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Axioms for preferences revealing subjective uncertainty and uncertainty aversion

Jens Tapking*

European Central Bank, Kaiserstrasse 29, 60311 Frankfurt/Main, Germany

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Abstract

This article analyzes a decision maker's preferences and their updating in situations with uncertainty. Axioms for a list containing a prior preference relation and an updated preference relation for different information are presented, such that (1) each preference relation in this list is a Choquet expected utility preference relation as axiomatized by [Econometrica 57 (1989) 571] and (2) the list reveals both the decision maker's subjective uncertainty and his uncertainty aversion.

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1. Introduction

1.1. From expected utility theory to choquet expected utility theory

Savage (1954) proposed axioms for preferences on sets of uncertain acts. A decision maker fulfilling these axioms can be characterized by (a) a utility function on the set of consequences and (b) a unique subjective probability measure on the set of states such that he prefers one act f over another act g if and only if f yields a higher expected utility than g. Savage's subjective expected utility theory has become the best known and most often used theory of decision making under uncertainty. However, real economic behavior often violates the subjective expected utility theory. Consider for example the Ellsberg (1961) experiment:

* Tel.: +49-69-13448478; fax: +49-69-13447488.
E-mail address: jens.tapking@ecb.int (J. Tapking).

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An urn is presented to a decision maker. He is informed that there are 90 balls in the urn, that 30 of the balls are red and the other 60 balls are white and black in some unknown proportion. Thus, the decision maker knows that a red ball will be drawn with probability 1/3, but he does not know the probability of a white and the probability of a black ball. A situation in which the probabilities of some events are (perceived to be) unknown is called a situation of (subjective) uncertainty. The decision maker is asked to choose one of the following two bets: (1) If a red balls is drawn, he gets US$ 100, otherwise nothing. (2) If a white ball is drawn, he gets US$ 100, otherwise nothing. Most decision makers prefer bet (1) over bet (2). Next, the decision maker is asked to choose one of the following two bets: (3) If a red or a black ball is drawn, he gets US$ 100, otherwise nothing. (4) If a white or black ball is drawn, he gets US$ 100, otherwise nothing. Most decision maker who preferred bet (1) over bet (2) now prefer bet (4) over bet (3). One explanation for this behavior is obvious: the probability of winning US$ 100 is known in bet (1) and (4), but not in bet (2) and (3). Decision makers apparently prefer bets in which the probabilities of the different consequences are known. They are uncertainty averse.

This behavior is in contradiction to Savage's expected utility theory. Denote by $u(100)$ and $u(0)$ the utility of winning US$ 100 and nothing respectively and by $\pi(\{r\})$, $\pi(\{w\})$ and $\pi(\{b\})$ the decision maker's probability judgement of a red, white and black ball being drawn. Since he prefers (1) over (2) and (4) over (3), the following must hold in expected utility theory:

$$\begin{aligned} &u(100)\pi(\{r\}) + u(0)[\pi(\{w\}) + \pi(\{b\})] > u(100)\pi(\{w\}) + u(0)[\pi(\{r\}) + \pi(\{b\})] \\ &\quad\Leftrightarrow u(100)\pi(\{r\}) + u(0)\pi(\{w\}) > u(100)\pi(\{w\}) + u(0)\pi(\{r\}) \end{aligned}$$

and

$$\begin{aligned} &u(100)[\pi(\{w\}) + \pi(\{b\})] + u(0)\pi(\{r\}) > u(100)[\pi(\{r\}) + \pi(\{b\})] + u(0)\pi(\{w\}) \\ &\quad\Leftrightarrow u(100)\pi(\{w\}) + u(0)\pi(\{r\}) > u(100)\pi(\{r\}) + u(0)\pi(\{w\}) \end{aligned}$$

The contradiction is obvious.

Schmeidler (1989) and Gilboa (1987) presented axioms implying that the decision maker has (a) a utility function on the set of consequences and (b) a unique non-additive measure, called capacity, on the set of states, such that he prefers act f over act g if and only if f yields a higher Choquet expected utility[1] than g. Let $S = \{r, w, b\}$ be the set of states of the world in the Ellsberg experiment. The following function on (the power set of) S is an example for a capacity:

Example 1.

$$\begin{aligned} &v(\{r\}) = \tfrac{1}{3}, v(\{w\}) = v(\{b\}) = \tfrac{1}{6} \\ &v(\{r, w\}) = v(\{r, b\}) = \tfrac{1}{2}, v(\{w, b\}) = \tfrac{2}{3} \\ &v(\{r, w, b\}) = 1, v(\varnothing) = 0 \end{aligned}$$

[1] See Choquet (1955). A formal definition of both a capacity and the Choquet utility are given in Section 2.

It is obviously not a probability measure, since for example $v(\{w\}) + v(\{r, b\}) \neq 1$. If we assume $u(100) > u(0)$ the Choquet expected utilities of Ellsberg's bets (1) to (4), denoted by $C_v(u \circ f^1)$ to $C_v(u \circ f^4)$, are given by

Example 2.

$$C_v(u \circ f^1) = u(100) \cdot v(\{r\}) + u(0) \cdot [1 - v(\{r\})]$$
$$C_v(u \circ f^2) = u(100) \cdot v(\{w\}) + u(0) \cdot [1 - v(\{w\})]$$
$$C_v(u \circ f^3) = u(100) \cdot v(\{r, b\}) + u(0) \cdot [1 - v(\{r, b\})]$$
$$C_v(u \circ f^4) = u(100) \cdot v(\{w, b\}) + u(0) \cdot [1 - v(\{w, b\})]$$

Using the capacity from Example 1 above we immediately arrive at $C_v(u \circ f^1) > C_v(u \circ f^2)$ and $C_v(u \circ f^4) > C_v(u \circ f^3)$. Thus the proband prefers bet (1) over (2) and (4) over (3) which is compatible with the Ellsberg findings. Thus, this Choquet expected utility theory is an alternative to the subjective expected utility theory which is not in contradiction to the typical behavior in the Ellsberg experiment.

1.2. Uncertainty and uncertainty aversion

However, the Choquet expected utility theory has a serious disadvantage. A satisfactory interpretation of a decision maker's capacity does not exist yet. But it is widely assumed that the capacity incorporates two elements, the decision maker's subjectively perceived uncertainty and his individual degree of uncertainty aversion.

Epstein and Zhang (2001) propose a definition of subjective uncertainty which can be applied to a very broad class of preferences including those of the Choquet expected utility type. One can conclude from a decision maker's preferences whether he perceives an event as uncertain or not according to their definition. However, it is not clear if it is really a definition of what is intuitively meant by uncertainty, i.e. a subjective feeling not to know the true probability of the event. Epstein and Zhang's definition implies that a decision maker who assigns a probability to every event[2] does not perceive any uncertainty at all. But intuition suggests that even a decision maker who assigns a probability to every event may have doubt that his probability judgement is correct, i.e. may perceive uncertainty. Even an expected utility maximizer may perceive uncertainty, i.e. may think not to know the true probabilities of the events, if he is uncertainty neutral. Moreover, it is desirable to determine whether one of two subjectively uncertain events is perceived as more or as less uncertain than the other. Epstein and Zhang's definition only provides a distinction between uncertain events and not uncertain events.

Kelsey and Nandeibam (1998), Ghiradato and Marinacci (2002) define the term "more uncertainty averse than". Their definitions in principle imply that some decision maker 1 is more uncertainty averse than some other decision maker 2, both being Choquet expected utility maximizer, if 1's capacity (v^1) is not higher than 2's capacity ($v^2(E) \geq v^1(E)$ for all events E). However, this notion is not absolutely intuitive. Even if a decision maker is

[2] i.e. who is probabilistic sophisticated, see Machina and Schmeidler (1992).

very uncertainty averse, he might have a very high capacity because he does not perceive any uncertainty. Epstein (1999) pays attention to this point. He assumes an exogenously given set of events which are not uncertain. His definition of "more uncertainty averse than" implies that a Choquet expected utility maximizing decision maker 1 is more uncertainty averse than another Choquet expected utility maximizing decision maker 2, if 1's capacity on uncertain events is not higher than 2's capacity on uncertain events. However, decision maker 1 may have a lower capacity on uncertain events than decision maker 2 just because he is less informed about the probabilities of the uncertain events.

In this article it is shown how both a measure of subjectively perceived uncertainty for every event and the decision maker's uncertainty aversion can be unequivocally concluded from his preferences, if his preferences satisfy Schmeidler's (1989) axioms for Choquet expected utility preferences and some further conditions. To achieve this aim we have to consider not only the prior preference relation of the decision maker, but also some of his updated preference relations to get enough information about the decision maker. Thus we also have to deal with the problem of updating Choquet expected utility preferences. The basic idea of this approach can be explained by means of the Ellsberg experiment:

Assume that we know that a decision maker in the Ellsberg experiment is uniquely characterized by the capacity v on $\{r, w, b\}$ given in Example 1.[3] Moreover, assume that we know that he is uniquely characterized by the capacity $v_{\{r,w\}}$ on $\{r, w\}$ with $v_{\{r,w\}}(r) = 5/12$ and $v_{\{r,w\}}(w) = 1/4$ after he receives the information $B = \{r, w\}$, i.e. the information that the ball drawn from the urn will not be black. We then ask whether it is possible to write these two capacities v and $v_{\{r,w\}}$ as functions of (i) a probability measure π on $\{r, w, b\}$, (ii) a lower bound capacity v^L $(\leq \pi)$ on $\{r, w, b\}$[4] and a number $\alpha \in [0, 1]$ such that $v = \alpha v^L + (1-\alpha)\pi$ and $v_{\{r,w\}} = \alpha v^L_{\{r,w\}} + (1-\alpha)\pi_{\{r,w\}}$. Here, $\pi_{\{r,w\}}$ is the updated probability measure on $\{r, w\}$ derived from π by Bayes' rule and $v^L_{\{r,w\}}$ is the updated lower bound on $\{r, w\}$ derived from v^L by Jaffray's (1992) rule

$$v^L_B(E) = \frac{v^L(E)}{v^L(E) + 1 - v^L(E \cup B^c)}, E \subseteq B$$

It is very easy to check that v and $v_{\{r,w\}}$ can indeed be written as functions in the way described above of the probability measure:

Example 3.

$$\pi(\{r\}) = \pi(\{w\}) = \pi(\{b\}) = \tfrac{1}{3}$$
$$\pi(\{r, w\}) = \pi(\{r, b\}) = \pi(\{w, b\}) = \tfrac{2}{3}$$
$$\pi(\{r, w, b\}) = 1, \pi(\emptyset) = 0$$

the lower bound capacity:

[3] The information on the decision maker revealed by his behaviour in the Elsberg experiment is not sufficient to derive a unique capacity characterising the decision maker. However, it is in principle possible to derive a unique capacity from a decision maker's preferences (see for example Schmeidler, 1989).

[4] See Definition 7 below.

Example 4.

$$v^L(\{r\}) = \tfrac{1}{3},\ v^L(\{w\}) = v^L(\{b\}) = 0$$
$$v^L(\{r, w\}) = v^L(\{r, b\}) = \tfrac{1}{3},\ v^L(\{w, b\}) = \tfrac{2}{3}$$
$$v^L(\{r, w, b\}) = 1,\ v^L(\emptyset) = 0$$

and $\alpha = 1/2$. If v and $v_{\{r,w\}}$ cannot be written as functions in the way described above of any other probability measure, lower bound and number in $[0, 1]$, i.e. if we have uniqueness, then we suggest the following interpretation: π is the decision maker's subjective probability judgement and the lower bound v^L gives his subjective uncertainty: If $v^L(A) + v^L(A^c) > v^L(E) + v^L(E^c)$, then the decision maker perceives the event A as less uncertain than the event E. And α is a parameter of the decision maker's uncertainty aversion: If $\alpha = 0$, his capacity is a probability measure ($v = \pi$) and he is uncertainty neutral. If $\alpha = 1$, his capacity is equal to his lower bound ($v = v^L$) and his uncertainty aversion is maximal. If $\alpha = 1/2$ as in the example above, his uncertainty aversion is between these two extremes. We assume that this parameter does not change when preferences are updated.

The aim of this article is to describe preferences from which a unique probability measure, lower bound and parameter of uncertainty aversion in the sense described above can be derived. Note that the reason way we need to consider updating in this article is that we cannot achieve uniqueness of the probability measure, the lower bound and the parameter of uncertainty aversion without information on updated preferences. If a prior capacity v can at all be written as a linear combination of a probability measure and a lower bound, it can (almost) always be written as a linear combination of many other probability measures and lower bounds as well. However, this is true only in the setting of our paper. In a quite different approach, Gajdos et al. (2004) show how a unique parameter of uncertainty aversion α can be derived from preferences without information on updating. Their approach differs from ours in that these authors assume that a set of possible prior probability measures Π and a reference probability measure $\pi \in \Pi$ are objectively given. In this setting, preferences of the multiple prior type as introduced by Gilboa and Schmeidler (1989) are axiomatised. The unique set of priors characterising the preferences is the set $\{p^\alpha | p^\alpha = \alpha p + (1 - \alpha)\pi,\ p \in co(\Pi)\}$. Here, $co(\Pi)$ is the closed and convex hull of Π and α is a unique number in $[0, 1]$ that can be interpreted as a parameter of uncertainty aversion. We will discuss this approach and how it is related to ours in Section 7.

