2010 American Control Conference
Marriott Waterfront, Baltimore, MD, USA
June 30-July 02, 2010

ThC05.6

Exactness Verification of Sum-of-Squares Approximations to Robust Semidefinite Programs with Functional Variables

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***Abstract*— Robust semidefinite programs (robust SDPs in short) with functional variables are revisited in this paper. We firstly consider the approximate approach suggested by Jennawasin and Oishi (in Proceedings of the 17th IFAC World Congress, Seoul, Korea, July 2008), and then provide a numerically computable condition to verify when the optimal value of an approximate problem is actually equal to that of the original robust SDP. The idea is based on capturing some special structure of a dual feasible solution of the approximate problem.**

I. Introduction

Many important problems in control can be formulated as an optimization problem described as follows:

$$\left.\begin{array}{lc} \text{minimize} & c^{\mathrm{T}}x \\ \text{subject to} & \mathcal{F}(x,\phi(\theta),\theta) \succeq O, \quad \forall \theta \in \Theta, \end{array}\right\} \quad (1)$$

where the optimization variables are a vector $x \in \mathbb{R}^n$ and a function ϕ. The function ϕ belongs to the space of piecewise continuous functions which map from $\Theta \in \mathbb{R}^p$ to $\mathbb{R}^{n_\phi}$. A parameter θ, which represents the uncertainties in the given system, can take any value in the set Θ. We assume throughout this paper that the set Θ is a given multi-dimensional interval in $\mathbb{R}^p$. The function $\mathcal{F}(x,a,\theta)$ is an $m \times m$ symmetric-matrix-valued function which is affine in $x \in \mathbb{R}^n$ and $a \in \mathbb{R}^{n_\phi}$, and depends polynomially on θ. An application of this problem is robustness analysis/synthesis for a parameter-dependent linear systems with parameter-dependent Lyapunov functions [5], [8]. The problem (1) is called a *robust semidefinite program (robust SDP) with functional variable* because the constraint has to be satisfied for all possible values of $\theta \in \Theta$. Note here that the standard robust SDPs in the literature [1], [2], [7], [21], [9] can be considered as (1) without the functional variable ϕ. A robust SDP in the form (1) is difficult to solve due to its infinite-dimensional nature caused by the functional variable ϕ. One way to deal with this problem is to reduce the problem into a finite-dimensional one by choosing an appropriate functional basis; for example, polynomial basis, for ϕ. Once the finite-dimensional robust SDP, i.e., robust SDP without the functional variable is obtained, one can apply various approximation schemes [22], [9], [3], [20], [16], [17] to approximate it into a standard SDP in asymptotically exact fashions. The gap between the resulting finite-dimensional robust SDP and the original problem (1) can be arbitrarily reduced using several existing methods [4], [18], [10].

In many examples (See [18], [10], [6]), however, the optimal values of the resulting approximate problems seem to actually equal to that of the original robust SDP (1). When such the situation occurs, it is said that the approximation is exact, or there is no gap between the approximate problem and the original problem. In the context of robust SDPs without functional variables, Several approaches [20], [15] have been suggested to verify whether an approximation is exact. The approach is based on considering a dual feasible solution of the approximate problem. If the dual solution satisfies some certain conditions, we can conclude that the optimal values of the two problems are actually equal, and hence there is no gap between the two problems. Extensions to the case of robust SDPs with functional variables have recently been proposed in [18], [6]. In [18], an approach for exactness verification has been provided for approximate problems constructed by the matrix-dilation technique [16]. Ebihara et al. [6] provided rank conditions on the dual solution to guarantee exactness of approximations to certain classes of robust SDPs (1) in robustness analysis problems.

In this paper, we revisit the approach of [10] which constructs approximate problems of (1) using the sum-of-squares (SOS) technique [19], [13], [12], [22]. The approximation gap is arbitrarily reduced by dividing the region Θ. Inspired by the work of Scherer [20], we observe a dual feasible solution of the approximate problem. If the dual solution contains some specific structure, we can conclude that the optimal values of the two problems are actually equal, i.e., the approximation is exact, and further division on the parameter region is not necessary any more. Moreover, the worst-case parameter θ which achieves the optimal value of (1) can be extracted by solving a linear program. Finally, we will show that the dual variable contains some block-matrix structure relevant with that of the moment matrices in [13], [12].

Contributions of our approach are summarized as follows. (I) Comparing with [6] which also applies to SOS approximations, the current approach provides a difference condition for the exactness verification of the SOS approximations. Our framework, however, can be applied not only to the robust SDPs arising from robustness analysis problems but also to the more general class of robust SDPs of the form (1). In addition, we show that the block-matrix structure of the dual variables is still preserved when the SOS approach is applied to a general robust SDP of the form (1).
(II) Unlike the conventional approaches in [20], [22], [6]

This work was not supported by any organization
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which increase the degrees of relevant polynomials to reduce the gap between the approximate problem and the original problem, the current approach reduces the gap by dividing the parameter region Θ without increasing the degrees of the polynomials. This leads to an approximate SDP with several constraints whose dual variables are uncoupled from one another. This problem structure allows us to apply a parallel computation to check the dual variables for the exactness verification.

II. A Region-Dividing Approach to Robust SDPs

A region-dividing approach in [10] to the robust SDP (1) is summarized in this section. In this approach, we make the problem finite-dimensional by choosing ϕ as a polynomial with fixed degree. In order to improve the quality of approximation, we divide the parameter set Θ into several subregions and allow ϕ to be a piecewise polynomial consistent to the division. Then we have a finite-dimensional robust SDP with several semi-infinite constraints corresponding to the division. To deal with the semi-infinite constraints, we apply the notion of sum-of-squares (SOS) matrices in [12], [22]. Finally, we obtain an approximate problem which is a standard SDP. The optimal value of the approximate problem converges to that of the original problem, as the resolution of the division becomes higher.

