of a regular n-dimensional simplex. For any quadruples of indices (i,j,k,l) with $i\neq j$ and $k\neq l$, the angle between the lines P_iP_j and P_kP_l is either 0, 60° or 90°.*

Proof. The set $\{P_i,P_j,P_k,P_l\}$ is the set of vertices of a one-, two-, or three-dimensional regular simplex. Our assertion is clear in either of these cases. $\square$

Definition 3.2. A self-similar set K is said to satisfy the *strong separation condition* if there exist similarities $S_0,\dots,S_k$ such that $K=S_0(K)\cup\dots\cup S_k(K)$ and the sets $S_i(K)$ are pairwise disjoint.

We say that the transformation $f:\mathbb{R}^n\to\mathbb{R}^n$ is a *homothety* if f is the identity or if f has exactly one fixed point (say O), and there exists a nonzero real number r such that for any point P we have $f(P)-O=r(P-O)$. The point O is called the *center of the homothety*, and r is called the *ratio of magnification*. We call K *homothetic* if S_i is a homothety for $i=1,\dots,k$.

Theorem 3.3. *For any $\alpha\in(0,180^\circ)\setminus\{60^\circ,90^\circ,120^\circ\}$ there exists a constant $c(\alpha)$ such that for every $n\geq 2$ there exists a compact homothetic self-similar set $K\subset\mathbb{R}^n$ with $\dim(K)\geq c(\alpha)\log n$ that does not contain the angle α. Consequently, for any such α we have $C(n,\alpha)\geq c(\alpha)\log n$.*

In fact, for any $\varepsilon>0$ we construct a set of dimension $c_\varepsilon\log n$ with the property that all angles occurring in the set fall into the ε-neighborhood of the special angles $\{0,60^\circ,90^\circ,120^\circ,180^\circ\}$.

Proof. Our set K will be a modified version of the Sierpiński gasket. Take a regular n-dimensional simplex with unit edge length in $\mathbb{R}^n$, denote its vertices by $P_0,\dots,P_n$ and let $K_1\stackrel{\text{def}}{=}\operatorname{conv}(\{P_0,\dots,P_n\})$. Fix a $0<\delta<1/2$ and denote by S_i the homothety of ratio δ centered at P_i $(i=0,\dots,n)$.

The similarities S_i $(i = 0, \dots, n)$ uniquely determine a self-similar set K which can also be written in the following form:

$$K \stackrel{\text{def}}{=} \bigcap_{k=1}^{\infty} \bigcup_{(i_1,\dots,i_k)\in\{0,\dots,n\}^k} S_{i_1}\Big(S_{i_2}\big(\cdots S_{i_k}(K_1)\big)\Big).$$

The set K clearly satisfies the strong separation condition. By [8, Theorem 4.14], the dimension of K is the unique positive number s for which $(n+1)\delta^s = 1$, therefore

$$\dim(K) = \frac{\log(n+1)}{\log\frac{1}{\delta}}.$$

We say that a *direction* $V \in G(n,1)$ *occurs in a set* $H \subset \mathbb{R}^n$ if there are $x, y \in H$, $x \neq y$ such that $x - y$ is parallel to V. We will show that the directions occurring in K are actually *close* to the directions occurring in $\{P_0, \dots, P_n\}$.

Let $V \in G(n,1)$ which occurs in K and let $x_0, x_1 \in K$, $x_0 \neq x_1$ such that $x_0 - x_1$ is parallel to V. We claim that there exist $y_0, y_1 \in K$, $y_0 \neq y_1$ such that $y_0 - y_1$ is also parallel to V and there exists an $i \neq j$ with $y_0 \in S_i(K)$ and $y_1 \in S_j(K)$.

As K satisfies the strong separation condition, there exist unique sequences $i_{0,1}, i_{0,2}, \dots$ and $i_{1,1}, i_{1,2}, \dots$ such that

$$x_0 \in S_{i_{0,1}}\Big(S_{i_{0,2}}\big(\cdots S_{i_{0,k}}(K)\big)\Big) \quad \text{and} \quad x_1 \in S_{i_{1,1}}\Big(S_{i_{1,2}}\big(\cdots S_{i_{1,k}}(K_1)\big)\Big)$$

for any positive integer k.

Let k be the smallest positive integer such that $i_{0,k} \neq i_{1,k}$ (such a k exists else x_0 and x_1 would coincide). Set

$$S \stackrel{\text{def}}{=} S_{i_{0,1}}\Big(S_{i_{0,2}}\big(\cdots S_{i_{0,k-1}}(\cdot)\big)\Big).$$

There exist $y_0 \in S_{i_{0,k}}(K)$ and $y_1 \in S_{i_{1,k}}(K)$ such that $x_0 = S(y_0)$ and $x_1 = S(y_1)$. Since S is also a homothety, $y_0 - y_1$ is parallel to $x_0 - x_1$.