From the example above, it is clear that we restrict our analysis to a certain class of preferences and updating procedures, i.e. we do not define subjective uncertainty and uncertainty aversion for any preference relation. The analysis is restrictive in three ways: Firstly, we consider only preferences of the Choquet expected utility type. Secondly, not every capacity can be written as a linear combination of a probability measure and a lower bound. For example, so called concave capacities cannot, but all convex and some other capacities can. I.e. we do not consider all preferences of the Choquet expected utility type. Thirdly, we assume that preferences are updated in a certain way. The following example may illustrate the last point:

A decision maker in the context of the Ellsberg experiment is indifferent between a bet on red, a bet on white and a bet on black. He is also indifferent between a bet on red or white, a bet on red or black and a bet on white or black. Finally, after receiving the information $\{r, w\}$, he is indifferent between a bet on red, a bet on white and a certain gain of US\$ x, after receiving the information $\{r, b\}$, he is indifferent between a bet on red, a bet on black and a certain gain of US\$ x, but after receiving the information $\{w, b\}$, he is not indifferent between a bet on white and a bet on black. The prior and updated capacities describing these preferences need to fulfill $v(\{r\}) = v(\{w\}) = v(\{b\}) \equiv A$, $v(\{r, w\}) = v(\{r, b\}) = v(\{w, b\}) \equiv B$, $v_{\{r,w\}}(\{r\}) = v_{\{r,w\}}(\{w\}) = v_{\{r,b\}}(\{r\}) = v_{\{r,b\}}(\{b\}) \equiv C$, $v_{\{w,b\}}(\{w\}) \neq v_{\{w,b\}}(\{b\})$. If there were a lower bound v^L, a probability measure π and a number $\alpha \in [0, 1]$ such that $v = \alpha v^L + (1-\alpha)\pi$ and $v_B = \alpha v^L_B + (1-\alpha)\pi_B$ for $B = \{r, w\}$, $B = \{r, b\}$ and $B = \{w, b\}$, then

$$\alpha[v^L(\{r\}) + v^L(\{w, b\})] + 1 - \alpha = \alpha[v^L(\{w\}) + v^L(\{r, b\})] + 1 - \alpha = A + B$$

and

$$\alpha\left[\frac{v^L(\{r\})}{v^L(\{r\}) + 1 - v^L(\{r, b\})} + \frac{v^L(\{w\})}{v^L(\{w\}) + 1 - v^L(\{w, b\})}\right] + 1 - \alpha = 2C$$

Since $v(\{w\}) = v(\{b\})$ and $v_{\{w,b\}}(\{w\}) \neq v_{\{w,b\}}(\{b\})$, the preferences are not of the expected utility type, i.e. $\alpha > 0$. With this, we easily get from the above

$$\alpha[v^L(\{r\}) + v^L(\{w\})] = \frac{(2C + \alpha - 1)(1 - A - B)}{1 - 2C} \equiv k$$

Analogously, we get $\alpha[v^L(\{r\})+v^L(\{b\})] \equiv k$, i.e. $v^L(\{w\}) = v^L(\{b\})$. Because of $v(\{w\}) = v(\{b\})$, we get $\pi(\{w\}) = \pi(\{b\})$. With $v(\{r, w\}) = v(\{r, b\})$, it follows that $v^L(\{r, w\}) = v^L(\{r, b\})$, thus $v_{\{w,b\}}(\{w\}) = v_{\{w,b\}}(\{b\})$. This is in contradiction to one of our assumption we were starting with.[5]

In Section 2 we define the two main mathematical concepts of the Choquet expected utility theory, the capacity and the Choquet integral. The following sections deal with lists $(\succsim_0, \succsim_1, \ldots, \succsim_n)$ containing a prior preference relation $\succsim_0$ and an updated preference relation $\succsim_i$ for n different information $i = 1, \ldots, n$. Section 3 presents well known axioms for $(\succsim_0, \succsim_1, \ldots, \succsim_n)$ implying that $\succsim_i$ $(i = 0, 1, \ldots, n)$ is an expected utility preference relation with associated probability measure π_i and π_i is derived from π_0 by Bayes rule. In Section 4 we introduce axioms for $(\succsim_0, \succsim_1, \ldots, \succsim_n)$ implying that $\succsim_i$ $(i = 0, 1, \ldots, n)$ is a Choquet expected utility preference relation with the associated capacity v^L_i being a lower bound and v^L_i is derived from v^L_0 by Jaffray's rule. Section 5 combines the results gained in Section 3 and Section 4. We present axioms for $(\succsim_0, \succsim_1, \ldots, \succsim_n)$ implying that (1) $\succsim_i$ is a Choquet expected utility preference relation with associated capacity v_i and (2) there is a unique number $\alpha \in [0, 1]$, a unique list $(\pi_0, \pi_1, \ldots, \pi_n)$ of probability measures with π_i derived from π_0 by Bayes rule and a unique list $(v^L_0, v^L_1, \ldots, v^L_n)$ of lower bounds with v^L_i derived from v^L_0 by Jaffray's rule satisfying $v_i(E) = \alpha v^L_i(E) + (1-\alpha)\pi_i(E)$ for all events E and all $i = 0, 1, \ldots, n$. In Section 6, we will discuss an example of preferences that satisfy all of our axioms but the main one. Finally, the paper by Gajdos, Tallon and

[5] Note that in general the updating procedure discussed in this paper is relatively flexible. For example both $v(\{r\}) > v(\{w\})$, $v(\{r, b\}) > v(\{w, b\})$, $v_{\{r,w\}}(\{r\}) > v_{\{r,w\}}(\{w\})$ and $v(\{r\}) > v(\{w\})$, $v(\{r, b\}) > v(\{w, b\})$, $v_{\{r,w\}}(\{r\}) < v_{\{r,w\}}(\{w\})$ is possible within our assumptions, the latter for example with $v_L(\{r\}) = 1/8$, $v_L(\{w\}) = 1/4$, $v_L(\{w, b\}) = 1/2$, $\pi(\{r\}) = 1/2$, $\pi(\{w\}) = 1/3$, $\pi(\{b\}) = 1/6$ and $\alpha = 14/25$.

Vergnaud is discussed in Section 7. We work with the Anscombe and Aumann (1963) frame which has been used by Schmeidler (1989).

2. Mathematical preliminaries

In this section the basic concepts of the Choquet expected utility theory, the capacity and the Choquet integral, are introduced as pure mathematical concepts. We start with

Definition 5. Let M be some set and denote by 2^M the set of all subsets of M. A capacity on M is a function $v : 2^M \to [0, 1]$ satisfying

(i) $v(\varnothing) = 0$, $v(M) = 1$
(ii) $A \subseteq B \Rightarrow v(A) \leq v(B)$.

The set functions on the set $\{r, w, b\}$ presented in the Examples 1, 3 and 4 are capacities according to this definition. Moreover, every probability measure is a capacity, but not vice versa:

Definition 6. Let M be some set. A capacity π on M is called a probability measure, if and only if

$$A \cap B = \emptyset \Rightarrow \pi(A) + \pi(B) = \pi(A \cup B)$$

Of special interest in this article is a special group of capacities, so called lower bounds:

Definition 7. Let M be some set. A capacity v on M is called a lower bound, if and only if there exists a set Π of probability measures on M such that

$$v(A) = \min\{\pi(A) | \pi \in \Pi\}$$

for all $A \subseteq M$.

Notable is the relation between lower bounds on the one hand and convex and superadditive capacities on the other. A capacity v on some set M is called convex, if and only if $v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$ for all $A, B \subseteq M$. It is supperadditive if and only if $v(A) + v(B) \leq v(A \cup B)$ for all $A, B \subseteq M$, $A \cap B = \emptyset$. Obviously, every convex capacity is a superadditive capacity, but not vice versa.

Proposition 8. *Every convex capacity is a lower bound, but not every lower bound is a convex capacity.*

Proposition 9. *Every lower bound is a superadditive capacity, but not every superadditive capacity is a lower bound.*

Given a capacity v on some set M and a random variable $\hat{u}$ we define something like an expected value, the so called Choquet integral:

Definition 10. Let v be a capacity on some set M and $\hat{u} : M \to \mathbb{R}$ be a function. The Choquet integral of $\hat{u}$ given v is defined by

$$C_v(\hat{u}) = \int_{-\infty}^{0} [v(\{m|\hat{u}(m) \geq t\}) - 1]\mathrm{d}t + \int_{0}^{\infty} [v(\{m|\hat{u}(m) \geq t\})]\mathrm{d}t$$

In Example 2, $C_v(u \circ f^i)$ is the Choquet integral of a function $u \circ f^i : \{r, w, b\} \to \mathbb{R}$ given a capacity v on $\{r, w, b\}$, where $f^i : \{r, w, b\} \to \{\text{US\$ 0}, \text{US\$ 100}\}$ is a function with $f^i(c)$ the gain of bet (i) $(i = 1, \ldots, 4)$ in the Ellsberg experiment, if the color of the ball being drawn is c. If the capacity v is a probability measure, then the Choquet integral $C_v(\hat{u})$ is the ordinary expected value of $\hat{u}$ given v. This can easily be shown.

3. Expected utility theory and updating

The frame we are using consists of a set of states S_0, some sequence $S_1, \ldots, S_n$ of nonempty subsets $S_i \subset S_0$, $i = 1, \ldots, n$, and a nonempty set of consequences X. Both S_0 and X are assumed to be finite. Moreover, we have the set Y of all probability measures on X, i.e. so called roulette lotteries. For the sake of simplicity, $x \in X$ also denotes the element in Y which assigns probability 1 to x.

Preferences will be defined on sets of acts. An act on S_i, $i \in \{0, 1, \ldots, n\}$, is a function $f_i : S_i \to Y$. The set of all acts on S_i is F_i. Given two acts f_i, g_i on S_i, the act $\lambda f_i + (1-\lambda)g_i$ for some $\lambda \in [0, 1]$ is defined by $(\lambda f_i + (1-\lambda)g_i)(s) = \lambda f_i(s) + (1-\lambda)g_i(s)$ for all $s \in S_i$. For the sake of simplicity $f_i(s)$ denotes both the probability measure assigned to state s given act f_i and the act which yields the probability measure $f_i(s)$ for all $s \in S_i$, i.e. a constant act. Analogously, y denotes both an element in Y and the constant act which yields y in all states. As usual $(f_i, E; g_i, S_i/E)$ denotes the act which yields the same probability measure on X as act f_i, if $s \in E$, and the same probability measure on X as act g_i, if $s \in S_i/E$.

We now consider some preference relation on F_i for some $i \in \{0, 1, \ldots, n\}$ and the following properties:

(i) A preference relation $\succsim_i$ on F_i satisfies weak order, if: (a) For all $f_i, g_i \in F_i$: $f_i \succsim_i g_i$ or $g_i \succsim_i f_i$. (b) For all $f_i, g_i, h_i \in F_i$: $(f_i \succsim_i g_i$ and $g_i \succsim_i h_i) \Rightarrow f_i \succsim_i h_i$.
(ii) $\succsim_i$ satisfies independence, if: For all $f_i, g_i, h_i \in F_i$ and all $\lambda \in]0, 1[$: $f_i \succ_i g_i \Rightarrow \lambda f_i + (1-\lambda)h_i \succ_i \lambda g_i + (1-\lambda)h_i$.
(iii) $\succsim_i$ satisfies continuity, if: For all $f_i, g_i, h_i \in F_i$: If $f_i \succ_i g_i$ and $g_i \succ_i h_i$, then there are $\lambda, \mu \in]0, 1[$ such that $\lambda f_i + (1-\lambda)h_i \succ_i g_i$ and $g_i \succ_i \mu f_i + (1-\mu)h_i$.
(iv) $\succsim_i$ satisfies monotonicity, if: For all $f_i, g_i \in F_i$: If $f_i(s) \succsim_i g_i(s)$ for all $s \in S_i$, then $f_i \succsim_i g_i$.
(v) $\succsim_i$ satisfies nondegeneracy, if: There are $f_i, g_i \in F_i$ such that $f_i \succ_i g_i$.

The following theorem is due to Anscombe and Aumann (1963) and the proof can be found in Fishburn (1970)[6]:

[6] Proof of theorem 13.3. See also Schmeidler (1989) for the relation between (i), (iv) and the property of strict monotonicity used in Fishburn (1970).

Theorem 11. *A preference relation $\succsim_i$ on F_i for some $i \in \{0, 1, \ldots, n\}$ satisfies* (i), (ii), (iii), (iv) *and* (v), *if and only if there are* (1) *an affine*[7] *function $u : Y \to \mathbb{R}$ unique up to positive linear transformations and* (2) *a unique probability measure π_i on S_i such that for all $f_i, g_i \in F_i$*

$$f_i \succsim_i g_i \Leftrightarrow C_{\pi_i}(u \circ f_i) \geq C_{\pi_i}(u \circ g_i)$$

We now consider a list of preference relations on F_i for all $i \in \{0, 1, \ldots, n\}$ and the following property:

(u-i) A list $(\succsim_0, \succsim_1, \ldots, \succsim_n)$ of preference relations $\succsim_i$ on F_i for all $i \in \{0, 1, \ldots, n\}$ satisfies Bayesian updating, if: for all $f_0, g_0, h_0 \in F_0$ and $f_i, g_i \in F_i$ with $f_i(s) = f_0(s)$ and $g_i(s) = g_0(s)$ for all $s \in S_i$ and all $i = 1, \ldots, n$: $f_i \succsim_i g_i \Leftrightarrow (f_0, S_i; h_0, S_0 \setminus S_i) \succsim_0 (g_0, S_i; h_0, S_0 \setminus S_i)$.