For the functional variable $\phi(\theta)$, we use a fixed-degree polynomial $\sum_{\alpha\in V} u_\alpha \theta^\alpha$ for some finite set $V \subset \mathbb{Z}_+^p$. Here the symbol θ^α stands for the product $\theta_1^{\alpha_1}\theta_2^{\alpha_2}\cdots\theta_p^{\alpha_p}$ with $\alpha = \begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_p\end{bmatrix} \in \mathbb{Z}_+^p$. We use the coefficients $u = (u_\alpha) \in \mathbb{R}^{n_u}$ to characterize the polynomial and write $\phi_u(\theta) = \sum_{\alpha\in V} u_\alpha\theta^\alpha$. Substitution of $\phi_u(\theta)$ into $\mathcal{F}$ makes this function dependent on finite-dimensional variables x and u. In particular, we define the notation

$$F(x,u,\theta) := \mathcal{F}(x,\phi_u(\theta),\theta).$$

Note that F is affine in x and u while polynomial in θ. Precisely, $F(x,u,\theta)$ is written as:

$$\begin{aligned} F(x,u,\theta) \;\; &= \;\; F_{00\cdots 0}(x,u) + F_{10\cdots 0}(x,u)\theta_1 + \cdots \\ &\qquad + F_{d_1 d_2\cdots d_p}(x,u)\theta_1^{d_1}\theta_2^{d_2}\cdots\theta_p^{d_p}. \end{aligned}$$

The definition of *a division* Δ of the parameter set Θ is given here for the succeeding discussion. We define a *division* $\Delta = \{\Theta^{[j]}\}_{j=1}^J$ of Θ as a set of closed convex polytopes such that $\Theta = \cup_{j=1}^J \Theta^{[j]}$ holds and $\Theta^{[j]}\cap\Theta^{[k]}$ has no interior point whenever $j\neq k$. Each element $\Theta^{[j]}$ of a division Δ is called a *subregion*. We assume that each $\Theta^{[j]}$ is a p-dimensional interval $\Pi_{i=1}^p[\underline{\theta}_i^{[j]},\overline{\theta}_i^{[j]}]$. Here, the coefficients u is allowed to take a difference value $u^{[j]}$ depending on the subregion $\Theta^{[j]}$, for each $j=1,2,\ldots,J$. Hence, the function ϕ is a piecewise polynomial.

We then consider the following finite-dimensional problem:

$$P_0(\Delta):\quad \left.\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x \\ \text{subject to} & F(x,u^{[j]},\theta)\succeq O, \quad \forall\theta\in\Theta^{[j]}, \\ & \forall j=1,2,\ldots,J, \end{array}\right\}$$

where the optimization variables are $x \in \mathbb{R}^n$ and $u^{[1]},u^{[2]},\ldots,u^{[j]}\in\mathbb{R}^{n_u}$. Since only a specific class of function are considered for ϕ in this problem, we immediately have $\inf P_0(\Delta)\geq v_{\mathrm{opt}}$, where $\inf P_0(\Delta)$ and v_{opt} denote the optimal values of $P_0(\Delta)$ and (1) respectively.

The problem $P_0(\Delta)$ is still difficult to solve due to its semi-infinite constraint. Here, we apply the result of Proposition 1 to overcome this difficulty. In particular, an approximate problem for $P_0(\Delta)$ with the notion of SOS matrices:

$$P(\Delta):\quad \left.\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x \\ \text{subject to} & F(x,u^{[j]},\theta) = S_0^{[j]}(\theta) \\ & \quad + \displaystyle\sum_{i=1}^p(\theta_i-\underline{\theta}_i^{[j]})(\overline{\theta}_i^{[j]}-\theta_i)S_i^{[j]}(\theta), \\ & \forall j=1,2,\ldots,J, \end{array}\right\}$$

where $S_0^{[j]},S_1^{[j]},\ldots,S_p^{[j]}$ are SOS matrices, for all $j = 1,2,\ldots,J$. In our setting, we use the same *monomial basis*, say $z_i(\theta)$, for the SOS matrices $S_i^{[j]}(\theta)$, for all $j=1,2,\ldots,J$. This leads to the parameterization $S_i^{[j]} = (z_i(\theta)\otimes I_m)^{\mathrm{T}}Q_i^{[j]}(z_i(\theta)\otimes I_m)$, $\forall j = 1,2,\ldots,J$, for some positive semidefinite matrices $Q_i^{[j]}$'s (See Appendix for details). With this parameterization, it is not difficult to express the problem $P(\Delta)$ as an SDP in the decision variables x, $u^{[j]}$, and $Q_0^{[j]},Q_1^{[j]},\ldots,Q_p^{[j]}$, for $j=1,2,\ldots,J$, using the idea discussed in [19], [22].

For each j, the existence of the SOS matrices $S_0^{[j]},S_1^{[j]},\ldots,S_p^{[j]}$ implies that $F(x,u^{[j]},\theta)\succeq O,\forall\theta\in\Theta^{[j]}$. This is immediately obtained from the definition of SOS matrices and the assumption on $\Theta^{[j]}$. Hence the feasible region of $P(\Delta)$ projected into the space of x and $u^{[1]},u^{[2]},\ldots,u^{[J]}$ is included in the feasible region of $P_0(\Delta)$. In particular, $\inf P(\Delta)\geq\inf P_0(\Delta)\geq v_{\mathrm{opt}}$.