We may assume without loss of generality that $y_0 \in S_0(K)$, $y_1 \in S_1(K)$. We will show that the angle φ between $y_0 - y_1$ and $P_0 - P_1$ is small, which is equivalent with $\cos\varphi$ being close to 1. Let $h_i = y_i - P_i$. We have $||h_i|| \leq \delta$ $(i = 0, 1)$, hence

$$\cos\varphi = \frac{\langle y_0 - y_1, P_0 - P_1\rangle}{||y_0 - y_1|| \cdot ||P_0 - P_1||} = \frac{1 + \langle h_0 - h_1, P_0 - P_1\rangle}{||(P_0 - P_1) + (h_0 - h_1)||} \geq \frac{1 - 2\delta}{1 + 2\delta}.$$

Set $\varepsilon(\delta) = 2\arccos(\frac{1-2\delta}{1+2\delta})$. Lemma 3.1 implies that the angles occurring in K are in the union of the following intervals: $[0, \varepsilon]$, $[60^\circ - \varepsilon, 60^\circ + \varepsilon]$, $[90^\circ - \varepsilon, 90^\circ + \varepsilon]$, $[120^\circ - \varepsilon, 120^\circ + \varepsilon]$, $[180^\circ - \varepsilon, 180^\circ]$. If δ, and therefore ε is sufficiently small, then neither of these intervals contain α. $\square$

The first author improved this result by showing that for any $\alpha \in (0, 180^\circ) \setminus \{60^\circ, 90^\circ, 120^\circ\}$ we have $C(n, \alpha) \geq c(\alpha)n$. Moreover, even for the angles 60° and 120° it is possible to construct large dimensional homothetic self-similar sets avoiding these angles. In fact, $C(n, 60^\circ) \geq c\sqrt[3]{n}/\log n$.

However, as the next theorem shows, one cannot avoid the right angle with similar constructions.

Theorem 3.4. *Let $K \subset \mathbb{R}^n$ be a compact self-similar set. Suppose that we have homotheties $S_0, \ldots, S_k$ with ratios less than 1 such that $K = S_0(K) \cup S_1(K) \cup \cdots \cup S_k(K)$ and the sets $S_i(K)$ are pairwise disjoint (that is, the strong separation condition is satisfied). Then K contains four points that form a non-degenerate rectangle given that* $\dim(K) > 1$.

Proof. We begin the proof by defining the following map:

$$D: \ K \times K \setminus \{(x,x) : x \in K\} \to S^{n-1}; \quad (x,y) \mapsto \frac{x-y}{||x-y||}.$$

We denote the range of D by $\mathrm{Range}(D)$. The set $\mathrm{Range}(D)$ can be considered as the set of directions in K. First we are going to prove that if K is such a self-similar set then $\mathrm{Range}(D)$ is closed.

By the method used in the proof of the previous theorem, we can prove that if $x, y \in K$, $x \neq y$ then there exist $x' \in S_i(K)$ and $y' \in S_j(K)$ for some i and j such that $x = S(x')$ and $y = S(y')$ where S is the composition of finitely many S_i's. The important thing for us is that $x - y$ is parallel to $x' - y'$. If $d(\cdot, \cdot)$ denotes the Euclidean distance then

$$\min_{0 \le i < j \le k} d(S_i(K), S_j(K)) = c > 0,$$

so $\mathrm{Range}(D)$ actually equals to the image of D restricted to the set $K \times K \setminus \{(x,y) : d(x,y) < c\}$. As this is a compact set, the continuous image is also compact. That is what we wanted to prove.

Next we show that for any $v \in S^{n-1}$ there exist $x, y \in K$, $x \neq y$ such that the vectors v and $D(x,y)$ are perpendicular. If this was not true, the compactness of $\mathrm{Range}(D)$ would imply that the orthogonal projection p to a line parallel to v would be a one-to-one map on K with p^{-1} being a Lipschitz map on $p(K)$. This would imply $\dim(K) \le 1$, which is a contradiction.

For simplifying our notation, let $f \stackrel{\text{def}}{=} S_0$, $g \stackrel{\text{def}}{=} S_1$. The homotheties $f \circ g$ and $g \circ f$ have the same ratio. Denote their fixed points by P and Q, respectively. Since $P \neq Q$, there are $x, y \in K$, $x \neq y$ such that $x - y$ is perpendicular to $P - Q$. It is easy to check that the points $f(g(x))$, $f(g(y))$, $g(f(y))$ and $g(f(x))$ form a non-degenerate rectangle. $\square$

4 Finding angles close to a given angle

We start this section by proving that a set that does not contain angles near to 90° must be very small, it cannot have Hausdorff dimension bigger than 1. This makes 90° very special since, as we will see later, the analogous statement would be false for any other angle.