The theorem we get is:

Theorem 12. *A list $(\succsim_0, \succsim_1, \ldots, \succsim_n)$ of preference relations $\succsim_i$ on F_i for $i = 0, 1, \ldots, n$ satisfies* (u-i) *and is such that $\succsim_i$ satisfies* (i), (ii), (iii), (iv) *and* (v) *for all $i \in \{0, 1, \ldots, n\}$, if and only if there are* (1) *an affine function $u : Y \to \mathbb{R}$ unique up to positive linear transformations and* (2) *a unique probability measure π* on *S_0 such that for all $f_i, g_i \in F_i$*

$$f_i \succsim_i g_i \Leftrightarrow C_{\pi_i}(u \circ f_i) \geq C_{\pi_i}(u \circ g_i)$$

with π_i defined by $\pi_i(E) = \pi(E)/\pi(S_i)$ for all $E \subseteq S_i$ (for $i = 0, 1, \ldots, n$).

If a list $(\succsim_0, \succsim_1, \ldots, \succsim_n)$ satisfies (u-i) and is such that $\succsim_0$ satisfies (i), (ii), (iii), (iv) and (v), then $\succsim_i$ does not necessarily satisfy (i), (ii), (iii), (iv) and (v) for all $i \in \{1, \ldots, n\}$. If for example $\succsim_0$ is associated with a probability measure $\pi_0(= \pi)$ with $\pi_0(S_i) = 0$ for some i, then (u-i) would imply $f_i \sim_i g_i$ for all $f_i, g_i \in F_i$, thus $\succsim_i$ would not satisfy (v). Therefore, if some list $(\succsim_0, \succsim_1, \ldots, \succsim_n)$ satisfies (u-i) and is such that $\succsim_i$ satisfies (i), (ii), (iii), (iv) and (v) for all $i \in \{0, 1, \ldots, n\}$, then the probability measure π associated with this list satisfies $\pi(S_i) > 0$ for all $i \in \{0, 1, \ldots, n\}$. This is implicitly mentioned in Theorem 12, since otherwise π_i would not be defined for some $i \in \{1, \ldots, n\}$.

4. Choquet expected utility and lower bound capacities

The following concept of comonotonicity is crucial in Choquet expected utility theory:

Definition 13. Two acts $f_i, g_i \in F_i$ are called comonotonic given the preference relation $\succsim_i$on F_i, if for no $s, s' \in S_i$: $f_i(s) \succ_i f_i(s')$ and $g_i(s') \succ_i g_i(s)$.

In order to obtain the Choquet expected utility theory, we just have to replace the independence property (ii) by the following property:

[7] Affinity of u means $u(\lambda y + (1 - \lambda)y') = \lambda u(y) + (1 - \lambda)u(y')$ for every $\lambda \in]0, 1[$, implying $u(y) = \sum_{x \in X} y(x)u(x)$.

(vi) $\succsim_i$ satisfies comonotonic independence, if: For all pairwise comonotonic acts f_i, g_i, $h_i \in F_i$ and all $\lambda \in]0,1[$: $f_i \succ_i g_i \Rightarrow \lambda f_i + (1-\lambda)h_i \succ_i \lambda g_i + (1-\lambda)h_i$.

The following theorem is presented and proved in Schmeidler (1989):

Theorem 14. *A preference relation $\succsim_i$ on F_i for some $i \in \{0, 1, \ldots, n\}$ satisfies* (i), (iii), (iv), (v) *and* (vi), *if and only if there are* (1) *an affine function $u : Y \to \mathbb{R}$ unique up to positive linear transformations and* (2) *a unique capacity v_i* on S_i *such that for all f_i, $g_i \in F_i$*

$$f_i \succsim_i g_i \Leftrightarrow C_{v_i}(u \circ f_i) \geq C_{v_i}(u \circ g_i)$$

Next we compare two preference relations on the same set of acts:

Definition 15. A preference relation $\succsim_i'$ on F_i is called less constancy-loving than another preference relation $\succsim_i$ on F_i if and only if

$$y \succsim_i' f_i \Rightarrow y \succsim_i f_i \text{ and } y \succ_i' f_i \Rightarrow y \succ_i f_i$$

for all $y \in Y$ and $f_i \in F_i$.

It has been argued that "less constancy-loving" means "less uncertainty averse".[8] We reject this interpretation. Assume that a decision maker P has preferences $\succsim_i$ and decision maker P' has preferences $\succsim_i'$ and that $\succsim_i'$ is less constancy-loving than $\succsim_i$. This may be because P' perceives less uncertainty than P, though P' may be more uncertainty averse than P.[9]

Let $e(\succsim_i)$ denote the set of all preference relations on F_i which (1) satisfy (i), (ii), (iii), (iv) and (v) and (2) are less constancy-loving than the preference relation $\succsim_i$ on F_i. Let $F_i^E(\succsim_i) \equiv \{f_i \in F_i | \; f_i = (y, E; y', S_i \setminus E)$ for some $y, y' \in Y$ with $y \succsim_i y'\}$. Obviously, if two acts f_i, g_i are in $F_i^E(\succsim_i)$, then they are comonotonic given $\succsim_i$. Moreover, all constant acts are in $F_i^E(\succsim_i)$.

(vii) $\succsim_i$ satisfies expected utility relatedness, if: For every $E \subseteq S_i$ there is a $\succsim_i' \in e(\succsim_i)$ such that $\succsim_i$ and $\succsim_i'$ agree on $F_i^E(\succsim_i)$.

A decision maker who's preference relation $\succsim_i$ satisfy property (vii) reveals that he considers a set $e(\succsim_i)$ of preference relations somehow sensible. But he does not know which preference relation in $e(\succsim_i)$ he should choose, since he faces uncertainty. He finally chooses a preference relation $\succsim_i$ which is not in $e(\succsim_i)$. But it is closely connected with $e(\succsim_i)$, since (1) every preference relation in $e(\succsim_i)$ is less constancy-loving than $\succsim_i$and (2) the restriction of $\succsim_i$ on a set $F_i^E(\succsim_i)$ agrees with some preference relation in $e(\succsim_i)$ for every $E \subseteq S_i$ and is in this sense justified by a preference relation which is deemed sensible.

[8] See Kelsey and Nandeibam (1998), Ghiradato and Marinacci (2002).

[9] This view is partly supported by Epstein (1999), how associates less constance loving with less uncertainty averse only if the two preference relation $\succsim_i$ and $\succsim_i'$ which are compared reveal at least up to a certain degree the same level of uncertainty.

Theorem 16. *A preference relation* $\succsim_i$ *on* F_i *for some* $i \in \{0, 1, \dots, n\}$ *satisfies* (i), (iii), (iv), (v), (vi) *and* (vii), *if and only if there are* (1) *an affine function* $u : Y \to \mathbb{R}$ *unique up to positive linear transformations and* (2) *a unique lower bound* v_i on S_i *such that for all* $f_i, g_i \in F_i$

$$f_i \succsim_i g_i \Leftrightarrow C_{v_i}(u \circ f_i) \geq C_{v_i}(u \circ g_i)$$

Let $E(\succsim_0, \succsim_1, \dots, \succsim_n)$ be the set of all lists of preference relations $(\succsim_0', \succsim_1', \dots, \succsim_n')$ which satisfy (1) property (u-i) and (2) $\succsim_i' \in e(\succsim_i)$ for all $i \in \{0, 1, \dots, n\}$. That means, if some preference relation $\succsim_i'$ is part of some list $(\succsim_0', \succsim_1', \dots, \succsim_n')$ in $E(\succsim_0, \succsim_1, \dots, \succsim_n)$, then $\succsim_i'$satisfy (i), (ii), (iii), (iv) and (v) and is less constancy-loving than $\succsim_i$. The set $E(\succsim_0, \succsim_1, \dots, \succsim_n)$ is the natural generalization of the set $e(\succsim_i)$ on lists of preference relations.

(u-ii) $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfies expected utility relatedness, if: For every $E \subseteq S_i$ and $i \in \{0, 1, \dots, n\}$ there is a $(\succsim_0', \succsim_1', \dots, \succsim_n') \in E(\succsim_0, \succsim_1, \dots, \succsim_n)$ such that $\succsim_i$ and $\succsim_i'$ agree on $F_i^E(\succsim_i)$.

Clearly, if $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfies (u-ii), then $\succsim_i$ satisfies (vii) for all $i \in \{0, 1, \dots, n\}$.

(u-iii) $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfies completeness, if: $\succsim_0^* \in e(\succsim_0) \Rightarrow$ there is a $(\succsim_0', \succsim_1', \dots, \succsim_n') \in E(\succsim_0, \succsim_1, \dots, \succsim_n)$ such that $\succsim_0' = \succsim_0^*$.

If some list $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfies (u-iii), then every preference relation $\succsim_0^*$ which satisfies (i), (ii), (iii), (iv) and (v) and which is less constancy-loving then $\succsim_0$ appears in some list in $E(\succsim_0, \succsim_1, \dots, \succsim_n)$.

Theorem 17. *A list* $(\succsim_0, \succsim_1, \dots, \succsim_n)$ *of preference relations* $\succsim_i$ *on* F_i *for* $i = 0, 1, \dots, n$ *satisfies* (u-ii) *and* (u-iii) *and is such that* $\succsim_i$ *satisfies* (i), (iii), (iv), (v) *and* (vi) *for alli* $\in \{0, 1, \dots, n\}$, *if and only if there are* (1) *an affine function* $u : Y \to \mathbb{R}$ *unique up to positive linear transformations and* (2) *a unique lower bound* v *on* S_0 *with* $v(S_i) > 0$ *for all* $i \in \{1, \dots, n\}$ *such that for all* $f_i, g_i \in F_i$

$$f_i \succsim_i g_i \Leftrightarrow C_{v_i}(u \circ f_i) \geq C_{v_i}(u \circ g_i)$$

with v_i *defined by* $v_i(E) = v(E)/v(E) + 1 - v(E \cup S_0 \backslash S_i)$ *for all* $E \subseteq S_i$ *(for* $i = 0, 1, \dots, n$).

The updating rule for lower bounds stated in Theorem 17 has been proposed by Jaffray (1992).[10] The proof of the theorem makes direct use of Jaffray's argumentation.

In this section we have presented properties of a very extreme list of preference relations $(\succsim_0, \succsim_1, \dots, \succsim_n)$. It is extreme in so far as we assume that the decision maker deviates from the set $E(\succsim_0, \succsim_1, \dots, \succsim_n)$ very far into the direction of constance love. The other extreme is given, if he just chooses one of the lists in $E(\succsim_0, \succsim_1, \dots, \succsim_n)$, i.e. a list with the properties stated in Theorem 12, instead of deviating from $E(\succsim_0, \succsim_1, \dots, \succsim_n)$. In the next section we present properties of lists between these two extremes.

[10] Some other updating rules for capacity are proposed in Gilboa and Schmeidler (1993).

5. Uncertainty and uncertainty aversion revealing preferences

We now combine the lists of preference relations described in the previous two sections to define one more list of preference relations. We first need a definition:

Definition 18. Let $\succsim_i^1, \succsim_i, \succsim_i^2$ be preference relations on F_i . If there exists an act $f_i \in F_i$ such than $f_i \sim_i^1 y_1$, $f_i \sim_i y$ and $f_i \sim_i^2 y_2$ for some $y_1, y, y_2 \in Y$ (neither $y_1 \sim_i y$ nor $y_1 \sim_i y_2$ nor $y \sim_i y_2$), then the ordered triple (y_1, y, y_2) is called a description of $\succsim_i$ relative to $\succsim_i^1$ and $\succsim_i^2$.

The main property in this section is

(u-iv) $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfies constant relative description, if: There exists (1) a list $(\succsim_0^E, \succsim_1^E, \dots, \succsim_n^E)$ which satisfies (u-i) and is such that $\succsim_i^E$ satisfies (i), (ii), (iii), (iv) and (v) for all $i \in \{0, 1, \dots, n\}$ and (2) a list $(\succsim_0^L, \succsim_1^L, \dots, \succsim_n^L)$ which satisfies (u-ii) and (u-iii) and is such that $\succsim_i^L$ satisfies (i), (iii), (iv), (v) and (vi) for all $i \in \{0, 1, \dots, n\}$, such that: (I) For all $i \in \{0, 1, \dots, n\}$: $\succsim_i^E$ is less constancy-loving than $\succsim_i$ and $\succsim_i$ is less constancy-loving than $\succsim_i^L$, (II) for all $f_i, g_i \in F_i$ and all $i \in \{0, 1, \dots, n\}$: $(f_i \sim_i^E g_i$ and $f_i \sim_i^L g_i) \Rightarrow f_i \sim_i g_i$ and (III) There is an ordered triple (y_E, y, y_L) which is a description of $\succsim_i$ relative to $\succsim_i^E$ and $\succsim_i^L$ for all $i \in \{0, \dots, n\}$ for which not $\succsim_i^E = \succsim_i = \succsim_i^L$.

If a list $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfies (u-iv), then it is somehow located between two extremes, a list $(\succsim_0^E, \succsim_1^E, \dots, \succsim_n^E)$ which is not uncertainty averse at all and a list $(\succsim_0^L, \succsim_1^L, \dots, \succsim_n^L)$ which is extremely uncertainty averse. The nearer $(\succsim_0, \succsim_1, \dots, \succsim_n)$ is located to $(\succsim_0^E, \succsim_1^E, \dots, \succsim_n^E)$, the less uncertainty averse is it. Part (I) of (u-iv) makes sure that there is a probability measure π_i and a lower bound v_i^L on S_i such that the capacity characterizing the preference relation $\succsim_i$ is between both, i.e. $\pi_i(E) \geq v_i(E) \geq v_i^L(E)$ for all $E \subseteq S_i$ (see Lemma 20 in the appendix). Part (I) and (II) together imply that there is a $\alpha_i \in [0, 1]$ such that $v_i(E) = \alpha_i v_i^L(E) + (1-\alpha)\pi_i(E)$. On top of part (I) and (II), part (III) ensures that $\alpha_i = \alpha$ for all $i \in \{0, \dots, n\}$.