We now have an approximate problem, which is a standard SDP, for (1). In order to improve the approximation, we make subdivision on Δ and solve again the new approximate problem $P(\Delta)$. This procedure is repeatedly performed until the obtained optimal value $\inf P(\Delta)$ is satisfactory. In fact, $\inf P(\Delta)$ converges to v_{opt} as the resolution of the division Δ becomes higher if the monomial bases $z_0(\theta)$, $z_1(\theta)$, ..., $z_p(\theta)$ are chosen as (See [10] for the proof)

$$\begin{aligned} z_0(\theta) &= \begin{bmatrix}1 & \theta_1 & \theta_2 & \cdots & \theta_1^{d_1+1}\theta_2^{d_2+1}\cdots\theta_p^{d_p+1}\end{bmatrix}^{\mathrm{T}}, \\ z_i(\theta) &= \begin{bmatrix}1 & \theta_1 & \theta_2 & \cdots & \theta_1^{d_1}\theta_2^{d_2}\cdots\theta_p^{d_p}\end{bmatrix}^{\mathrm{T}},\ i=1,\ldots,p, \end{aligned} \tag{2}$$

where d_1,..., d_p are the highest degrees of variables θ_1, ..., θ_p, respectively, in the polynomial expression of $F(x,u^{[j]},\theta)$. In many cases, however, the approximate optimal value $\inf P(\Delta)$ turns out to be equal to the true optimal value v_{opt}. In the next section, we discuss how to verify whether $\inf P(\Delta) = v_{\mathrm{opt}}$. In such case, no further improvement on the approximation is needed any more.

III. Main Result

Exactness verification of the approximate problem $P(\Delta)$ is discussed in this section.

We start from a simple idea, which was suggested by [18], that if an optimal solution $(\hat{x}, \{\hat{u}^{[j]}\}, \{\hat{Q}_i^{[j]}\}_{i=0}^p)$ is obtained, and there exist some j and $\hat{\theta} \in \Theta^{[j]}$ such that $(\hat{x}, \hat{u}^{[j]})$ is optimal for the problem

$$P(\hat{\theta}): \quad \left.\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x \\ \text{subject to} & F(x, u^{[j]}, \hat{\theta}) \succeq O, \end{array}\right\}$$

then $\inf P(\Delta) = v_{\mathrm{opt}}$. To see this, note that $\inf P(\hat{\theta}) \leq v_{\mathrm{opt}} \leq \inf P(\Delta) = c^{\mathrm{T}}\hat{x}$ in general, and $\inf P(\hat{\theta}) = c^{\mathrm{T}}\hat{x}$. This implies that these three values are equal to each other. The parameter $\hat{\theta} \in \Theta^{[j]}$ satisfying the above condition is understood as the worst-case parameter value.

Optimality of $(\hat{x}, \hat{u}^{[j]})$ can be verified by the existence of a dual feasible solution satisfying the complementary slackness condition [1]. More precisely, $(\hat{x}, \hat{u}^{[j]})$ is optimal for $P(\hat{\theta})$ if and only if there exists a matrix $\widehat{W} \in \mathbb{R}^{m\times m}$ such that

$$\begin{aligned} \widehat{W} &\succeq O, & (3)\\ \langle \widehat{W}, F(e_i^n, 0, \hat{\theta}) - F(0,0,\hat{\theta})\rangle &= c_i,\ i = 1,2,\ldots,n, & (4)\\ \langle \widehat{W}, F(0, e_i^{n_u}, \hat{\theta}) - F(0,0,\hat{\theta})\rangle &= 0,\ i = 1,2,\ldots,n_u, & (5)\\ \langle \widehat{W}, F(\hat{x}, \hat{u}^{[j]}, \hat{\theta})\rangle &= 0, & (6) \end{aligned}$$

where $\langle A, B\rangle := \mathrm{Trace}(AB)$ for symmetric matrices A and B, and e_i^n be the i^{th} column of the $n \times n$-identity matrix. Since direct computation of $\widehat{W}$ and $\hat{\theta}$ satisfying (3)-(6) is difficult, we next provide a computable sufficient condition to guarantee the existence of such $\widehat{W}$ and $\hat{\theta}$. The sufficient condition can be derived by examining some special structure of the dual solution of $P(\Delta)$.

Recall that the constraints of $P(\Delta)$ are represented as

$$\begin{aligned} F(x, u^{[j]}, \theta) &= (z_0(\theta)\otimes I_m)^{\mathrm{T}} Q_0^{[j]} (z_0(\theta)\otimes I_m) \\ &\quad + \sum_{i=1}^{p} (\theta_i - \underline{\theta}_i^{[j]})(\overline{\theta}_i^{[j]} - \theta_i) \\ &\quad (z_i(\theta)\otimes I_m)^{\mathrm{T}} Q_i^{[j]} (z_i(\theta)\otimes I_m), \\ &\quad \forall j = 1,2,\ldots,J, \end{aligned} \tag{7}$$

with the monomial bases $z_0, z_1, \ldots, z_p$ chosen from (2) and $Q_0^{[j]} \succeq O, Q_1^{[j]} \succeq O, \ldots, Q_p^{[j]} \succeq O$. We see from (7) that, for each j, the matrix $Q_0^{[j]}$ depends affinely on $Q_1^{[j]}, \ldots, Q_p^{[j]}$ and matrix coefficients of $F(x, u^{[j]}, \theta)$. If we set $Q_1^{[j]}, \ldots, Q_p^{[j]}, u^{[j]}, x$ as free variables, then $P(\Delta)$ can be represented as the following SDP:

$$P(\Delta): \quad \left.\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x \\ \text{subject to} & Q_0^{[j]} = \sum_{r=1}^{n_q} q_r^{[j]} B_r + \sum_{s=1}^{n_k} k_s^{[j]} C_s \\ & +E(x, u^{[j]}) \succeq O, \\ & Q_i^{[j]} \succeq O, \quad i = 1,\ldots,p, \\ & \forall j = 1,2,\ldots,J, \end{array}\right\}$$