Theorem 4.1. *Any analytic (compact) set A in $\mathbb{R}^n$ ($n \ge 2$) with Hausdorff dimension greater than 1 contains angles arbitrarily close to the right angle.*

Proof. We can assume that $0 < \mathcal{H}^s(A) < \infty$ for some $s > 1$ (see the comments after Notation 1.4). Applying Theorem 1.6 for $m = 1$ we obtain that for $\mathcal{H}^s$ almost all $x \in A$ the set $A \cap (W + x)$ has positive dimension for $\gamma_{n,n-1}$ almost all $W \in G(n, n-1)$. Let us fix a point x with this property and let $y \neq x$ be an arbitrary point in A.

For any $\delta > 0$ it holds that $\gamma_{n,n-1}(\{W \in G(n,n-1) : \angle(n_W, xy) < \delta\}) > 0$ where n_W denotes the normal vector of W. It follows that

$$\gamma_{n,n-1}\left(\{W \in G(n,n-1) : \dim(A \cap (W+x)) > 0 \text{ and } \angle(n_W, xy) < \delta\}\right) > 0,$$

which clearly implies the statement. ☐

Now we prove the same result for upper Minkowski dimension instead of Hausdorff dimension. It is well-known that the upper Minkowski dimension is always greater or equal than the Hausdorff dimension. Hence the following theorem is stronger than the previous one.

Theorem 4.2. *Any set A in $\mathbb{R}^n$ ($n \geq 2$) with upper Minkowski dimension greater than 1 contains angles arbitrarily close to the right angle.*

The upper Minkowski dimension can be defined in many different ways, we will use the following definition (see [8, Section 5.3] for details).

Definition 4.3. By $B(x, r)$ we denote the closed ball with center $x \in \mathbb{R}^n$ and radius r. For a non-empty bounded set $A \subset \mathbb{R}^n$ let $P(A, \varepsilon)$ denote the greatest integer k for which there exist disjoint balls $B(x_i, \varepsilon)$ with $x_i \in A$, $i = 1, \ldots, k$. The *upper Minkowski dimension* of A is defined as

$$\overline{\dim}_{\mathrm{M}}(A) \stackrel{\text{def}}{=} \sup\{s : \limsup_{\varepsilon \to 0+} P(A, \varepsilon)\varepsilon^s = \infty\}.$$

Note that we get an equivalent definition if we consider the lim sup for ε's only in the form $\varepsilon = 2^{-k}$, $k \in \mathbb{N}$.

The next lemma is mainly technical. It roughly says that in a set of large upper Minkowski dimension one can find many points such that the distance of each pair is more or less the same.

Lemma 4.4. *Suppose that $\overline{\dim}_{\mathrm{M}}(A) > t$ for a set $A \subset \mathbb{R}^n$ and a positive real t. Then for infinitely many positive integers k it holds that for any integer $0 < l < k$ there are more than $2^{(k-l)t}$ points in A with the property that the distance of any two of them is between 2^{-k+1} and 2^{-l+2}.*

Proof. Let

$$r_k = P(A, 2^{-k})2^{-kt}.$$

Due to the previous definition $\limsup_{k\to\infty} r_k = \infty$. It follows that there are infinitely many values of k such that $r_k > r_l$ for all $l < k$. Let us fix such a k and let $0 < l < k$ be arbitrary.

By the definition of r_k, there are $r_k 2^{kt}$ disjoint balls with radii 2^{-k} and centers in A. Let $\mathcal{S}$ denote the set of the centers of these balls. Clearly the distance of any two of them is at least 2^{-k+1}.

Similarly, we can find a maximal system of disjoint balls $B(x_i, 2^{-l})$ with $x_i \in A$, $i = 1, \ldots, r_l 2^{lt}$. Consider the balls $B(x_i, 2^{-l+1})$ of doubled radii. These doubled balls are covering the whole A (otherwise the original system would not be maximal). By the pigeonhole principle, one of these doubled balls contains at least

$$\frac{r_k 2^{kt}}{r_l 2^{lt}} = \frac{r_k}{r_l} 2^{(k-l)t} > 2^{(k-l)t}$$

points of $\mathcal{S}$. These points clearly have the desired property. ☐

Now we are in the position to prove the theorem.

Proof of Theorem 4.2. We can assume that $\mathrm{diam}(A) > 2$. Fix a t such that $\overline{\dim}_{\mathrm{M}}(A) > t > 1$.