We finally need a technical property:

(u-v) $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfies richness, if: $n \geq 3$ and there is a partition (A, B, C) of S_0 with $A, B, C \neq \emptyset$, such that (1) $S_1 = A \cup B$, $S_2 = A \cup C$, $S_3 = B \cup C$, (2) for all $y_1, y_2, y_3 \in Y$ with $y_1, y_2 \succ_0 y_3$: $(y_1, A; y_3, S_0 \setminus A) \sim_0 (y_2, A; y_3, S_0 \setminus A)$, (3) not for all $y_1, y_2, y_3 \in Y$ with $y_1 \succ_0 y_2, y_3$: $(y_1, B \cup C; y_2, A) \sim_0 (y_1, B \cup C; y_3, A)$ and (4) for all $\lambda_1, \lambda_2, \lambda_3 \in [0, 1]$ with $1 - \lambda_1 = \lambda_1 - \lambda_2 - \lambda_3$ and $y_1, y_2 \in Y$ with $y_1 \succ_0 y_2$: If $(y_1, B \cup C; y_2, A) \sim_0 \lambda_1 y_1 + (1 - \lambda_1)y_2$, then not both $(y_1, B; y_2, A \cup C) \sim_0 \lambda_2 y_1 + (1 - \lambda_2)y_2$ and $(y_1, C; y_2, A \cup B) \sim_0 \lambda_3 y_1 + (1 - \lambda_1)y_3$.

Part (2) of the property is the most restrictive. It implies that the capacity v_0 associated with $\succsim_0$ satisfies $v_0(A) = 0$ for some $A \subseteq S_0$. Part (3) guarantees that $v_0(B \cup C) < 1$, i.e. v_0 is not a probability measure. Finally, because of part (4) we get $1 - v_0(B \cup C) \neq v_0(B \cup C) - v_0(B) - v_0(C)$. Property (u-v) is helpful, though in many cases not necessary to get uniqueness of a number α, a lower bound v^L and a probability measure π as described in the following theorem, which is the main theorem of this article:

Theorem 19. *A list* $(\succsim_0, \succsim_1, \dots, \succsim_n)$ *of preference relations* $\succsim_i$ *on* F_i *for* $i = 0, 1, \dots, n$ *satisfies* (u-iv) *and* (u-v) *and is such that* $\succsim_i$ *satisfies* (i), (iii), (iv), (v) *and* (vi) *for all* $i \in \{0, 1, \dots, n\}$, *if and only if there are* (1) *an affine function* $u : Y \to \mathbb{R}$ *unique up to positive linear transformations,* (2) *a unique number* $\alpha \in]0, 1]$, (3) *a unique lower bound* v^L on S_0 *with* $v^L(S_i) > 0$ *for all* $i \in \{1, \dots, n\}$ *and* (4) *a probability measure* π on S_0 *with* $\pi(E) \geq v^L(E)$ *for all* $E \subseteq S_0$ *and unique if and only if* $\alpha < 1$, *such that for all* $f_i, g_i \in F_i$

$$f_i \succsim_i g_i \Leftrightarrow C_{v_i}(u \circ f_i) \geq C_{v_i}(u \circ g_i)$$

with $v_i(E) = \alpha v_i^L(E) + (1 - \alpha)\pi_i(E)$, $v_i^L(E) = \frac{v^L(E)}{v^L(E)+1-v^L(E\cup S_0 \setminus S_i)}$ *and* $\pi_i(E) = \pi(E)/\pi(S_i)$ *for all* $E \subseteq S_i$ *and all* $i \in \{0, 1, \dots, n\}$, $n \geq 3$, $v_0(A) = 0$, $v_0(B \cup C) < 1$, $1 - v_0(B \cup C) \neq v_0(B \cup C) - v_0(B) - v_0(C)$ *for some partition* (A, B, C) *of* S_0 *with* $A, B, C \neq \emptyset$, $S_1 = A \cup B$, $S_2 = A \cup C$ *and* $S_3 = B \cup C$.

The interpretation of the unique number α and the unique lower bound v^L has already been given in Section 1.2: The lower bound is a measure of the decision maker's subjectively perceived uncertainty: If $v^L(A) + v^L(S_0 \setminus A) < v^L(B) + v^L(S_0 \setminus B)$, then the decision maker perceives the event A as more uncertain than the event B. And α is the parameter of the decision maker's individual uncertainty aversion. If α is close to 0, the decision maker is almost uncertainty neutral, if $\alpha = 1$, he is completely uncertainty averse.

6. An example

We are now going to discuss an example of a list of a prior and some updated preference relations which does not satisfy part (III) of (u-iv). We start with some general remarks on how to test whether preferences satisfy part (III) of (u-iv). Then we discuss our example.

Assume that all preference relations in $(\succsim_0, \succsim_1, \dots, \succsim_n)$ are of the Choquet expected utility type (characterized by the same utility function u) and that $(v_0, \dots v_n)$ is the related list of capacities. Moreover assume that part (I) and (II) of (u-iv) are satisfied. Then there are a lower bound v^L, a probability measure $\pi(\geq v^L)$ and a number $\alpha_i \in [0, 1]$ for all $i = 0, \dots, n$ such that $v_i = \alpha_i v_i^L + (1 - \alpha_i)\pi_i$ for all $i = 0, \dots, n$. Only if we additionally have $\alpha_i = \alpha$ for all $i = 0, \dots, n$, then part (iii) of (u-iv) is also satisfied. This is shown in the appendix. One important implication is the following: Take any two subsets $\overline{I}$ and $\hat{I}$ of $\{0, \dots, n\}$. If $v_i = \alpha v_i^L + (1 - \alpha)\pi_i$ for all $i \in \overline{I}$ has a unique solution with $\alpha = \overline{\alpha}$, $v_i = \alpha v_i^L + (1 - \alpha)\pi_i$ for all $i \in \hat{I}$ has a unique solution with $\alpha = \hat{\alpha}$ and $\overline{\alpha} \neq \hat{\alpha}$, then part (III) of (u-iv) is not satisfied.

In our example, we assume that there are five events $A, \overline{B}, \overline{C}, \hat{B}, \hat{C} \subset S_0$ such that $\overline{B} \cup \overline{C} = \hat{B} \cup \hat{C} = S_0 \setminus A$ and $\overline{B} \cap \overline{C} = \hat{B} \cap \hat{C} = \emptyset$. For technical reasons, we assume that part (1) and (2) of (u-v) are satisfied. Moreover, we assume $n = 4$, where $S_1 = A \cup \overline{B}$,

$S_2 = A \cup \overline{C}$, $S_3 = A \cup \hat{B}$ and $S_4 = A \cup \hat{C}$. The decision maker prefers a bet on $\overline{B}$ over a bet on $\hat{B}$ and a bet on $\overline{C}$ over a bet on $\hat{C}$, i.e. for any y_1, y_2 with $y_1 \succ_0 y_2$, we have $(y_1, \overline{B}; y_2, S_0 \setminus \overline{B}) \succ_0 (y_1, \hat{B}; y_2, S_0 \setminus \hat{B})$ and $(y_1, \overline{C}; y_2, S_0 \setminus \overline{C}) \succ_0 (y_1, \hat{C}; y_2, S_0 \setminus \hat{C})$. Furthermore, we assume that there are y_3, y_4, y_5 such that $(y_3, \overline{B}; y_4, S_1 \setminus \overline{B}) \prec_1 y_5 \prec_3 (y_3, \hat{B}; y_4, S_3 \setminus \hat{B})$ and that there are y_6, y_7, y_8 such that $(y_6, \overline{C}; y_7, S_2 \setminus \overline{C}) \prec_2 y_8 \prec_4 (y_6, \hat{C}; y_7, S_4 \setminus \hat{C})$. In our setting, all this requires $v_0(\overline{B}) > v_0(\hat{B})$, $v_0(\overline{C}) > v_0(\hat{C})$, $v_1(\overline{B}) < v_3(\hat{B})$ and $v_2(\overline{C}) < v_4(\hat{C})$.

In the appendix (proof of Theorem 19, part (1.2)) we show that for any partition $\{B, C\}$ of $S_0 \setminus A$, our setting requires

$$\alpha = [1 - v_{A\cup B}(B)] + \frac{[1 - v_{A\cup B}(B)]}{[1 - v_0(B \cup C)]}\alpha v^L(B)$$

and

$$\alpha = [v_0(B \cup C) - v_0(B) - v_0(C)]\frac{[1 - v_{A\cup C}(C)]}{[v_0(B \cup C) - v_{A\cup C}(C)]} + \frac{[1 - v_{A\cup C}(C)]}{[v_0(B \cup C) - v_{A\cup C}(C)]}\alpha v^L(B)$$

This is a system of two equations that are linear in the variables α and $\alpha v^L(B)$. The system normally has a unique solution for α. It is obvious that the solution for α that we get if we replace B and C by $\overline{B}$ and $\overline{C}$ is not the same as the solution for α that we get if we replace B and C by $\hat{B}$ and $\hat{C}$. Thus, the list of preferences of our example does not satisfy part (III) of (u-iv).

7. Gajdos, Tallon and Vergnaud (2004)

The paper by Gajdos, Tallon and Vergnaud is important for our model in two ways: Firstly, it axiomatizes preferences with a representation very similar to ours. And secondly, it indirecly supports our interpretation of the parameter α as a parameter of uncertainty aversion. It is therefore discussed here in more details. Let Ω be the set of all pairs (Π, π) with Π a closed set of probability distributions on S_0 and $\pi \in \Pi$. A pair (Π, π) is called a situation. Assume that decision makers have preferences $\succsim$on the set $F_0 \times \Omega$. In their Theorem 2, Gajdos, Tallon and Vergnaud describe preferences characerised by a function $u : Y \to \mathbb{R}$ unique up to positive linear transformations and a unique number $\alpha \in [0, 1]$ such that for all $f, g \in F_0$, $(\Pi_1, \pi_1), (\Pi_2, \pi_2) \in \Omega$:

$$[f, (\Pi_1, \pi_1)] \succsim [g, (\Pi_2, \pi_2)] \Leftrightarrow \min_{q\in\tilde{\Pi}(\alpha,\Pi_1,\pi_1)} C_q(u \circ f) \geq \min_{q\in\tilde{\Pi}(\alpha,\Pi_2,\pi_2)} C_q(u \circ g)$$

where $\tilde{\Pi}(\alpha, \Pi, \pi) = \{p^\alpha | p^\alpha = \alpha p + (1-\alpha)\pi,\ p \in co(\Pi)\}$ and $co(\Pi)$ is the closed convex hull of Π. Thus, the decision maker has preferences of the multiple prior type (see Gilboa and Schmeidler (1989)) that are characterised by the set of (additive) priors $\tilde{\Pi}(\alpha, \Pi, \pi)$ for a given situation (Π, π). Note that $\tilde{\Pi}(\alpha, \Pi, \pi)$ is a linear combination of $co(\Pi)$ and π with α being the linear weight.

To understand how this result relates to our paper, recall the relation between the Choquet expected utility and the multiple prior theory as for example described in the proposition in Schmeidler (1989). If a preference relation of the multiple prior type is characterised by a closed convex set of priors (i.e. the lower bound of this set is a convex capacity), then it is also of the Choquet expected utility type and the capacity characterising the preferences is the lower bound of the set of priors. Thus, the preferences described by Gajdos, Tallon and Vergnaud are for a given situation (Π, π) of the Choquet expected utility type with the associated capacity v defined by

$$\begin{aligned} v(E) &= \min\{p^{\alpha}(E) | p^{\alpha}(E) = \alpha p(E) + (1-\alpha)\pi(E),\ p \in co(\Pi)\} \\ &= \alpha v_{\Pi}^{L}(E) + (1-\alpha)\pi(E) \end{aligned}$$

for all $E \subseteq S_0$ with v_{Π}^{L} defined by $v_{\Pi}^{L}(E) = \min\{p(E) | p \in co(\Pi)\}$. Since v_{Π}^{L} is a lower bound, the structure of the decision maker's capacity coincides with the one assumed in our paper.

In the setting of Gajdos, Tallon and Vergnaud, π and Π, thus the lower bound v_{Π}^{L} are objectively given, while we derive π and the lower bound from preferences. Since we derive more information from preferences, we need to take updating of preferences into consideration. While we assume that the parameter of uncertainty aversion α does not change when preferences are updated, Gajdos, Tallon and Vergnaud assume that α does not change when the situation (Π, π) changes. Most important for our model: The approach of Gajdos, Tallon and Vergnaud leaves no room for subjective uncertainty, since the uncertainty is objectively given by the set Π. It is therefore clear that α is not a parameter of subjective uncertainty, but of uncertainty aversion. Since their model and our model lead to the same structure of the decision maker's capacity, the model of Gajdos, Tallon and Vergnaud supports our view that α can be interpreted as a parameter of uncertainty aversion in our model as well.