where $q^{[j]} = (q_1^{[j]}, q_2^{[j]}, \ldots, q_{n_q}^{[j]})$ is a vector containing all the elements of $Q_1^{[j]}, \ldots, Q_p^{[j]}$, $k^{[j]} = (k_1^{[j]}, \ldots, k_{n_k}^{[j]})$ is a vector containing the remaining free variables, and $B_1, \ldots, B_{n_q}$, $C_1, \ldots, C_{n_k}$ are constant matrices. The matrix $E(x, u^{[j]})$ contains matrix coefficients of $F(x, u^{[j]}, \theta)$, and satisfies $(z_0(\theta)\otimes I_m)^{\mathrm{T}} E(x, u^{[j]}) (z_0(\theta)\otimes I_m) = F(x, u^{[j]}, \theta)$. Note here that $E(x, u^{[j]})$ depends affinely on x and $u^{[j]}$. Moreover, such decomposition for $F(x, u^{[j]}, \theta)$ is possible as shown in [16].

The following theorem states that if a dual feasible solution of $P(\Delta)$ contains some special structure then the approximate problem $P(\Delta)$ is exact, i.e., $\inf P(\Delta) = v_{\mathrm{opt}}$.

Theorem 1: Let $(\hat{x}, \{\hat{u}^{[j]}\}, \{\hat{Q}_i^{[j]}\}_{i=1}^p, \{\hat{k}^{[j]}\})$ be an optimal solution of $P(\Delta)$ for a division Δ. Consider a subregion $\Theta^{[j]}$, and dual feasible solutions $\widehat{Y}_0^{[j]}$ associated with the constraint $\sum_{r=1}^{n_q} q_r^{[j]} B_r + \sum_{s=1}^{n_k} k_s^{[j]} C_s + E(x, u^{[j]}) \succeq O$, and $\widehat{Y}_i^{[j]}$ associated with $\hat{Q}_i^{[j]} \succeq O$, for $i = 1, \ldots, p$. If $\widehat{Y}_0^{[j]}, \widehat{Y}_1^{[j]}, \ldots, \widehat{Y}_p^{[j]}$ satisfy the complementary slackness condition with $(\hat{x}, \hat{u}^{[j]}, \hat{Q}_0^{[j]}, \hat{Q}_1^{[j]}, \ldots, \hat{Q}_p^{[j]})$, and there exist a point $\hat{\theta} \in \Theta^{[j]}$, a matrix $\widehat{W} \succeq O$, and positive numbers $a_0, a_1, \ldots, a_p$ such that

$$\widehat{Y}_i^{[j]} = a_i (z_i(\hat{\theta})\otimes I_m) \widehat{W} (z_i(\hat{\theta})\otimes I_m)^{\mathrm{T}}, \quad i = 0,1,\ldots,p,$$

Then $\widehat{W}$ and $\hat{\theta}$ satisfy (3)–(6).

Proof: Suppose the dual feasible solutions $\widehat{Y}_0^{[j]}, \widehat{Y}_1^{[j]}, \ldots, \widehat{Y}_p^{[j]}$ satisfy the dual feasibility conditions

$$\widehat{Y}_i^{[j]} \succeq O, \quad i = 0,1,\ldots,p, \tag{8}$$

$$\langle \widehat{Y}_0^{[j]}, E(e_i^n, 0) - E(0,0)\rangle = c_i, \quad i = 1,2,\ldots,n, \tag{9}$$

$$\langle \widehat{Y}_0^{[j]}, E(0, e_i^{n_u}) - E(0,0)\rangle = 0, \quad i = 1,2,\ldots,n_u, \tag{10}$$

$$\langle \widehat{Y}_0^{[j]}, \sum_{r=1}^{n_q} q_r B_r + \sum_{s=1}^{n_k} k_s C_s\rangle + \sum_{i=1}^{p} \langle \widehat{Y}_i^{[j]}, Q_i\rangle = 0, \quad \text{for any } Q_1, \ldots, Q_p, k, \tag{11}$$

and the complementary slackness conditions

$$\begin{aligned} \langle \widehat{Y}_0^{[j]}, \hat{Q}_0^{[j]}\rangle &= 0, & (12)\\ \langle \widehat{Y}_i^{[j]}, \hat{Q}_i^{[j]}\rangle &= 0, \ i = 1,\ldots,p. & (13) \end{aligned}$$

Substituting $\widehat{Y}_0^{[j]} = a_0 (z_0(\hat{\theta})\otimes I_m) \widehat{W} (z_0(\hat{\theta})\otimes I_m)^{\mathrm{T}}$ into (9) yields (4), due to the fact that $(z_0(\hat{\theta})\otimes I_m)^{\mathrm{T}} E(x,u) (z_0(\hat{\theta})\otimes I_m) = F(x, u, \hat{\theta})$. We also obtain (5) from (10) by a similar reason. Finally, the complementary slackness conditions (12)-(13) imply that

$$\langle \widehat{W}, (z_0(\hat{\theta})\otimes I_m)^{\mathrm{T}} \hat{Q}_0^{[j]} (z_0(\hat{\theta})\otimes I_m)\rangle = 0,$$

and

$$\begin{aligned} &\langle \widehat{W}, (\hat{\theta}_i - \underline{\theta}_i^{[j]})(\overline{\theta}_i^{[j]} - \hat{\theta}_i)(z_i(\hat{\theta})\otimes I_m)^{\mathrm{T}} \hat{Q}_i^{[j]} (z_i(\hat{\theta})\otimes I_m)\rangle, \\ &\quad = 0, \quad i = 1,\ldots,p. \end{aligned}$$