Lemma 4.4 tells us that there are arbitrarily large integers k such that for any $l < k$ one can have more than $2^{(k-l)t}$ points in A such that each distance is between 2^{-k+1} and 2^{-l+2}. Let $\mathcal{S}$ be a set of such points and pick an arbitrary point $O \in \mathcal{S}$. Since $\mathrm{diam}(A) > 2$, there exists a point

$P \in A$ with $OP \geq 1$. Now we project the points of $\mathcal{S}$ to the line OP. There must be two distinct points $Q_1, Q_2 \in \mathcal{S}$ such that the distance of their projection is at most

$$\frac{2^{-l+2}}{2^{(k-l)t}} = 2^{-l+2-(k-l)t},$$

It follows that

$$\cos \angle(\overrightarrow{Q_1Q_2}, \overrightarrow{OP}) \leq \frac{2^{-l+2-(k-l)t}}{2^{-k+1}} = 2^{-(k-l)(t-1)+1}.$$

Since $Q_1O \leq 2^{-l+2}$ and $OP \geq 1$, the angle of the lines OP and Q_1P is at most $C_1 2^{-l}$ with some constant C_1. Combining the previous results we get that

$$|\angle PQ_1Q_2 - 90°| \leq C_1 2^{-l} + C_2 2^{-(k-l)(t-1)}$$

with some constants C_1, C_2. The right hand side can be arbitrarily small since $t-1$ is positive and both l and $k-l$ can be chosen to be large. □

Now we try to find angles close to $60°$. We will do that by finding three points forming an *almost regular* triangle provided that the dimension of the set is sufficiently large.

We will need a simple result from Ramsey theory. Let $R_r(3)$ denote the least positive integer k for which it holds that no matter how we color the edges of a complete graph on k vertices with r colors it contains a monochromatic triangle. The next inequality can be obtained easily:

$$R_r(3) \leq r \cdot R_{r-1}(3) - (r-2).$$

(A more general form of the above inequality can be found in e.g. [3, p. 90, Eq. 2].) It readily implies the following upper bound for $R_r(3)$.

Lemma 4.5. *For any positive integer $r \geq 2$*

$$R_r(3) \leq 3r!,$$

that is, any complete graph on at least $3r!$ vertices edge-colored by r colors contains a monochromatic triangle.

Using this lemma we can prove the following theorem.

Theorem 4.6. *There exists an absolute constant C such that whenever $\overline{\dim}_{\mathrm{M}}(A) > \frac{C}{\delta}\log(\frac{1}{\delta})$ for some set $A \subset \mathbb{R}^n$ and $\delta > 0$ the following holds: A contains three points that form a δ-almost regular triangle, that is, the ratio of the length of the longest and shortest sides is at most $1+\delta$.*

As an immediate consequence, we can find angles close to $60°$.

Corollary 4.7. *Suppose that $\overline{\dim}_{\mathrm{M}}(A) > \frac{C}{\delta}\log(\frac{1}{\delta})$ for some set $A \subset \mathbb{R}^n$ and $\delta > 0$. Then A contains angles from the interval $(60° - \delta, 60°]$ and also from $[60°, 60° + \delta)$.*

Remark 4.8. The above theorem and even the corollary is essentially sharp: the first author constructed a set with Hausdorff dimension $\frac{c}{\delta}/\log(\frac{1}{\delta})$ and without any angles from the interval $(60° - \delta, 60° + \delta)$.

Proof of Theorem 4.6. Let $t = \frac{C}{\delta}\log(\frac{1}{\delta})$ and apply Lemma 4.4 for $l = k-1$. We obtain at least 2^t points in A such that each distance is in the interval $[2^{-k+1}, 2^{-k+3}]$. Let $a = 2^{-k+1}$ and divide $[a, 4a]$ into $N = \lceil \frac{3}{\delta} \rceil$ disjoint intervals of length at most δa. Regard the points of A as the vertices of a graph. Color the edges of this graph with N colors according to which interval contains the distance of the corresponding points.

Easy computation shows that $2^t > 3N!$ (with a suitable choice of C). Therefore the above graph contains a monochromatic triangle by Lemma 4.5. It easily follows that the three corresponding points form a δ-almost regular triangle in $\mathbb{R}^n$. $\square$

Remark 4.9. The same proof yields the following: for any positive integer d and positive real δ there is a number $K(d,\delta)$ such that whenever $\overline{\dim}_{\mathrm{M}}(A) > K(d,\delta)$ for some set A, one can find d points in A with the property that the ratio of the largest and the smallest distance among these points is at most $1+\delta$. (One needs to use the fact that the Ramsey number $R_r(d)$ is finite.)

In order to derive similar results for 120° instead of 60° we show that if large Hausdorff dimension implies the existence of an angle near α, then it also implies the existence of an angle near $180^\circ - \alpha$.