8. Conclusion

In this article it has been shown how both a decision maker's subjectively perceived uncertainty and his individual degree of uncertainty aversion can be concluded unequivocally from his prior and updated preferences, if his preferences satisfy certain properties. We do not claim that most decision maker's usually satisfy these properties. Therefore we are reluctant to call them axioms. But if they are satisfied, then there are good reasons for a distinction between subjective uncertainty and individual uncertainty aversion.

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Appendix

Proof of Proposition 8. The first statement of Proposition 8 follows directly from proposition 11 in Chateauneuf and Jaffray (1989). We give an example of a lower bound on some set $M = \{m_1, m_2, m_3, m_4\}$, which is not convex. Consider the set $\Pi = \{\pi^1, \pi^2\}$ where π^1 and π^2 are the probability measures given by

	m_1	m_2	m_3	m_4
$\pi^1(\{m_j\})$	1/4	1/4	1/4	1/4
$\pi^2(\{m_j\})$	1/2	0	1/2	0

The lower bound $v(A) = \min\{\pi(A)|\pi \in \Pi\}$ satisfies $v(\{m_2\}) = 0$, $v(\{m_1, m_2\}) = 1/2$, $v(\{m_2, m_3\}) = 1/2$, $v(\{m_1, m_2, m_3\}) = 3/4$. Let $A = \{m_1, m_2\}$ and $B = \{m_2, m_3\}$, i.e. $v(A \cup B)+ v(A \cap B) = 3/4 \leq 1 = v(A) + v(B)$, thus v is not convex. □

Proof of Proposition 9. Let $v(A) = \min\{\pi(A)|\pi \in \Pi\}$ for some set Π of probability measures on M. If v were not superadditive, then there would be events $A, B \subseteq M$ with $A \cap B = \emptyset$, such that $v(A \cup B) < v(A) + v(B)$. Because $v(A) = \min\{\pi(A)|\pi \in \Pi\}$, there are $\pi, \hat{\pi}, \overline{\pi} \in \Pi$ such that $v(A) = \pi(A)$, $v(B) = \hat{\pi}(B)$, $v(A \cup B) = \overline{\pi}(A \cup B) = \overline{\pi}(A) + \overline{\pi}(B)$. Thus, if v were not superadditive, then $\overline{\pi}(A) + \overline{\pi}(B) < \pi(A) + \hat{\pi}(B)$. But this is not possible since $\pi(A) \leq \overline{\pi}(A)$ and $\hat{\pi}(B) \leq \overline{\pi}(B)$. Thus every lower bound is superadditive.

Now consider the following superadditive capacity on $M = \{m_1, m_2, m_3\}$:

$$\left.\begin{array}{c} v(\{m_i\}) = \frac{1}{8},\ v(\{m_i, m_j\}) = \frac{5}{8}\ \text{für alle}\ i, j \in \{1, 2, 3\}, i \neq j, \\ v(M) = 1,\ v(\emptyset) = 0 \end{array}\right\}$$

If this capacity were a lower bound, then there would exist a set Π of probability measures on M and a $\pi \in \Pi$ with $\pi(\{m_1\}) = 1/8$, $\pi(\{m_1\}) + \pi(\{m_2\}) \geq 5/8$ and $\pi(\{m_1\}) + \pi(\{m_3\}) \geq 5/8$. From that we get $\pi(\{m_2\}) \geq 5/8 - 1/8 = 4/8$ and $\pi(\{m_3\}) \geq 5/8 - 1/8 = 4/8$. Thus $\pi(\{m_1\}) + \pi(\{m_2\}) + \pi(\{m_3\}) \geq 9/8$ and π would be no probability measure. Thus not every superadditive capacity is a lower bound. □

Proof of Theorem 12.

(1) If $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfies (u-i) and is such that $\succsim_i$ satisfies (i), (ii), (iii), (iv) and (v) for all $i \in \{0, 1, \dots, n\}$, then there are u and π as described in the theorem: Since $\succsim_i$ satisfies (i), (ii), (iii), (iv) and (v), there is according to Theorem 11 a utility function $u_i : Y \to \mathbb{R}$ unique up to positive linear transformations and a unique probability measure π_i on S_i associated with $\succsim_i$ for all $i \in \{0, 1, \dots, n\}$. From (u-i) it follows that for all $y_k, y_l \in Y$ we have $y_k \succsim_i y_l \Leftrightarrow y_k \succsim_0 y_l$, i.e. all preference relations $\succsim_i$, $i = 0, 1, \dots, n$ agree on the set of constant acts. By the von Neumann–Morgenstern theorem (see for example Schmeidler (1989), p. 577) we therefore get $u_0 = u_1 = \dots = u_n \equiv u$. Clearly, we have $\pi_0(S_i) > 0$ for all $i \in \{0, 1, \dots, n\}$. Otherwise (u-i) would

imply $f_i \sim_i g_i$ for all $f_i, g_i \in F_i$ and $\succsim_i$ would not satisfy (v). Thus the condition in (u-i) can be translated into

$$C_{\pi_i}(u \circ f_i) = \sum_{s\in S_i} u[f_i(s)]\pi_i(\{s\}) \geq \sum_{s\in S_i} u[g_i(s)]\pi_i(\{s\}) = C_{\pi_i}(u \circ g_i)$$
$$\Leftrightarrow \sum_{s\in S_i} u[f_i(s)]\frac{\pi_0(\{s\})}{\pi_0(S_i)} \geq \sum_{s\in S_i} u[g_i(s)]\frac{\pi_0(\{s\})}{\pi_0(S_i)}$$

Since u is affine and unique up to positive linear transformations, it has a nondegenerated convex range and we can assume that $[-1, 1]$ is a subset of the range of u. Thus for every $E \subset S_i$ there exist acts $f_i, g_i \in F_i$ with

$$u[f_i(s)] = \begin{cases} -\pi_i(S_i \setminus E) \text{ for } s \in E \\ 1 \text{ for } s \in S_i \setminus E \end{cases} \text{ and } u[g_i(s)] = \begin{cases} 0 \text{ for } s \in E \\ 1 - \pi_i(E) \text{ for } s \in S_i \setminus E \end{cases}$$

Thus $C_{\pi_i}(u \circ f_i) = C_{\pi_i}(u \circ g_i)$. To meet (u-i) we therefore have to ensure

$$\sum_{s\in S_i} u[f_i(s)]\frac{\pi_0(\{s\})}{\pi_0(S_i)} = -\pi_i(S_i \setminus E)\frac{\pi_0(E)}{\pi_0(S_i)} + 1\frac{\pi_0(S_i \setminus E)}{\pi_0(S_i)}$$
$$= 0 + [1 - \pi_i(E)]\frac{\pi_0(S_i \setminus E)}{\pi_0(S_i)}$$
$$= \sum_{s\in S_i} u[g_i(s)]\frac{\pi_0(\{s\})}{\pi_0(S_i)} \Leftrightarrow \pi_i(E) = \frac{\pi_0(E)}{\pi_0(S_i)}$$

With $\pi_0 \equiv \pi$ the first part of the theorem is proved.

(2) If there are u and π as described in the theorem, then $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfies (u-i) and is such that $\succsim_i$ satisfies (i), (ii), (iii), (iv) and (v) for all $i \in \{0, 1, \dots, n\}$: Since π_i defined by $\pi_i(E) = \pi(E)/\pi(S_i)$ for all $E \subseteq S_i$ is a probability measure on S_i, we know from Theorem 11 that $\succsim_i$ satisfies (i), (ii), (iii), (iv) and (v) for all $i \in \{0, 1, \dots, n\}$. Since

$$\sum_{s\in S_i} u[f_i(s)]\pi_i(s) \geq \sum_{s\in S_i} u[g_i(s)]\pi_i(s) \Leftrightarrow \sum_{s\in S_i} u[f_0(s)]\pi(s) + \sum_{s\in S_0\setminus S_i} u[h_0(s)]\pi(s)$$
$$\geq \sum_{s\in S_i} u[g_0(s)]\pi(s) + \sum_{s\in S_0\setminus S_i} u[h_0(s)]\pi(s)$$

for all $f_0, g_0, h_0 \in F_0$ and $f_i, g_i \in F_i$ with $f_i(s) = f_0(s)$ and $g_i(s) = g_0(s)$ for all $s \in S_i$ and all $i = 1, \dots, n$, it is clear that $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfies (u-i). □

For the next step we need to prove two lemmas:

Lemma 20. *Let $\succsim_i'$ and $\succsim_i$ be two preference relations on F_i satisfying* (i), (iii), (iv), (v) *and* (vi). *Let v_i' and u' be the capacity and utility function associated with $\succsim_i'$ and v_i and*

u be the capacity and utility function associated with $\succsim_i$. Then $\succsim_i'$ is less constancy-loving than $\succsim_i$ if and only if $v_i'(E) \geq v_i(E)$ for all $E \subseteq S_i$ and $u' = u$.

(1) If $\succsim_i'$ is less constancy-loving than $\succsim_i$, then $v_i'(E) \geq v_i(E)$ for all $E \subseteq S_i$ and $u' = u$: If $\succsim_i'$ is less constancy-loving than $\succsim_i$, then we have by Definition 15 for all $y_k, y_l \in Y$: $y_k \succsim_i' y_l \Leftrightarrow y_k \succsim_i y_l$, i.e. $\succsim_i'$ and $\succsim_i$agree on the set of constant acts. By the von Neumann-Morgenstern theorem (see for example Schmeidler (1989), page 577) we therefore get $u = u'$. Fix some $E \subseteq S_i$ and some $y_k, y_l \in Y$ with $y_k \succ_i' y_l$, i.e. $y_k \succ_i y_l$. Because of (v) such elements in Y exist. Because $\succsim_i'$ is less constancy-loving than $\succsim_i$, we have for every $y \in Y$ by Definition 15

$$y \sim_i (y_k, E; y_l, S_i \setminus E) \Rightarrow (y_k, E; y_l, S_i \setminus E) \succsim_i' y$$

which can be translated into

$$u(y) = u(y_k)v_i(E) + u(y_l)[1 - v_i(E)] \Rightarrow u(y) \leq u(y_k)v_i'(E) + u(y_l)[1 - v_i'(E)]$$

Since u is affine, it has a convex range and there exists a $y \in Y$ defined by $u(y) = u(y_k)v_i(E) + u(y_l)[1 - v_i(E)]$. Thus

$$u(y_k)v_i(E) + u(y_l)[1 - v_i(E)] \leq u(y_k)v_i'(E) + u(y_l)[1 - v_i'(E)]$$

which is (because $u(y_k) > u(y_l)$) equivalent to $v_i(E) \leq v_i'(E)$.

(2) If $v_i'(E) \geq v_i(E)$ for all $E \subseteq S_i$ and $u' = u$, then $\succsim_i'$ is less constancy-loving than $\dot{\succsim}$: This follows directly from the obvious fact that $v_i'(E) \geq v_i(E)$ for all $E \subseteq S_i$ implies $C_{v_i'}(u \circ f_i) \geq C_{v_i}(u \circ f_i)$ for all $f_i \in F_i$. □

Lemma 21. *Let $\succsim_i$ satisfy* (i), (iii), (iv), (v) *and* (vi) *and let $\succsim_i' \in e(\succsim_i)$. Let v_i be the capacity associated with $\succsim_i$ and π_i be the probability measure associated with $\succsim_i'$. Then $\succsim_i$ and $\succsim_i'$ agree on $F_i^E(\succsim_i)$ for some $E \in S_i$ if and only if $v_i(E) = \pi_i(E)$.*

(1) If $\succsim_i$ and $\succsim_i'$ agreeing on $F_i^E(\succsim_i)$, then $v_i(E) = \pi_i(E)$: Since $\succsim_i'$ is less constancy-loving than $\succsim_i$, the utility function associated with $\succsim_i(u)$ and the utility function associated with $\succsim_i'$ (u') are according to Lemma 20 identical ($u = u'$). Since u is affine and unique up to positive linear transformations, it has a nondegenerated convex range and we can assume that $[-1, 1]$ is a subset of the range of u. Thus there are acts $f_i, g_i \in F_i^E(\succsim_i)$ with

$$u[f_i(s)] = \begin{cases} v_i(E) \text{ for } s \in E \\ 0 \text{ for } s \in S_i \setminus E \end{cases} \text{ and } u[g_i(s)] = \begin{cases} 1 \text{ for } s \in E \\ -v_i(E) \text{ for } s \in S_i \setminus E \end{cases}$$

We obviously get $C_{v_i}[u \circ f_i] = C_{v_i}[u \circ g_i]$ from this. Since $\succsim_i$ and $\succsim_i'$ agree on $F_i^E(\succsim_i)$, we also have to ensure $C_{\pi_i}[u \circ f_i] = C_{\pi_i}[u \circ g_i]$, i.e.

$$v_i(E)\pi_i(E) = \pi_i(E) - v_i(E)[1 - \pi_i(E)]$$

which is equivalent to $\pi_i(E) = v_i(E)$.

(2) If $v_i(E) = \pi_i(E)$, then $\succsim_i$ and $\succsim_i'$ agreeing on $F_i^E(\succsim_i)$: trivial since the utility function associated with $\succsim_i$and the utility function associated with $\succsim_i'$ are identical. □

Proof of Theorem 16.