Summing the above equations yields (6). ■

For a given $Y \succeq O$, there exist $W \succeq O$ and $a \geq 0$ such that $Y = a(z(\hat{\theta})\otimes I_m) W (z(\hat{\theta})\otimes I_m)^{\mathrm{T}}$ if and only if $H(\hat{\theta})^{\mathrm{T}} Y = 0$, where $H(\theta)$ is an orthogonal complement of $(z(\theta)\otimes I_m)$, i.e., $H(\theta)^{\mathrm{T}}(z(\theta)\otimes I_m) = 0$. Here, the upper-left $m \times m$ submatrix of Y is equal to aW. Note that $H(\hat{\theta})$ can be chosen to be affine in $\hat{\theta}$ [16]. Once we obtain the dual solutions $\widehat{Y}_i$ whose the upper-left $m \times m$ submatrices are all in the same structure, the parameter $\hat{\theta}$ can be computed by solving a linear programming problem. Note here that the condition $H(\hat{\theta})^{\mathrm{T}} Y = 0$ is consistent with those for exactness verification in the literature [20], [21], [16], [18].

It can be seen that the constraints (8)–(13) for the dual variables $\widehat{Y}_0^{[j]}, \widehat{Y}_1^{[j]}, \ldots, \widehat{Y}_p^{[j]}$ are uncoupled, with respect to each subregion $\Theta^{[j]}$, from one another. Therefore, this allows us to simply apply a parallel computation, for example [11], for simultaneously solving $\widehat{Y}_0^{[j]}, \widehat{Y}_1^{[j]}, \ldots, \widehat{Y}_p^{[j]}$ in every subregion. Reduction of the overall computational time can be expected from such parallel computation, even in the case of a large number of subregions.

Remark: If the above condition fails to verify the exactness of the approximate optimal value, we can compute a lower bound on the actual optimal value $v_{\rm opt}$ by randomly sampling in Θ, and solve an SDP with constraints corresponding to the sampled points. If the lower bound and the upper bound $\inf P(\Delta)$ are close to each other, then a good approximate optimal value can be obtained from $\inf P(\Delta)$.

We close this section with the discussion on the structure of the dual variables $\widehat{Y}_0^{[j]}, \widehat{Y}_1^{[j]}, \ldots, \widehat{Y}_p^{[j]}$. In particular, we will show that the matrices $\widehat{Y}_0^{[j]}, \widehat{Y}_1^{[j]}, \ldots, \widehat{Y}_p^{[j]}$ which have the moment-matrix structure [13], [12] satisfy the dual constraint (11).

Before going ahead, the following definition will be useful for the succeeding discussion.

Definition 1: Let A be a matrix of dimension $lm \times lm$ and A is partitioned into block matrices of dimension $m \times m$ as follows,

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1l} \\ A_{21} & A_{22} & \cdots & A_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ A_{l1} & A_{l2} & \cdots & A_{ll} \end{bmatrix}.$$

Define $\mathrm{Trace}_m(A)$ to be the $m \times m$ matrix obtained from A as follows,

$$\mathrm{Trace}_m(A) \triangleq A_{11} + A_{22} + \cdots + A_{ll}.$$

■

For illustrating the idea, let us consider a robust SDP of the form (1) with the parameter $-\beta_1 \leq \theta_1 \leq \beta_1$, and $-\beta_2 \leq \theta_2 \leq \beta_2$, where β_1 and β_2 are given positive numbers. Suppose that the corresponding SOS-approximate problem is constructed with the following SOS constraint using the coarsest division $\Delta = [-\beta_1, \beta_1] \times [-\beta_2, \beta_2]$:

$$\begin{aligned} F(x,u,\theta) \;=\; & (z_0(\theta) \otimes I_m)^{\mathrm{T}} Q_0 (z_0(\theta) \otimes I_m) \\ & + (\beta_1^2 - \theta_1^2)(z_1(\theta) \otimes I_m)^{\mathrm{T}} Q_1 (z_1(\theta) \otimes I_m) \\ & + (\beta_2^2 - \theta_2^2)(z_2(\theta) \otimes I_m)^{\mathrm{T}} Q_2 (z_2(\theta) \otimes I_m). \end{aligned} \tag{14}$$

Suppose that $F(x,u,\theta)$ is a polynomial matrix of degree 2 in θ_1, and θ_2, the monomail bases z_0, z_1 and z_2 are given, for example, by

$$\begin{aligned} z_0(\theta) &= [1 \;\; \theta_1 \;\; \theta_2 \;\; \theta_1^2 \;\; \theta_1\theta_2 \;\; \theta_2^2], \\ z_1(\theta) &= z_2(\theta) = [1 \;\; \theta_1 \;\; \theta_2]. \end{aligned}$$

Using Definition 1 and the decomposition $(z_0(\theta) \otimes I_m)^{\mathrm{T}} E(x,u)(z_0(\theta) \otimes I_m) = F(x,u,\theta)$, the SOS constraint (14) can be rewritten as

$$\begin{aligned} & \mathrm{Trace}_m(E(x,u)(z_0(\theta) \otimes I_m)(z_0(\theta) \otimes I_m)^{\mathrm{T}}) = \\ & \mathrm{Trace}_m(Q_0(z_0(\theta) \otimes I_m)(z_0(\theta) \otimes I_m)^{\mathrm{T}}) \\ & + \mathrm{Trace}_m(Q_1(\beta_1^2 - \theta_1^2)(z_1(\theta) \otimes I_m)(z_1(\theta) \otimes I_m)^{\mathrm{T}}) \\ & + \mathrm{Trace}_m(Q_2(\beta_2^2 - \theta_2^2)(z_2(\theta) \otimes I_m)(z_2(\theta) \otimes I_m)^{\mathrm{T}}). \end{aligned} \tag{15}$$