Proposition 4.10. *Suppose that $s = s(\alpha,\delta,n)$ is a positive real number such that any analytic set $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) > 0$ contains an angle from the interval $(\alpha-\delta, \alpha+\delta)$. Then any analytic set $B \subset \mathbb{R}^n$ with $\mathcal{H}^s(B) > 0$ contains an angle from the interval $(180^\circ - \alpha - \delta', 180^\circ - \alpha + \delta')$ for any $\delta' > \delta$.*

Proof. Again, we can assume that $0 < \mathcal{H}^s(B) < \infty$. It is well-known that for $\mathcal{H}^s$ almost all $x \in B$ the set $B \cap B(x,r)$ has positive $\mathcal{H}^s$ measure for any $r > 0$ [8, Theorem 6.2]. If we omit the exceptional points from B, this will be true for every point of the obtained set. Assume that B had this property in the first place. Then, by the assumptions of the proposition, any ball around any point of B contains an angle from the δ-neighborhood of α.

We define the points $P_m, Q_m, R_m \in B$ recursively in the following way. Fix a small ε. First take P_0, Q_0, R_0 such that the angle $\angle P_0Q_0R_0$ falls into the interval $(\alpha-\delta, \alpha+\delta)$. If the points P_m, Q_m, R_m are given, then choose points $P_{m+1}, Q_{m+1}, R_{m+1}$ from the $\varepsilon \cdot \min(Q_mP_m, Q_mR_m)$-neighborhood of P_m such that $\angle P_{m+1}Q_{m+1}R_{m+1} \in (\alpha-\delta, \alpha+\delta)$.

We can find two indices $k > l$ such that the angle enclosed by the vectors $\overrightarrow{Q_lP_l}$ and $\overrightarrow{Q_kP_k}$ is less than ε. It is clear that if we choose ε sufficiently small, then $\angle(Q_l, Q_k, R_k) \in (180^\circ - \alpha - \delta', 180^\circ - \alpha + \delta')$. $\square$

Remark 4.11. Proposition 4.10 holds for $\delta' = \delta$ as well. Surprisingly, it even holds for some $\delta' < \delta$. The reason behind is the following. If every analytic set $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) > 0$ contains an angle from the interval $(\alpha-\delta, \alpha+\delta)$, then there necessarily exists a closed subinterval $[\alpha-\gamma, \alpha+\gamma]$ ($\gamma < \delta$) such that every analytic set $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) > 0$ contains an angle from the interval $[\alpha-\gamma, \alpha+\gamma]$. We prove this statement in the Appendix (Theorem 5.1).

Theorem 4.12. *There exists an absolute constant C such that any analytic set $A \subset \mathbb{R}^n$ with $\dim(A) > \frac{C}{\delta}\log(\frac{1}{\delta})$ contains an angle from the δ-neighborhood of 120°.*

Proof. The claim readily follows from Corollary 4.7, Proposition 4.10 and the fact that the upper Minkowski dimension is greater or equal than the Hausdorff dimension. $\square$

To find angles arbitrarily close to 0 and 180°, it suffices to have infinitely many points.

Proposition 4.13. *Any $A \subset \mathbb{R}^n$ of infinite cardinality contains angles arbitrarily close to* 0 *and angles arbitrarily close to* 180°.

Sketch of the proof. We claim that given N points in $\mathbb{R}^n$ they must contain an angle less than $\delta_1 = \frac{C}{\sqrt[n-1]{N}}$ and an angle greater than $180^\circ - \delta_2$ with $\delta_2 = \frac{C}{\sqrt[n-1]{\log N}}$. The former follows easily from the pigeonhole principle. The latter is a result of Erdős and Füredi [1, Theorem 4.3]. □

In this section we have seen results saying that large dimensional sets contain angles close to a given angle $\alpha \in \{0, 60^\circ, 90^\circ, 120^\circ, 180^\circ\}$. Note that in these results the dimension of the Euclidean space (n) did not play any role. To sum up the results we introduce the following function $\widetilde{C}$ depending on an angle $\alpha \in [0, 180^\circ]$ and a small positive δ.

$$\widetilde{C}(\alpha, \delta) \stackrel{\text{def}}{=} \sup\{\dim(A) : A \subset \mathbb{R}^n \text{ for } \textbf{some } n; A \text{ is analytic};$$
$$A \text{ does not contain any angle from } (\alpha - \delta, \alpha + \delta)\}.$$

Remark 4.11 implies that $\widetilde{C}$ satisfies the symmetry property

$$\widetilde{C}(\alpha, \delta) = \widetilde{C}(180^\circ - \alpha, \delta).$$

In the previous section we constructed sets of arbitrarily large dimension such that all the angles fall into the ε-neighborhood of the special angles 0, 60°, 90°, 120°, 180° (Theorem 3.3). So for any angle α other than the special angles $\widetilde{C}(\alpha, \delta) = \infty$ if δ is smaller than the distance of α from the special angles. Therefore this construction and the results of this section give essentially all the values of $\widetilde{C}(\alpha, \delta)$, see the table below.