(1) If $\succsim_i$ satisfies (i), (iii), (iv), (v), (vi) and (vii), then there are u and v_i as described in the theorem: From Theorem 14we know that there is a utility function u as described in Theorem 16and a unique capacity v_i associated with $\succsim_i$. We therefore just have to show that (vii) implies that v_i is a lower bound. But this is now obvious: With Lemma 20 we know that the set of probability measures Π_i associated with $e(\succsim_i)$ contains all probability measures π_i satisfying $\pi_i(E) \geq v_i(E)$ for all $E \subseteq S_i$. From Lemma 21 we learn that there is at least one probability measure π_i for every $E \subseteq S_i$ in Π_i, such that $\pi_i(E) = v_i(E)$. Thus $v_i(E) = \min\{\pi_i(E)|\pi_i \in \Pi_i\}$. A look at Definition 7 completes the proof.

(2) If there are u and v_i as described in the theorem, then $\succsim_i$ satisfies (i), (iii), (iv), (v), (vi) and (vii): Since every lower bound is a capacity, we know from Theorem 14 that $\succsim_i$ satisfies (i), (iii), (iv), (v) and (vi). Since v_i is a lower bound on S_i, there is according to Definition 7 a set Π_i of probability measures on S_i such that $v_i(A) = \min\{\pi_i(A)|\pi_i \in \Pi_i\}$ for all $A \subseteq S_i$. Thus for every $E \subseteq S_i$ there is some $\pi_i' \in \Pi_i$ with $v_i(E) = \pi_i'(E)$ and $v_i(A) \leq \pi_i'(A)$ for all $A \subseteq S_i$. The preference relation $\succsim_i'$ associated with π_i' and u according to Theorem 11is according to Lemma 20 in $e(\succsim_i)$ and agrees according to Lemma 21 with $\succsim_i$ on $F_i^E(\succsim_i)$. Thus $\succsim_i$ satisfies (vii). □

The following lemma is due to Jaffray (1992):

Lemma 22. *Let v be a lower bound on S_0 with $v(S_i) > 0$ for all $i \in \{0, 1, \ldots, n\}$. Let $\Pi = \{\pi|\pi$ is a probability measure on S_0 and $\pi(E) \geq v(E)$ for all $E \subseteq S_0\}$, $\Pi_i = \{\pi_i|\pi_i$ is a probability measure on S_i and there is a $\pi \in \Pi$ such that $\pi_i(E) = \pi(E)/\pi(S_i)$* for all *$E \subseteq S_i\}$ and $v_i(E) = \min\{\pi_i(E)|\pi_i \in \Pi_i\}$ for all $E \subseteq S_i$ and all $i \in \{0, 1, \ldots, n\}$. Then* $v_i(E) = v(E)/v(E) + 1 - v(E \cup S_0 \setminus S_i)$ for all $E \subseteq S_i$ *and all $i \in \{0, 1, \ldots, n\}$.*

Clearly, $v_i(E) = \min\{\pi(E)/\pi(S_i)|\pi \in \Pi\} = \min\{\pi(E)/\pi(E) + 1 - \pi(E \cup S_0 \setminus S_i)|\pi \in \Pi\}$. If $v(E) > 0$, then $\pi(E) > 0$ for all $\pi \in \Pi$ and $v_i(E) = (1 + \max\{1 - \pi(E \cup S_0 \setminus S_i)/\pi(E)|\pi \in \Pi\})^{-1}$. Because of the definition of Π in the lemma and because v is a lower bound, there is a $\pi' \in \Pi$ with $\pi'(E) = v(E)$ and $\pi'(E \cup S_0 \setminus S_i) = v(E \cup S_0 \setminus S_i)$. Since $1 - \pi(E \cup S_0 \setminus S_i)/\pi(E) \leq 1 - v(E \cup S_0 \setminus S_i)/v(E) = 1 - \pi'(E \cup S_0 \setminus S_i)/\pi'(E)$ for all $\pi \in \Pi$, we get $v_i(E) = (1+1-\pi'(E \cup S_0 \setminus S_i)/\pi'(E))^{-1} = v(E)/v(E)+1-v(E \cup S_0 \setminus S_i)$. If $v(E) = 0$, then there is a $\pi \in \Pi$ with $\pi(E) = 0$ and therefore $v_i(E) = 0 = v(E)/v(E) + 1 - v(E \cup S_0 \setminus S_i)$. (Note that $v(E) + 1 - v(E \cup S_0 \setminus S_i) = 0$ implies $v(E) = 0$ and $v(E \cup S_0 \setminus S_i) = 1$. Since v is a lower bound, it follows that there is a probability measure π with $\pi(E) = 0$ and $\pi(A) \geq v(A)$ for all $A \subseteq S_0$, i.e. $\pi(E \cup S_0 \setminus S_i) = 1$ and therefore $\pi(S_i \setminus E) = 0$, implying $\pi(S_i) = 0 < v(S_i)$. Thus $v(E) + 1 - v(E \cup S_0 \setminus S_i) > 0$.) □

Proof of Theorem 17.

(1) If $(\succsim_0, \succsim_1, \ldots, \succsim_n)$ satisfies (u-ii) and (u-iii) and is such that $\succsim_i$ satisfies (i), (iii), (iv), (v), and (vi) for all $i \in \{0, 1, \ldots, n\}$, then there are u and v as described in the

theorem: Since all $\succsim_i$ in $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfy (i), (iii), (iv), (v) and (vi), there is according to Theorem 14 a capacity v_i on S_i and a utility function u_i associated with every $\succsim_i$. Because of (u-ii), $E(\succsim_0, \succsim_1, \dots, \succsim_n)$ is not empty. Take some list $(\succsim'_0, \succsim'_1, \dots, \succsim'_n)$ in $E(\succsim_0, \succsim_1, \dots, \succsim_n)$. According to Theorem 12 there is a utility function u' associated with $(\succsim'_0, \succsim'_1, \dots, \succsim'_n)$. Since $\succsim'_i$ is less constancy-loving than $\succsim_i$, we get from Lemma 20 $u_i = u' \equiv u$ for all $i \in \{0, 1, \dots, n\}$. (This implies that the utility function associated with some list in $E(\succsim_0, \succsim_1, \dots, \succsim_n)$ is equal to the utility function associated with another list in $E(\succsim_0, \succsim_1, \dots, \succsim_n)$.)

Moreover there is according to Theorem 12 a list $(\pi_0, \pi_1, \dots, \pi_n)$ with a probability measure π_i on S_i for all $i \in \{0, 1, \dots, n\}$ associated with $(\succsim'_0, \succsim'_1, \dots, \succsim'_n)$, such that $\pi_i(E) = \pi_0(E)/\pi_0(S_i)$ for all $E \subseteq S_i$. (The list of probability measures associated with one list in $E(\succsim_0, \succsim_1, \dots, \succsim_n)$ is not equal to the list of probability measures associated with another list in $E(\succsim_0, \succsim_1, \dots, \succsim_n)$.) Thus $E(\succsim_0, \succsim_1, \dots, \succsim_n)$ generates a list $(\Pi \equiv \Pi_0, \Pi_1, \dots, \Pi_n)$ such that $\Pi_i = \{\pi_i | \pi_i$ is a probability on S_i and there is a $\pi \in \Pi$ such that $\pi_i(E) = \pi(E)/\pi(S_i)$ for all $E \subseteq S_i\}$ for all $i \in \{0, 1, \dots, n\}$. From Lemma 20 we know that $v_i(E) \leq \pi_i(E)$ for all $E \subseteq S_i$, all $\pi_i \in \Pi_i$ and all $i \in \{0, 1, \dots, n\}$. From (u-ii) we know that there is a list $(\succsim'_0, \succsim'_1, \dots, \succsim'_n)$ in $E(\succsim_0, \succsim_1, \dots, \succsim_n)$ such that $\succsim'_i$ and $\succsim_i$ agree on $F_i^E(\succsim_i)$. Thus (because of Lemma 21) there is a $\pi_i \in \Pi_i$ for every $E \subseteq S_i$ such that $\pi_i(E) = v_i(E)$. It follows that for all $i \in \{0, 1, \dots, n\}$ and all $E \subseteq S_i$ we have $v_i(E) = \min\{\pi_i(E) | \pi_i \in \Pi_i\}$, i.e. v_i is a lower bound for all $i \in \{0, 1, \dots, n\}$. Since $\pi(S_i) > 0$ for all $\pi \in \Pi$, we know $v_0(S_i) > 0$ for all $i \in \{0, 1, \dots, n\}$. With Lemma 20 we learn from (u-iii) that $\Pi = \{\pi | \pi$ is a probability measure on S_0 and $\pi(E) \geq v_0(E)$ for all $E \subseteq S_0\}$. From Lemma 22 we conclude $v_i(E) = v_0(E)/(v_0(E) + 1 - v_0(E \cup S_0 \setminus S_i))$ for all $E \subseteq S_i$ and all $i \in \{0, 1, \dots, n\}$. With $v \equiv v_0$ the first part of the theorem is proved.

(2) If there are u and v as described in the theorem, then $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfies (u-ii) and (u-iii) and is such that $\succsim_i$ satisfies (i), (iii), (iv), (v), and (vi) for all $i \in \{0, 1, \dots, n\}$: From Lemma 22 we know that v_i defined by $v_i(E) = v(E)v(E) + 1 - v(E \cup S_0 \setminus S_i)$ for all $E \subseteq S_i$ and all $i \in \{0, 1, \dots, n\}$ is a lower bound and therefore a capacity, thus $\succsim_i$ satisfies (i), (iii), (iv), (v), and (vi) for all $i \in \{0, 1, \dots, n\}$ according to Theorem 14.

Let $\Pi = \{\pi | \pi$ is a probability measure on S_0 and $\pi(E) \geq v(E)$ for all $E \subseteq S_0\}$. If $\succsim^*_0 \in e(\succsim_0)$, then the probability measure π^* associated with $\succsim^*_0$ according to Theorem 11 is according to Lemma 20 in Π. Let $(\succsim'_0, \succsim'_1, \dots, \succsim'_n)$ be the list associated with u and π^* according to Theorem 12. Of course we have $\succsim'_0 \ = \succsim^*_0$. Since $v(E)/(v(E)+1-v(E \cup S_0 \setminus S_i))$ is increasing in $v(E)$, $\pi^*(E)/\pi^*(S_i) < v(E)/v(E) + 1 - v(E \cup S_0 \setminus S_i)$ would imply $\pi^*(E)/\pi^*(S_i) < \pi^*(E)/\pi^*(E) + 1 - v(E \cup S_0 \setminus S_i)$, which is equivalent to $v(E \cup S_0 \setminus S_i) > \pi^*(E \cup S_0 \setminus S_i)$ and thus a contradiction to $\pi^* \in \Pi$. Thus $\pi^*(E)/\pi^*(S_i) \geq v(E)/v(E) + 1 - v(E \cup S_0 \setminus S_i)$ for all $E \subseteq S_i$ and $i \in \{0, 1, \dots, n\}$, i.e. (because of Lemma 20) $(\succsim'_0, \succsim'_1, \dots, \succsim'_n) \in E(\succsim_0, \succsim_1, \dots, \succsim_n)$ and $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfies (u-iii).

Fix some $i \in \{0, 1, \dots, n\}$ and $E \subseteq S_i$. Obviously, there is a $\pi^* \in \Pi$ with $\pi^*(E) = v(E)$ and $\pi^*(E \cup S_0 \setminus S_i) = v(E \cup S_0 \setminus S_i)$. Let $(\succsim'_0, \succsim'_1, \dots, \succsim'_n)$ be the list associated with u and π^* according to Theorem 12. Because $\pi^*(A)/\pi^*(S_j) \geq v(A)/v(A) + 1 - v(A \cup S_0 \setminus S_j)$ for all $A \subseteq S_j$ and $j \in \{0, 1, \dots, n\}$ (see above), we know from

Lemma 20 that $(\succsim_0', \succsim_1', \ldots, \succsim_n') \in E(\succsim_0, \succsim_1, \ldots, \succsim_n)$. Since $\pi^*(E)/\pi^*(S_i) = \pi^*(E)/\pi^*(E) + 1 - \pi^*(E \cup S_0 \setminus S_i) = v(E)/v(E) + 1 - v(E \cup S_0 \setminus S_i) = v_i(E)$, $\succsim_i'$ and $\succsim_i$ agree according to Lemma 21 on $F_i^E(\succsim_i)$ and $(\succsim_0, \succsim_1, \ldots, \succsim_n)$ satisfies (u-ii). □

Proof of Theorem 19.