It is not difficult to see that equality in (15) is still valid if the matrices $(z_0(\theta) \otimes I_m)(z_0(\theta) \otimes I_m)^{\mathrm{T}}$, $(\beta_1^2 - \theta_1^2)(z_1(\theta) \otimes I_m)(z_1(\theta) \otimes I_m)^{\mathrm{T}}$, and $(\beta_2^2 - \theta_2^2)(z_2(\theta) \otimes I_m)(z_2(\theta) \otimes I_m)^{\mathrm{T}}$ are replaced by

$$\begin{aligned} Y_0 &= \begin{bmatrix} Y_{00} & Y_{10} & Y_{01} & Y_{20} & Y_{11} & Y_{02} \\ Y_{10} & Y_{20} & Y_{11} & Y_{30} & Y_{21} & Y_{12} \\ Y_{01} & Y_{11} & Y_{02} & Y_{21} & Y_{12} & Y_{03} \\ Y_{20} & Y_{30} & Y_{21} & Y_{40} & Y_{31} & Y_{22} \\ Y_{11} & Y_{21} & Y_{12} & Y_{31} & Y_{22} & Y_{13} \\ Y_{02} & Y_{12} & Y_{03} & Y_{22} & Y_{13} & Y_{04} \end{bmatrix}, \\ Y_1 &= \beta_1^2 \begin{bmatrix} Y_{00} & Y_{10} & Y_{01} \\ Y_{10} & Y_{20} & Y_{11} \\ Y_{01} & Y_{11} & Y_{02} \end{bmatrix} - \begin{bmatrix} Y_{20} & Y_{30} & Y_{21} \\ Y_{30} & Y_{40} & Y_{31} \\ Y_{21} & Y_{31} & Y_{22} \end{bmatrix}, \\ Y_2 &= \beta_2^2 \begin{bmatrix} Y_{00} & Y_{10} & Y_{01} \\ Y_{10} & Y_{20} & Y_{11} \\ Y_{01} & Y_{11} & Y_{02} \end{bmatrix} - \begin{bmatrix} Y_{02} & Y_{12} & Y_{03} \\ Y_{12} & Y_{22} & Y_{13} \\ Y_{03} & Y_{13} & Y_{04} \end{bmatrix}, \end{aligned} \tag{16}$$

respectively. In particular, the matrices Y_0, Y_1, and Y_2 above satisfy the equation

$$\begin{aligned} \mathrm{Trace}_m(E(x,u)Y_0) \;=\; & \mathrm{Trace}_m(Q_0 Y_0) + \mathrm{Trace}_m(Q_1 Y_1) \\ & + \mathrm{Trace}_m(Q_2 Y_2). \end{aligned}$$

Since $\mathrm{Trace}(\mathrm{Trace}_m(A)) = \mathrm{Trace}(A)$, the above equation implies

$$\langle Y_0, Q_0 - E(x,u) \rangle + \sum_{i=1}^{2} \langle Y_i, Q_i \rangle = 0,$$

which is nothing but the dual constraint (11) on the variables Y_0, Y_1, and Y_2, because of the relation $Q_0 = \sum_{r=1}^{n_q} q_r B_r + \sum_{s=1}^{n_k} k_s C_s + E(x,u)$.

It is notable that Y_0, Y_1, and Y_2 in (16) are constructed from $(z_0(\theta) \otimes I_m)(z_0(\theta) \otimes I_m)^{\mathrm{T}}$, $(\beta_1^2 - \theta_1^2)(z_1(\theta) \otimes I_m)(z_1(\theta) \otimes I_m)^{\mathrm{T}}$, and $(\beta_2^2 - \theta_2^2)(z_2(\theta) \otimes I_m)(z_2(\theta) \otimes I_m)^{\mathrm{T}}$ by replacing the block $\theta_1^{\alpha_1}\theta_2^{\alpha_2} I_m$ with the new variable $Y_{\alpha_1\alpha_2} \in \mathbb{R}^{m \times m}$. Such construction is consistent with the construction of moment matrices in [13], [12]. It is notable that the moment-matrix structure of Y_0, Y_1, and Y_2 does not depend on $F(x,u,\theta)$ but depends only on the choice of z_0, z_1 and z_2. This idea can be extended more generally; that is we can use the moment-matrix construction to extract the particular structure of the dual variables $\widehat{Y}_0^{[j]}, \widehat{Y}_1^{[j]}, \ldots, \widehat{Y}_p^{[j]}$ associated with the constraints of $P(\Delta)$ for any p and for any choices of the monomial bases $z_0(\theta), z_1(\theta), \ldots, z_p(\theta)$.

IV. Examples

Two numerical examples are provided in this section to illustrate the idea. The software YALMIP [14] with SeDuMi [23] as an SDP solver is used for the computation.