Table 1: Smallest dimensions that guarantee angle in the δ-neighborhood of α

α	$\widetilde{C}(\alpha, \delta)$	
$0, 180^\circ$	$= 0$	
90°	$= 1$	
$60^\circ, 120^\circ$	$\approx 1/\delta$	apart from a multiplicative error $C \cdot \log(1/\delta)$
other angles	$= \infty$	provided that δ is sufficiently small

5 Appendix

Our first goal is to prove the following theorem, which was claimed in Remark 4.11.

Theorem 5.1. *Suppose that $s = s(\alpha, \delta, n)$ is a positive real number such that every analytic set $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) > 0$ contains an angle from the interval $(\alpha - \delta, \alpha + \delta)$. Then there exists a closed subinterval $[\alpha - \gamma, \alpha + \gamma]$ ($\gamma < \delta$) such that every analytic set $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) > 0$ contains an angle from the interval $[\alpha - \gamma, \alpha + \gamma]$.*

To prove this theorem, we need two lemmas. For $r \in (0, \infty]$ let

$$\mathcal{H}^s_r(A) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^s \,:\, \operatorname{diam}(U_i) \le r,\ A \subset \cup_{i=1}^{\infty} U_i\right\},$$

thus $\mathcal{H}^s(A) = \lim_{r \to 0+} \mathcal{H}^s_r(A)$.

Lemma 5.2. *Let A_i be a sequence of compact sets converging in the Hausdorff metric to a set A. Then the following two statements hold.*

(i) $\mathcal{H}^s_\infty(A) \geq \limsup_{i\to\infty} \mathcal{H}^s_\infty(A_i)$.

(ii) Suppose that for every $i = 1, 2, \ldots$ the set A_i does not contain an angle from $[\alpha - \delta + 1/i,\ \alpha + \delta - 1/i]$. Then A does not contain an angle from $(\alpha - \delta,\ \alpha + \delta)$.

Proof. The first statement is well-known and easy. To prove the second, notice that for any three points x, y, z of A there exist three points in A_i arbitrarily close to x, y, z, for sufficiently large i. ☐

The next lemma follows easily from [2, Theorem 2.10.17 (3)]. For the sake of completeness, we give a short direct proof.

Lemma 5.3. *Let $A \subset \mathbb{R}^n$ be a compact set satisfying $\mathcal{H}^s(A) > 0$. Then there exists a ball B such that $\mathcal{H}^s_\infty(A \cap B) \geq c\, diam(B)^s$, where $c > 0$ depends only on s.*

Proof. We may suppose without loss of generality that $\mathcal{H}^s(A) < \infty$. (Otherwise we choose a compact subset of A with positive and finite $\mathcal{H}^s$ measure. If the theorem holds for a subset of A then it clearly holds for A as well.)

Choose $r > 0$ so that $\mathcal{H}^s_r(A) > \mathcal{H}^s(A)/2$. Cover A by sets U_i of diameter at most $r/2$ such that $\sum_i \operatorname{diam}(U_i)^s \leq 2\mathcal{H}^s(A)$. Cover each U_i by a ball B_i of radius at most the diameter of U_i. Then the balls B_i cover A, have diameter at most r, and $\sum_i \operatorname{diam}(B_i)^s \leq 2^{1+s}\mathcal{H}^s(A)$.

We claim that one of these balls B_i satisfies the conditions of the Lemma for $c = 2^{-2-s}$. Otherwise we have

$$\mathcal{H}^s_\infty(A \cap B_i) < 2^{-2-s} \operatorname{diam}(B_i)^s$$

for every i. Since the sets $A \cap B_i$ have diameter at most r, clearly $\mathcal{H}^s_r(A \cap B_i) = \mathcal{H}^s_\infty(A \cap B_i)$. Therefore

$$\mathcal{H}^s_r(A) \leq \sum_i \mathcal{H}^s_r(A \cap B_i) < \sum_i 2^{-2-s} \operatorname{diam}(B_i)^s \leq 2^{-2-s} 2^{1+s} \mathcal{H}^s(A) = \mathcal{H}^s(A)/2,$$

which contradicts the choice of r. ☐

Proof of Theorem 5.1. Suppose on the contrary that there exist compact sets $K_i \subset \mathbb{R}^n$ with $\mathcal{H}^s(K_i) > 0$ such that K_i does not contain an angle from $[\alpha - \delta + 1/i,\ \alpha + \delta - 1/i]$. Choose a ball B_i for each compact set K_i according to Lemma 5.3. Let B be a ball of diameter 1. Let K_i' be the image of $K_i \cap B_i$ under a similarity transformation which maps B_i to the ball B. Thus $\mathcal{H}^s_\infty(K_i') \geq c$. Let K denote the limit of a convergent subsequence of the sets K_i. We can apply Lemma 5.2 to this subsequence and obtain $\mathcal{H}^s_\infty(K) \geq c$, implying $\mathcal{H}^s(K) > 0$. Also, K does not contain any angle from the interval $(\alpha - \delta,\ \alpha + \delta)$, which is a contradiction. ☐

Now we show that if we allowed arbitrary sets in Definition 1.3 then $C(n, \alpha)$ would be n.