(1) If $(\succsim_0, \succsim_1, \ldots, \succsim_n)$ satisfies (u-iv) and (u-v) and is such that $\succsim_i$ satisfies (i), (iii), (iv), (v) and (vi) for all $i \in \{0, 1, \ldots, n\}$, then there are u, α, v^L and π as described in the theorem:

(1.1) Existence of u, α, v^L and π: Since $(\succsim_0, \succsim_1, \ldots, \succsim_n)$ is such that $\succsim_i$ satisfies (i), (iii), (iv), (v) and (vi) for all $i \in \{0, 1, \ldots, n\}$, there exist for every $i \in \{0, 1, \ldots, n\}$ a utility function u_i and a capacity v_i on S_i associated with $\succsim_i$ according to Theorem 14. Since $(\succsim_0, \succsim_1, \ldots, \succsim_n)$ satisfies (u-iv), there is a list $(\succsim_0^E, \succsim_1^E, \ldots, \succsim_n^E)$ with a utility function u and a probability measure π on S_0 associated with $(\succsim_0^E, \succsim_1^E, \ldots, \succsim_n^E)$ according to Theorem 12, such that $\succsim_i^E$ is less constancy-loving than $\succsim_i$ for all $i \in \{0, 1, \ldots, n\}$, thus according to Lemma 20 $v_i(E) \leq \pi(E)/\pi(S_i) \equiv \pi_i(E)$ and $u_i = u$ for all $i \in \{0, 1, \ldots, n\}$ and $E \subseteq S_i$. And there is a list $(\succsim_0^L, \succsim_1^L, \ldots, \succsim_n^L)$ with a utility function u^L and a lower bound v^L on S_0 associated with $(\succsim_0^L, \succsim_1^L, \ldots, \succsim_n^L)$ according to Theorem 17, such that $\succsim_i$ is less constancy-loving than $\succsim_i^L$ for all $i \in \{0, 1, \ldots, n\}$, thus according to Lemma 20 $v_i(E) \geq v^L(E)/(v^L(E) + 1 - v^L(E \cup S_0 \setminus S_i)) \equiv v_i^L(E)$ and $u_i = u = u^L$ for all $E \subseteq S_i$ and $i \in \{0, 1, \ldots, n\}$. We now prove:

Lemma 23. *For every pair $A, B \subseteq S_i$ with $\pi_i(A) > v_i^L(A)$, $\pi_i(B) > v_i^L(B)$ there are $y_1, y_2, y_3, y_4 \in Y$ such that $y_1 \succ_i^E y_2$, $y_3 \succ_i^E y_4$ (thus $y_1 \succ_i^L y_2$ and $y_3 \succ_i^L y_4$), $(y_1, A; y_2, S_i \setminus A) \sim_i^E (y_3, B; y_4, S_i \setminus B)$ and $(y_1, A; y_2, S_i \setminus A) \sim_i^L (y_3, B; y_4, S_i \setminus B)$.*

Since u is affine and unique up to positive linear transformations, it has a nondegenerated convex range and we can assume that $[-1, 1]$ is a subset of the range of u. Thus we have to prove the existence of numbers $u(y_1), \ldots, u(y_4)$ such that

$$\pi_i(A)u(y_1) + [1 - \pi_i(A)]u(y_2) = \pi_i(B)u(y_3) + [1 - \pi_i(B)]u(y_4)$$
$$v_i^L(A)u(y_1) + [1 - v_i^L(A)]u(y_2) = v_i^L(B)u(y_3) + [1 - v_i^L(B)]u(y_4)$$
$$1 \geq u(y_1) > u(y_2) \geq -1, 1 \geq u(y_3) > u(y_4) \geq -1$$

for all $\pi_i(A), \pi_i(B), v_i^L(A), v_i^L(B) \in [0, 1]$, $\pi_i(A) > v_i^L(A)$, $\pi_i(B) > v_i^L(B)$. Simple calculations show that

$$u(y_1) = \varepsilon \frac{v_i^L(B)[1 - \pi_i(A)] - \pi_i(B)[1 - v_i^L(A)]}{v_i^L(A) - \pi_i(A)},$$
$$u(y_2) = \varepsilon \frac{v_i^L(A)\pi_i(B) - \pi_i(A)v_i^L(B)}{v_i^L(A) - \pi_i(A)},$$
$$u(y_3) = \varepsilon, u(y_4) = 0$$

for a sufficiently small $\varepsilon > 0$ satisfy the above conditions.

Next we prove

Lemma 24. For all $f_i, g_i \in F_i$: $(f_i \sim_i^E g_i$ and $f_i \sim_i^L g_i) \Rightarrow f_i \sim_i g_i$, if and only if there is a $\alpha_i \in [0, 1]$ such that $v_i(E) = \alpha_i v_i^L(E) + (1 - \alpha_i)\pi_i(E)$ for all $E \subseteq S_i$.

(1) If for all $f_i, g_i \in F_i$: $(f_i \sim_i^E g_i$ and $f_i \sim_i^L g_i) \Rightarrow f_i \sim_i g_i$, then there is a $\alpha_i \in [0, 1]$ such that $v_i(E) = \alpha_i v_i^L(E) + (1-\alpha_i)\pi_i(E)$ for all $E \subseteq S_i$: Let $\succsim_i$ satisfy (i), (ii), (iii), (iv) and (v), i.e. v_i is a probability measure. Since $\succsim_i^E$ is less constancy-loving than $\succsim_i$, we learn from Lemma 20 that $v_i = \pi_i$ must hold, thus $\alpha_i = 0$ and the lemma is true. Now assume $\succsim_i$ does not satisfy (i), (ii), (iii), (iv) and (v), thus v_i is not a probability measure. Then there is either only one event $A \subseteq S_i$ with $\pi_i(A) > v_i^L(A)$ (i. e. $\pi_i(B) = v_i^L(B)$ for all $B \subseteq S_i$, $B \neq A$) and Lemma 24 is true with α_i (implicitly) defined by $v_i(A) = \alpha_i v_i^L(A) + (1-\alpha_i)\pi_i(A)$. Or there are at least two events $A, B \subseteq S_i$ such that $\pi_i(A) > v_i^L(A)$, $\pi_i(B) > v_i^L(B)$. In this case we use Lemma 23, from which we know that there are $y_1, y_2, y_3, y_4 \in Y$ such that $y_1 \succ_i^E y_2$, $y_3 \succ_i^E y_4$, $y_1 \succ_i^L y_2$, $y_3 \succ_i^L y_4$, $(y_1, A; y_2, S_i \setminus A) \sim_i^E (y_3, B; y_4, S_i \setminus B)$ and $(y_1, A; y_2, S_i \setminus A) \sim_i^L (y_3, B; y_4, S_i \setminus B)$. Now assume that $(y_1, A; y_2, S_i \setminus A) \sim_i (y_3, B; y_4, S_i \setminus B)$, thus

$$u(y_1)\pi_i(A) + u(y_2)[1 - \pi_i(A)] = u(y_3)\pi_i(B) + u(y_4)[1 - \pi_i(B)]$$
$$u(y_1)v_i^L(A) + u(y_2)[1 - v_i^L(A)] = u(y_3)v_i^L(B) + u(y_4)[1 - v_i^L(B)]$$
$$u(y_1)v_i(A) + u(y_2)[1 - v_i(A)] = u(y_3)v_i(B) + u(y_4)[1 - v_i(B)]$$

Define α_i implicitly by $v_i(A) = \alpha_i v_i^L(A) + (1-\alpha_i)\pi_i(A)$. By the three equations above it is easy to see that $v_i(B) = \alpha_i v_i^L(B) + (1 - \alpha_i)\pi_i(B)$ and $\alpha_i \in [0, 1]$.

(2) If there is a $\alpha_i \in [0, 1]$ such that $v_i(E) = \alpha_i v_i^L(E) + (1 - \alpha_i)\pi_i(E)$ for all $E \subseteq S_i$, then for or all $f_i, g_i \in F_i$: $(f_i \sim_i^E g_i$ and $f_i \sim_i^L g_i) \Rightarrow f_i \sim_i g_i$: It is easy to see that $v_i(E) = \alpha_i v_i^L(E) + (1 - \alpha_i)\pi_i(E)$ for all $E \subseteq S_i$ implies $C_{v_i}[u \circ f_i] = \alpha_i C_{v_i^L}[u \circ f_i] + (1 - \alpha_i)C_{\pi_i}[u \circ f_i]$ for all $f_i \in F_i$. Thus $(C_{v_i^L}[u \circ f_i] = C_{v_i^L}[u \circ g_i]$ and $C_{\pi_i}[u \circ f_i] = C_{\pi_i}[u \circ g_i]) \Rightarrow C_{v_i}[u \circ f_i] = C_{v_i}[u \circ g_i]$.

We now know that there is a $\alpha_i \in [0, 1]$ such that $v_i(E) = \alpha_i v_i^L(E) + (1-\alpha_i)\pi_i(E)$ for all $E \subseteq S_i$ and all $i \in \{0, 1, \dots, n\}$. Next we show that $\alpha_0 = \alpha_1 = \dots = \alpha_n$. If $\succsim_i = \succsim_i^E = \succsim_i^L$ for some $i \in \{1, \dots, n\}$, then $\pi_i = v_i = v_i^L$ and $v_i(E) = \alpha_i v_i^L(E) + (1-\alpha_i)\pi_i(E)$ holds for any number α_i. Thus we can set $\alpha_i = \alpha_0$. If not $\succsim_i = \succsim_i^E = \succsim_i^L$ for some $i \in \{1, \dots, n\}$, then we know from (u-iv) (III) that there is a description (y_E, y, y_L) of $\succsim_0$ relative to $\succsim_0^E$ and $\succsim_0^L$ with $y_E \succ_0 y_L$ which is also a description of $\succsim_i$ relative to $\succsim_i^E$ and $\succsim_i^L$. Thus there is a $f_0 \in F_0$ and a $f_i \in F_i$ such that $C_{\pi_j}[u \circ f_j] = u(y_E)$, $C_{v_j^L}[u \circ f_j] = u(y_L)$ and $C_{v_j}[u \circ f_j] = u(y)$ for $j = 0, i$. Because $v_j(E) = \alpha_j v_j^L(E) + (1 - \alpha_j)\pi_j(E)$ for all $E \subseteq S_j$, we know $C_{v_j}[u \circ f_j] = \alpha_j C_{v_j^L}[u \circ f_j] + (1 - \alpha_j)C_{\pi_j}[u \circ f_j]$, thus $u(y) = \alpha_j u(y_L) + (1 - \alpha_j)u(y_E)$ $\Leftrightarrow \alpha_j = u(y) - u(y_E)/u(y_L) - u(y_E)$ for $j = 0, i$, thus $\alpha_i = \alpha_0 \equiv \alpha$.

In the next part of the proof it will become apparent that $\alpha > 0$, $v_0(A) = 0$, $v_0(B \cup C) < 1$ and $1 - v_0(B \cup C) \neq v_0(B \cup C) - v_0(B) - v_0(C)$ for some partition (A, B, C) of S_0 with $A, B, C \neq \emptyset$, $S_1 = A \cup B$, $S_2 = A \cup C$ and $S_3 = B \cup C$.

(1.2) Uniqueness of α, v^L and π: We now know: If $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfies the properties mentioned in Theorem 19, then there are (1) a utility function $u : Y \to \mathbb{R}$ unique up to positive linear transformations, (2) a number $\alpha \in [0, 1]$, (3) a lower bound v^L on S_0 and (4) a probability measure π on S_0 with $\pi(E) \geq v^L(E)$ for all $E \subseteq S_0$, such that for all f_i, $g_i \in F_i$ □

$$f_i \succsim_i g_i \Leftrightarrow C_{v_i}(u \circ f_i) \geq C_{v_i}(u \circ g_i)$$

with $v_i(E) = \alpha v_i^L(E) + (1-\alpha)\pi_i(E)$, $v_i^L(E) = v^L(E)/v^L(E) + 1 - v^L(E \cup S_0 \setminus S_i)$ and $\pi_i(E) = \pi(E)/\pi(S_i)$ for all $E \subseteq S_i$ (for $i = 0, 1, \ldots, n$). We now prove uniqueness. According to Theorem 14 we have a unique capacity v_i on S_i for all $i \in \{0, 1, \ldots, n\}$. We get uniqueness of α, $\pi(A)$, $\pi(B)$, $\pi(C)$, $v^L(E)$ for $E = A, B, C, A \cup B, A \cup C, B \cup C$, if the following has a unique solution for these variables:

(1), (2), (3) $\quad v_0(E) = \alpha v^L(E) + (1-\alpha)\pi(E),\ E = A, B, C$

(4), (5), (6) $\quad v_0(E_1 \cup E_2) = \alpha v^L(E_1 \cup E_2) + (1-\alpha)[\pi(E_1) + \pi(E_2)],\ (E_1, E_2) = (A, B), (A, C), (B, C)$

(7), (8) $$v_1(E) = \alpha \frac{v^L(E)}{v^L(E) + 1 - v^L(E \cup C)} + (1-\alpha)\frac{\pi(E)}{\pi(A) + \pi(B)},\ E = A, B$$

(9), (10) $$v_2(E) = \alpha \frac{v^L(E)}{v^L(E) + 1 - v^L(E \cup B)} + (1-\alpha)\frac{\pi(E)}{\pi(A) + \pi(C)},\ E = A, C$$

(11), (12) $$v_3(E) = \alpha \frac{v^L(E)}{v^L(E) + 1 - v^L(E \cup A)} + (1-\alpha)\frac{\pi(E)}{\pi(B) + \pi(C)},\ E = B, C$$

$\alpha \in [0, 1]$, π a probability measure and v^{L} a lower bound on S_0

(If it has no solution for these variables at all, then $(\succsim_0, \succsim_1, \ldots, \succsim_n)$ by (1.1) does not satisfy the properties mentioned in Theorem 19.) Because of (u-v) (2), we know that $u(y_1)v_0(A) + u(y_3)[1 - v_0(A)] = u(y_2)v_0(A) + u(y_3)[1 - v_0(A)]$ for all $u(y_1), u(y_2) > u(y_3)$, thus $v_0(A) = 0$. Moreover we know that $u(y_1)v_0(B \cup C) + u(y_2)[1 - v_0(B \cup C)] = u(y_1)v_0(B \cup C) + u(y_3)[1 - v_0(B \cup C)]$ not for all $u(y_1) > u(y_2), u(y_3)$, i.e. $v_0(B \cup C) < 1$. Thus $v_0(A) < \pi(A)$ or $v_0(B \cup C) < \pi(B \cup C)$ or both. Since $\alpha = 0$ would imply $v_0(A) = \pi(A)$ and $v_0(B \cup C) = \pi(B \cup C)$, we know that $\alpha > 0$. This implies (with $v_0(A) = 0$ and equation (1)) $v^L(A) = 0$ and $\pi(A) = 0$ or $\alpha = 1$ and therefore $v_1(A) = v_2(A) = 0$. Summation of some of the above equations yields