Example 1: Consider the following uncertain system borrowed from [5]

$$\begin{aligned}\dot{x}(t) &= A(\theta)x(t) + Bw(t)\\ y(t) &= Cx(t),\end{aligned}$$

where $A(\theta) = A_0 + \theta A_1$. Constant matrices A_0, A_1, B, and C are given as

$$A_0 = \begin{bmatrix} 0.1 & 2.5 & 5 \\ 4.9 & -3 & 0.5 \\ -5.5 & -5 & -10.7 \end{bmatrix}, \ A_1 = \begin{bmatrix} -4.2 & -3 & -12 \\ -4.8 & 0 & 0 \\ 9 & 12 & 14.4 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

The parameter set is given by $\Theta = [0, 1]$. The maximum L_2-induced norm over Θ is computed by solving the following robust SDP:

$$\begin{aligned}&\text{minimize} \quad \gamma\\ &\text{subject to}\\ &X(\theta) \succeq O,\\ &\begin{bmatrix} -X(\theta)A(\theta) - A(\theta)^{\mathrm{T}}X(\theta) & -X(\theta)B(\theta) & C^{\mathrm{T}}(\theta) \\ -B(\theta)^{\mathrm{T}}X(\theta) & \gamma I & O \\ C(\theta) & O & \gamma I \end{bmatrix} \succeq O,\\ &\forall \theta \in \Theta\end{aligned}$$

where the decision variables are a real scalar γ and a symmetric-matrix-valued function $X(\theta)$. We first compute an upper bound on the maximum induced norm using $X(\theta) = X_0 + X_1\theta + X_2\theta^2$ and the coarsest division $\Delta = \{\Theta\}$. The computed upper bound is $\inf P(\Delta) = 6.2010$. By examining the dual optimal solution of $P(\Delta)$, the worst-case parameter $\hat{\theta} = 0.9732$ is computed. The L_2-induced norm of the system with respect to this parameter is 6.2010, which is actually equal to the computed upper bound. The exactness of $P(\Delta)$ is numerically verified, and hence further increase of the degree of $X(\theta)$ or division on Θ is not needed any more. ■

Example 2: We next apply our approach with a system with vector uncertain parameter. Consider the following system borrowed from [21]

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B(\theta)w(t)\\ y(t) &= C(\theta)x(t),\end{aligned}$$

with

$$\begin{aligned} A &= \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix},\\ B(\theta) &= \begin{bmatrix} -3\theta_2 \frac{(2a^2+a)\theta_1^2\theta_2 + 2a\theta_1^2 + 2a\theta_2 + \theta_1^2 - 2}{2 - 2a^2\theta_2^2 - \theta_1^2 + a^2\theta_1^2\theta_2^2} \\ 1 \end{bmatrix},\\ C(\theta) &= \begin{bmatrix} -1 & 0 \end{bmatrix}.\end{aligned}$$

The parameter set is $\Theta = [-0.7, 0.8]^2$, and a takes a value between 0 and 1. We again compute an upper bound the maximum L_2-induced norm of the system by solving a robust SDP. Rational dependence on the parameter can be made to polynomial dependence by multiplication of an appropriate polynomial.

We first solve the robust SDP, for each a, using $X(\theta) = X_{00} + X_{10}\theta_1 + X_{20}\theta_1^2 + X_{01}\theta_2$ and the coarsest division $\Delta = \{\Theta\}$. Our approach can guarantee the exactness of the computed upper bound for $a = [0, 0.4)$ and $a = (0.7, 1]$. For $a = [0.4, 0.7]$, we obtain a lower bound by gridding the region and computing the maximum norm over the grid points. The numerical result show that the upper bound and the lower bound are closed to each other.

The problem is then solved again using the same $X(\theta)$ and the division $\Delta = \{\Theta^{[1]}, \Theta^{[2]}\}$ with $\Theta^{[1]} = [-0.7, 0.05] \times [-0.7, 0.8]$ and $\Theta^{[2]} = [-0.05, 0.8] \times [-0.7, 0.8]$. The upper bound for each a in this case is the same as that of the coarsest division. However, the exactness of the lower bound is verified for $a = [0, 0.4)$ and $a = (0.5, 1]$. This reveals that subdivision on Θ also increases the chance of the exactness verification. ■

V. Conclusion

We provided a computational approach to verify the exactness of SOS approximations to the robust SDPs with the functional variable. A dual feasible solution of the approximate problem is considered in the current approach. The certain constraints on the dual solution guarantee there is no gap between the approximate problem and the original problem. The worst-case parameter can also be extracted by solving a linear program. Finally, the moment-matrix structure of the dual variables was also discussed. Since the constraint $\widehat{Y}_i = a_i(z_i(\hat{\theta}) \otimes I_m)\widehat{W}(z_i(\hat{\theta}) \otimes I_m)^{\mathrm{T}}$ in Theorem 1 implies some rank constraint on the dual variable $\widehat{Y}_i$, it is interesting to find some connections between the resulting rank constraint and the rank constraint suggested by [6], when applying the current approach to the robustness analysis problems.

Acknowledgement

This work is supported in part by the Hitech Research Center, Project for Private University from the Ministry of Education, Culture, Sports, Science and Technology. The authors would like to thank Yasuaki Oishi for helpful suggestions on the proof of the main theorem.

References

[1] A. Ben-Tal and A. Nemirovski. *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*. Philadelphia, USA: SIAM, 2001.

[2] A. Ben-Tal and A. Nemirovski. On tractable approximations of uncertain linear matrix inequality affected by interval uncertainty. *SIAM Journal on Optimization*, vol. 12, no. 3, pp. 811–833, 2002.

[3] P.-A. Bliman. On robust semidefinite programming. In *Proceeding of the 16th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2004)*, Leuven, Belgium, July 2004.

[4] P.-A. Bliman. An existence result for polynomial solutions of parameter-dependent LMIs. *Systems & Control Letters*, vol. 51, nos. 3–4, pp. 165–169, 2004.

[5] G. Chesi, A. Garulli, A. Tesi, and A. Vicino. Polynomially parameter-dependent Lyapunov functions for robust H_∞ performance analysis. In *Proceedings of the 16th IFAC World Congress*, Prague, Czech Republic, July 2005.

[6] Y. Ebihara, Y. Onishi and T. Hagiwara. Robust performance analysis of uncertain LTI systems: Dual LMI approach and verifications for exactness. *IEEE Transactions on Automatic Control*, vol. 54, no. 5, pp. 938–951, 2009.