Theorem 5.4. *Let $n \geq 2$. For any $\alpha \in [0, 180°]$ there exists $H \subset \mathbb{R}^n$ such that H does not contain the angle α, and H has positive Lebesgue outer measure. In particular,* $\dim(H) = n$.

Proof. Take a well-ordering $\{B_\beta : \beta < \mathfrak{c}\}$ of the Borel null-sets of $\mathbb{R}^n$ (with respect to the n-dimensional Lebesgue measure). We will construct a sequence of points $\{x_\beta : \beta < \mathfrak{c}\}$ of $\mathbb{R}^n$ using transfinite induction, and define H as $\{x_\beta : \beta < \mathfrak{c}\}$.

We introduce the following notation. If $y, z \in \mathbb{R}^n$ and $y \neq z$ then

$$C_{yz} \overset{\text{def}}{=} \{x \in \mathbb{R}^n \setminus \{y,z\} : \text{ the angle between } x-y \text{ and } z-y \text{ is } \alpha\} \cup \{y,z\};$$

$$D_{yz} \overset{\text{def}}{=} \{x \in \mathbb{R}^n \setminus \{y,z\} : \text{ the angle between } y-x \text{ and } z-x \text{ is } \alpha\} \cup \{y,z\}.$$

C_{yz} is a cone with vertex y, while D_{yz} has the property that a 2-dimensional plane containing y and z intersects it in the union of two circular arcs going between y and z. When $\alpha = 0$ or $180°$, both sets are degenerate: C_{yz} becomes a half-line, D_{yz} becomes a segment or the union of two half-lines.

First we show that if v is a vector such that the angle between v and $z-y$ is not α or $180° - \alpha$, then any line l parallel to v intersects C_{yz} in at most two points. Let $x \in l$ be arbitrary. Then $l = \{x + tv : t \in \mathbb{R}\}$. Suppose that $t_0 \in \mathbb{R}$ such that $x + t_0 v \in C_{yz}$. Then

$$\cos^2 \alpha = \frac{\langle (x+t_0v) - y, z-y\rangle^2}{||(x+t_0v)-y||^2||z-y||^2} = \frac{p_1(t_0)}{p_2(t_0)},$$

where $p_1(t)$ and $p_2(t)$ are polynomials of degree 2, with leading coefficients $\langle v, z-y\rangle^2$ and $||v||^2||z-y||^2$, respectively. The number t_0 is a root of

$$p(t) \overset{\text{def}}{=} p_2(t)\cos^2\alpha - p_1(t),$$

which has degree 2, as the coefficient of t^2 is

$$||v||^2||z-y||^2\cos^2\alpha - \langle v, z-y\rangle^2 \neq 0.$$

Hence $p(t)$ has at most two roots which means that l intersects C_{yz} in at most two points.

Similarly, we prove that if D_{yz} is non-degenerate, then any line l intersects it in at most four points. Let $l = \{x + tv : t \in \mathbb{R}\}$ again, and suppose that $x + t_0 v \in D_{yz}$ for some $t_0 \in \mathbb{R}$. Then

$$\cos^2\alpha = \frac{\langle y - (x+t_0v), z-(x+t_0v)\rangle^2}{||y-(x+t_0v)||^2||z-(x+t_0v)||^2} = \frac{p_1(t_0)}{p_2(t_0)},$$

where $p_1(t)$ and $p_2(t)$ now denote polynomials of degree 4. Again, t_0 is a root of the polynomial $p_2(t)\cos^2\alpha - p_1(t)$ which has degree exactly 4 as the leading coefficient of both p_1 and p_2 are $||v||^4$, and $\cos^2\alpha \neq 1$. As it has at most four roots, we are done. When D_{yz} is degenerate, any line that does not go through both y and z intersects D_{yz} in at most one point.

Now we move on to the construction. Suppose that $\beta < \mathfrak{c}$ and we have already defined x_γ for all $\gamma < \beta$. Let $H_\beta = \{x_\gamma : \gamma < \beta\}$.

We want the point x_β to satisfy the following properties:

(i) $x_\beta \notin C_{yz}$ for any $y, z \in H_\beta$ with $y \neq z$;

(ii) $x_\beta \notin D_{yz}$ for any $y, z \in H_\beta$ with $y \neq z$;

(iii) $x_\beta \notin B_\beta$.

If we prove that it is possible to define x_β this way, then we are done, because (i) and (ii) guarantee that the resulting set H will not contain the angle α, while (iii) ensures that H will not be a null set as each null set is contained by a Borel null set.