(1) + (2) + (3) $\quad v_0(B) + v_0(C) = \alpha[v^L(B) + v^L(C)] + (1-\alpha)$

(1) + (6) $\quad v_0(B \cup C) = \alpha v^L(B \cup C) + (1-\alpha)$

(7) + (8) $$v_1(B) = \alpha \frac{v^L(B)}{v^L(B) + 1 - v^L(B \cup C)} + (1-\alpha)$$

(9) + (10) $$v_2(C) = \alpha \frac{v^L(C)}{v^L(C) + 1 - v^L(B \cup C)} + (1-\alpha)$$

Elimination of $v^L(B \cup C)$ and rearranging yields

$$v_0(B) + v_0(C) = \alpha v^L(B) + \alpha v^L(C) + 1 - \alpha$$
$$\alpha[1 - v_0(B \cup C)] = \alpha v^L(B)[1 - v_1(B)] + [1 - v_0(B \cup C)][1 - v_1(B)]$$
$$\alpha[1 - v_0(B \cup C)] = \alpha v^L(C)[1 - v_2(C)] + [1 - v_0(B \cup C)][1 - v_2(C)]$$

Finally we eliminate $\alpha v^L(C)$ and get

$$\alpha = [1 - v_1(B)] + \frac{[1 - v_1(B)]}{[1 - v_0(B \cup C)]}\alpha v^L(B)\alpha[v_2(C) - v_0(B \cup C)]$$
$$= [v_0(B) + v_0(C) - v_0(B \cup C)][1 - v_2(C)] - [1 - v_2(C)]\alpha v^L(B)$$

This is a system of linear equations in the variables α and $\alpha v^l(B)$. Since $v_0(B \cup C) < 1$, we know $v^L(B \cup C) < 1$, thus with (7)+(8) and (9)+(10) $[1 - v_1(B)] > 0$ and $[1 - v_2(C)] > 0$. From (1)+(6) and (9)+(10) we easily get $[v_2(C) - v_0(B \cup C)] \leq 0$. If $[v_2(C) - v_0(B \cup C)] = 0$, the system obviously has a unique solution. If $[v_2(C) - v_0(B \cup C)] < 0$, the system has a unique solution, if both equations are not identical. They are identical only if both $[1 - v_1(B)] = [v_0(B \cup C) - v_0(B) - v_0(C)][1 - v_2(C)]/[v_0(B \cup C) - v_2(C)]$ and $[1 - v_1(B)]/[1 - v_0(B \cup C)] = [1 - v_2(C)]/[v_0(B \cup C) - v_2(C)]$. This implies $1 - v_0(B \cup C) = v_0(B \cup C) - v_0(B) - v_0(C)$. Now look at (u-v), part (4): Since for $y \succ_0 y'$ we have

$$(y, E; y', S_0 \setminus E) \sim_0 \lambda y + (1 - \lambda)y' \Leftrightarrow u(y)v_0(E) + u(y')[1 - v_0(E)]$$
$$= u(\lambda y + (1 - \lambda)y')$$

and (because u is affine according to Theorem 14) $u(\lambda y + (1 - \lambda)y') = \lambda u(y) + (1 - \lambda)u(y')$, we know $(y, E; y', S_0 \setminus E) \sim_0 \lambda y + (1 - \lambda)y'$ is equivalent to $v_0(E) = \lambda$. Thus we get from (u-v) (4) $1 - v_0(B \cup C) \neq v_0(B \cup C) - v_0(B) - v_0(C)$ and the above system has a unique solution in α and $\lambda v^l(B)$. With that we get a unique solution for α and $v^L(B)$ (since $\alpha > 0$) and (using (1)+(2)+(3) and (1)+(6)) also for $v^L(E)$, $E = C, B \cup C$. If $\alpha = 1$, we obviously can not conclude a unique π. If $\alpha < 1$, we get $\pi(E)$, $E = A, B, C$ from (1), (2) and (3). Finally, Eq. (4) yields a unique $v^L(A \cup B)$ and Eq. (5) yields a unique $v^L(A \cup C)$. Thus the system (1) to (12) has a unique solution.

If we found $\alpha = 1$, then it is clear that there is no unique π, but a unique v^L with $v^L = v$. If $\alpha < 1$, we first consider some $D \subset B$ (analogous for $D \subset A$ or $D \subset C$). The following must hold:

$$v_0(D) = \alpha v^L(D) + (1 - \alpha)\pi(D)$$
$$v_0(D \cup C) = \alpha v^L(D \cup C) + (1 - \alpha)[\pi(D) + \pi(C)]$$
$$v_1(D) = \alpha \frac{v^L(D)}{v^L(D) + 1 - v^L(D \cup C)} + (1 - \alpha)\frac{\pi(D)}{\pi(A) + \pi(B)}$$

where $\pi(A)$, $\pi(B)$ and α are already known. From the first two equations we get $v^L(D) - v^L(D \cup C) = \frac{1}{\alpha}[v_0(D) - v_0(D \cup C) + (1 - \alpha)\pi(C)] \equiv a$, which is known. We get

$$v_0(D) = \alpha v^L(D) + (1 - \alpha)\pi(D)$$
$$v_1(D) = \frac{\alpha}{1 + a}v^L(D) + \frac{1 - \alpha}{\pi(A) + \pi(B)}\pi(D)$$

We now have to show that this pair of equations has a unique solution for $v^L(D)$ and $\pi(D)$ for some $\alpha < 1$. It has a unique solution, if both equations are not identical, i.e. if not both $a = 0$ and $\pi(A) + \pi(B) = 1$. Since $\alpha < 1$ implies (because of $v_0(A) = 0$) $\pi(A) = 0$, $\pi(A) + \pi(B) = 1$ implies $\pi(A \cup C) = \pi(S_2) = 0$, which is not possible (see Theorem 12). Thus the system has a unique solution for $v^L(D)$ and $\pi(D)$.

Finally if neither $D \subset A$ nor $D \subset B$ nor $D \subset C$, then there are of course $D_1 \subset A$, $D_2 \subset B$, $D_3 \subset C$ such that $D = D_1 \cup D_2 \cup D_3$. Because of the above arguments we get unique $\pi(D_j)$ for $j = 1, 2, 3$, thus a unique $\pi(D)$. Since $v(D) = \alpha v^L(D) + (1-\alpha)\pi(D)$, we easily get a unique $v^L(D)$. This completes the proof of the uniqueness.

(2) If there are u, α, v^L and π as described in the theorem, then $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfies (u-iv) and (u-v) and is such that $\succsim_i$ satisfies (i), (iii), (iv), (v) and (vi) for all $i \in \{0, 1, \dots, n\}$:

We first prove that $\succsim_i$ satisfies (i), (iii), (iv), (v) and (vi) for all $i \in \{0, 1, \dots, n\}$. From Theorem 14 we know that we just have to prove, that a linear combination of a lower bound and a probability measure is always a capacity. Let $v_i(E) = \alpha v_i^L(E) + (1-\alpha)\pi_i(E)$ for all $E \subseteq S_i$, let v_i^L be a lower bound on S_i and π_i a probability measure on S_i. Of course we get $v_i(S_i) = 1$ and $v_i(\emptyset) = 0$. Let $A \subseteq B$. Of course we have $v_i^L(A) \le v_i^L(B)$ and $\pi_i(A) \le \pi_i(B)$, thus $v_i(A) \le v_i(B)$ and v_i is according to Definition 5 a capacity.

Next we prove that $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfies (u-iv). We just have to prove that the relations between $(\succsim_0, \succsim_1, \dots, \succsim_n)$, the list $(\succsim_0^E, \succsim_1^E, \dots, \succsim_n^E)$ associated with u and π according to Theorem 12 and the list $(\succsim_0^L, \succsim_1^L, \dots, \succsim_n^L)$ associated with u and v^L according to Theorem 17 satisfy part (I), (II) and (III) of (u-iv). From Lemma 20 we know that part (I) holds, if $v_i^L(E) \le v_i(E) \le \pi_i(E)$ for all $E \subseteq S_i$ and $i \in \{0, 1, \dots, n\}$. Since $v_i^L(E)$ is not decreasing in $v^L(E)$, we know that $v_i^L(E) > \pi_i(E)$ would imply (because $v^L(E) \le \pi(E)$)

$$\frac{\pi(E)}{\pi(E) + 1 - v^L(E \cup S_0 \setminus S_i)} > \frac{\pi(E)}{\pi(S_i)}$$

This is equivalent to $v^L(E \cup S_0 \setminus S_i) > \pi(E \cup S_0 \setminus S_i)$ which is in contradiction to the assumption $v^L \le \pi$. Thus $v_i^L(E) \le \pi_i(E)$ and, because $\alpha \in]0, 1]$, $v_i^L(E) \le v_i(E) \le \pi_i(E)$ for all $E \subseteq S_i$ and $i \in \{0, 1, \dots, n\}$. Part (II) holds because of Lemma 24. Now consider part (III). If $v_i = \pi_i = v_i^L$ for some $i \in \{1, \dots, n\}$, then clearly $\succsim_i = \succsim_i^E = \succsim_i^L$ and part (III) imposes no restriction on the relation between $\succsim_i$, $\succsim_i^E$ and $\succsim_i^L$. Now assume that not $v_i = \pi_i = v_i^L$. Then there is a $E \subset S_i$ with $\pi_i(E) > v_i^L(E)$. Since $v_0(A) + v_0(B \cup C) < 1$, we have $v^L(A) < \pi(A)$ or $v^L(B \cup C) < \pi(B \cup C)$. Let $D \equiv A$ if $v^L(A) < \pi(A)$ and $D \equiv B \cup C$ otherwise. Because u is affine and unique up to positive linear transformations, it has a nondegenerated convex range and we can assume that $[-1, 1]$ is a subset of the range of u. Thus, for a sufficiently small $\varepsilon > 0$ there are y_E, y, y_L with $u(y_E) = \varepsilon$, $u(y) = \varepsilon[1 - 2\alpha]$, $u(y_L) = -\varepsilon$ and acts $f_0 \in F_0$, $f_i \in F_i$ with

$$u[f_0(s)] = \begin{cases} \varepsilon \dfrac{2 - \pi(D) - v^L(D)}{\pi(D) - v^L(D)} & \text{for } s \in D \\[2ex] -\varepsilon \dfrac{\pi(D) + v^L(D)}{\pi(D) - v^L(D)} & \text{for } s \in S_0 \setminus D \end{cases}$$

and

$$u[f_i(s)] = \begin{cases} \varepsilon \dfrac{2 - \pi_i(E) - v_i^L(E)}{\pi_i(E) - v_i^L(E)} & \text{for } s \in E \\[2ex] -\varepsilon \dfrac{\pi_i(E) + v_i^L(E)}{\pi_i(E) - v_i^L(E)} & \text{for } s \in S_i \setminus E \end{cases}$$

Some simple calculations show that $C_\pi(u\circ f_0)=\varepsilon$, $C_{v_0}(u\circ f_0)=\varepsilon(1-2\alpha)$, $C_{v^L}(u\circ f_0)=-\varepsilon$, $C_{\pi_i}(u\circ f_i)=\varepsilon$, $C_{v_i}(u\circ f_i)=\varepsilon(1-2\alpha)$, $C_{v_i^L}(u\circ f_i)=-\varepsilon$, i.e. (y_E, y, y_L) is both a description of $\succsim_0$ relative to $\succsim_0^E$ and $\succsim_0^L$ and a description of $\succsim_i$ relative to $\succsim_i^E$ and $\succsim_i^L$.

Finally we prove that $(\succsim_0, \succsim_1, \dots, \succsim_n)$ satisfies (u-v). Part (1) of (u-v) obviously holds. Part (2) obviously holds because $v_0(A)=0$. Part (3) holds because $v_0(B\cup C)<1$ and (since u is affine and unique up to positive linear transformations) there are y_1, $y_2, y_3 \in Y$ with $u(y_1) > u(y_2), u(y_3)$. Part (4) holds, if the following is contradictory:

$$
\begin{aligned}
&1-\lambda_1=\lambda_1-\lambda_2-\lambda_3\\
&u(y_1)v_0(B\cup C)+u(y_2)[1-v_0(B\cup C)]=\lambda_1 u(y_1)+(1-\lambda_1)u(y_2)\\
&u(y_1)v_0(B)+u(y_2)[1-v_0(B)]=\lambda_2 u(y_1)+(1-\lambda_2)u(y_2)\\
&u(y_1)v_0(C)+u(y_2)[1-v_0(C)]=\lambda_3 u(y_1)+(1-\lambda_3)u(y_2)\\
&u(y_1)>u(y_2),\\
&1-v_0(B\cup C)\neq v_0(B\cup C)-v_0(B)-v_0(C)
\end{aligned}
$$

Summation of the third and fourth equation and elimination of $\lambda_2+\lambda_3$ by means of the first equation yields

$$
\begin{aligned}
&u(y_1)\tfrac{1}{2}[1+v_0(B)+v_0(C)]+u(y_2)\tfrac{1}{2}[1-v_0(B)-v_0(C)\\
&\quad=\lambda_1 u(y_1)+(1-\lambda_1)u(y_2)
\end{aligned}
$$

By using the second equation and $u(y_1)>u(y_2)$ we get $1-v_0(B\cup C)=v_0(B\cup C)-v_0(B)-v_0(C)$ and the contradiction is clear.

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