[7] L. El Ghaoui, F. Oustry, and H. Lebret. Robust solutions to uncertain semidefinite programs. *SIAM Journal on Optimization*, vol. 9, no. 1, pp. 33–52, 1998.

[8] D. Henrion, D. Arzelier, D. Peaucelle, and J. B. Lasserre. On parameter-dependent Lyapunov functions for robust stability of linear systems. In *Proceedings of the 43th IEEE Conference on Decision and Control*, Paradise Island, Bahamas, December 2004.
[9] T. Jennawasin and Y. Oishi. A region-dividing technique for constructing the sum-of-squares approximations to robust semidefinite programs. In *Proceedings of the 46th IEEE Conference on Decision and Control*, New Orleans, USA, December 2007.
[10] T. Jennawasin and Y. Oishi. Sum-of-squares approximations to robust semidefinite programs with functional variables: A region-dividing approach. In *Proceedings of the 17th IFAC World Congress*, Seoul, Korea, July 2008.
[11] M. Kawanishi and Y. Shibata. BMI global optimization using parallel branch and bound method with a novel branching method. In *Proceedings of the 2007 American Control Conference*, New York, USA, July 2007.
[12] M. Kojima. Sum of squares relaxations in polynomial semidefinite programs. *Research Reports on Mathematical and Computing Sciences*, Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, 2003.
[13] J. B. Lasserre. Global optimization with polynomials and the problems of moments. *SIAM Journal on Optimization*, vol. 11, no. 3, pp. 796–817, 2001.
[14] J. Löfberg. YALMIP: A Toolbox for Modeling and Optimization in MATLAB. In *Proceedings of the CACSD Conference*, Taipei, Taiwan, September 2004.
[15] I. Masubuchi and C. W. Scherer. A recursive algorithm of exactness verification of relaxations for robust SDPs. *Systems & Control Letters*, vol. 58, no. 8, pp. 592–601, 2009.
[16] Y. Oishi. A region-dividing approach to robust semidefinite programming and its error bound. In *Proceedings of the 2006 American Control Conference*, Minneapolis, USA, June 2006.
[17] Y. Oishi. Reduction of the number of constraints in the matrix-dilation approach to robust semidefinite programming. In *Proceedings of the 45th IEEE Conference on Decision and Control*, San Diego, USA, December 2006.
[18] Y. Oishi. An asymptotically exact approach to robust semidefinite programming problems with function variables. *IEEE Transactions on Automatic Control*, vol. 54, no. 5, pp. 1000–1006, 2009.
[19] P. A. Parrilo. *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. PhD Thesis, California Institute of Technology, May 2000.
[20] C. W. Scherer. Relaxations for robust linear matrix inequality problems with verifications for exactness. *SIAM Journal on Matrix Analysis and Applications*, vol. 27, no. 2, pp. 365–395, 2005.
[21] C. W. Scherer. LMI relaxations in robust control. *European Journal of Control*, vol. 12, no. 1, pp. 3–29, 2006.
[22] C. W. Scherer and C. W. J. Hol. Matrix sum-of-squares relaxations for robust semi-definite Programs. *Mathematical Programming, Series B*, vol. 107, nos. 1–2, pp. 189–211, 2006.
[23] J. F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, vols. 11–12, pp. 625–653, 1999.

Appendix

Sum-of-squares polynomial matrices

Let $\mathbb{R}[\theta]^{m\times n}$ denote the set of $m \times n$ polynomial matrices in $\theta \in \mathbb{R}^p$ and $\mathbb{S}[\theta]^n$ denote the set of $n \times n$ symmetric polynomial matrices. We use the notion of *sum-of-squares (SOS)* polynomial matrices as follows.

Definition 2: [12], [22] A polynomial matrix $S \in \mathbb{S}[\theta]^m$ is said to be a sum of squares (SOS) if there exists a polynomial matrix $T \in \mathbb{R}[\theta]^{q\times m}$ such that

$$S(\theta) = T(\theta)^{\mathrm{T}} T(\theta).$$

This is a generalization of the SOS representation for scalars [13], [19]. We use $\Sigma[\theta]^m$ to represent the set of $m\times m$ SOS polynomial matrices. It is clear that any polynomial matrix $S \in \Sigma[\theta]^m$ is globally positive semidefinite, but the converse is not true in general.

A computational procedure for verifying whether $S(\theta)$ is an SOS proceeds as follows. Choose pairwise different monomials $u_1(\theta), \dots, u_{n_u}(\theta)$ and search for the coefficient matrix Y in the representation

$$T(\theta) = Y(u(\theta) \otimes I_m)$$

with $Y = (Y_1, \dots, Y_{n_u})$ and $u(\theta) = (u_1(\theta), \dots, u_{n_u}(\theta))^{\mathrm{T}}$. Based on [22], the matrix $S(\theta)$ is said to be an SOS with respect to $u(\theta)$ if there exists some Y satisfying $S(\theta) = (u(\theta) \otimes I_m)^{\mathrm{T}}(Y^{\mathrm{T}}Y)(u(\theta) \otimes I_m)$. Substituting $Z = Y^{\mathrm{T}}Y$ yields the following result.

Proposition 1: [12], [22] A polynomial matrix $S \in \mathbb{S}[\theta]^m$ is an SOS with respect to the monomial basis $u(\theta)$ if and only if there exists a symmetric matrix $Z \succeq O$ with

$$S(\theta) = (u(\theta) \otimes I_m)^{\mathrm{T}} Z(u(\theta) \otimes I_m). \qquad (17)$$

The condition (17) can be interpreted as an affine constraint in Z. This implies that the problem to find $Z \succeq O$ with (17) can be formulated as an SDP. In other words, we can check whether $S \in \Sigma[\theta]^m$ with respect to some monomial basis by solving an SDP.