First we show that there is a direction $v \in S^{n-1}$ such that each line parallel to v intersects the set

$$A_\beta \overset{\text{def}}{=} \bigcup_{y,z\in H_\beta, y\neq z} (C_{yz} \cup D_{yz})$$

in less than $\mathfrak{c}$ points. We say that v is *good* for C_{yz} (or D_{yz}) if each line parallel to v intersects C_{yz} (or D_{yz}) in less than $\mathfrak{c}$ points. We have already shown that for each D_{yz} there are at most two $v \in S^{n-1}$ which are not good for D_{yz}. Therefore there are less than $\mathfrak{c}$ directions that are not good for some D_{yz}.

For a fixed y and z the set of directions which are not good for C_{yz} is

$$\{v \in S^{n-1} : \langle v, z-y\rangle = \pm||z-y|| \cos\alpha\} = S^{n-1} \cap (\Sigma_{y-z} \cup \Sigma_{z-y}),$$

where Σ_w denotes the hyperplane $\{x \in \mathbb{R}^n : \langle x, w\rangle = ||w|| \cos\alpha\}$ for $w \in \mathbb{R}^n \setminus \{0\}$. First, suppose that $\alpha \neq 90^\circ$, whence $0 \notin \Sigma_w$. Take arbitrary v_1 and v_2 with $v_2 \neq \pm v_1$, and denote the two-dimensional plane $\{sv_1 + tv_2 : s, t \in \mathbb{R}\}$ by F. The set $C \overset{\text{def}}{=} S^{n-1} \cap F$ is an ordinary circle. It is clear that the set $S^{n-1} \cap \Sigma_w \cap C = S^{n-1} \cap (\Sigma_w \cap F)$ has at most two elements for all $w \in \mathbb{R}^n \setminus \{0\}$, because $\Sigma_w \cap F$ is an at most one-dimensional affine subspace of $\mathbb{R}^n$ as $0 \notin \Sigma_w$. From this we can conclude that there are less than $\mathfrak{c}$ points on C which are not good for some C_{yz}, hence there is a point on C which is good for every C_{yz} and D_{yz}.

This method does not work if $\alpha = 90^\circ$. In this case take a subset V of S^{n-1} such that $\mathrm{card}(V) = \mathfrak{c}$ and no n distinct elements of V are linearly dependent. For example, the set $U = \{(1, t, \dots, t^{n-1}) : t \in [0, 1]\}$ does not contain n distinct points which are linearly dependent (their determinant is a Vandermonde determinant), so we may get a good V by normalizing each $u \in U$ to $u/||u||$. As Σ_w goes through the origin in this case, it can contain at most $n-1$ points of V. It follows that the union of the hyperplanes Σ_{y-z} and Σ_{z-y} can cover only less than $\mathfrak{c}$ points of V ($y, z \in H_\beta, y \neq z$). Hence there exists a $v \in V$ which is good for every C_{yz} and D_{yz} in this case, too.

Take such a v. The only thing we need to prove in order to finish the proof of the theorem is that $A_\beta \cup B_\beta \neq \mathbb{R}^n$. Taking a Cartesian coordinate system with one axis in the direction of v, and applying Fubini's Theorem for the characteristic function of B_β gives that $\mathcal{H}^1(l_x \cap B_\beta) = 0$ for almost all $x \in \{v\}^\perp$, where l_x denotes the line $\{x + tv : t \in \mathbb{R}\}$. We also have $\mathrm{card}(l_x \cap A_\beta) < \mathfrak{c}$ for all $x \in \{v\}^\perp$, therefore it remains to show that the complement of a null set of $\mathbb{R}$ has cardinality $\mathfrak{c}$. But this is clear, as the complement of a null set contains a compact set with positive measure, which is the union of a non-empty perfect set and a countable set. $\square$

References

[1] P. Erdős, Z. Füredi, The greatest angle among n points in the d-dimensional Euclidean space, *Annals of Discrete Mathematics* **17** (1983), 275–283.

[2] H. Federer, *Geometric Measure Theory*, Springer Verlag, 1969.

[3] R. Graham, B. L. Rothschild, J. H. Spencer, *Ramsey Theory*, Wiley, 2nd edition, 1990.

[4] A. Iosevich, M. Mourgoglou, S. Senger, On sets of directions and angles determined by subsets of $\mathbb{R}^d$, arXiv:1010.0749v1.

[5] T. Keleti, Construction of 1-dimensional subsets of the reals not containing similar copies of given patterns, *Anal. PDE* **1** (2008), 29–33.

[6] P. Maga, Full dimensional sets without given patterns, *Real Anal. Exchange*, to appear.

[7] J. M. Marstrand, Some fundamental geometrical properties of plane sets of fractional dimensions, *Proc. London Math. Soc.* **4** (1954) no. 3, 257–302.

[8] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.

[9] E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, *Acta Arithmetica* **27** (1975), 199–245.