# MULTIPOLE FRINGE FIELDS

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*Abstract*

When creating an initial model of an accelerator, one usually has to resort to a hard edge model for the quadrupoles and higher order multipoles at the start of the project. Ordinarily, it is not until much later on that one has a field map for the given multipoles. This can be rather inconvenient when one is dealing with particularly thin elements or elements which are rather close together in a beamline as the hard edge model may be inadequate for the level of precision desired. For example, in the EMMA project, the two types of quadrupoles used are so close together that they are usually described by a single field map or via hard edge models. The first method has the desired accuracy but was not available at the start of the project and the second is known to be a rough approximation. In this paper, an analytic expression is derived and presented for fringe fields for a multipole of any order with a view to applying it to cases like EMMA.

## FRINGE FIELDS FOR DIPOLES

In order to have fringe fields, given by a $\vec{B}$ which satisfy Maxwell's equations, it is important to write all equations down explicitly. For Dipoles, it is sufficient to consider a two dimensional version of the equations

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \cdot \vec{B} = 0.$$

Now, if we take $B_x = 0$, we are left with

$$\partial_y B_y + \partial_z B_z = \partial_y B_z - \partial_z B_y = 0 \tag{1}$$

together with

$$\partial_x B_z = \partial_x B_y = 0 \tag{2}$$

which excludes all dependence on $x$. Further, we seek fringe fields which have a possible fall-off on axis given by the six parameter Enge function [1]

$$F(z) = \frac{1}{1 + \exp\left[E(z)\right]}$$

with $E(z)$ given by

$$E(z) = a_1 + a_2 \left(\frac{z}{D}\right) + a_3 \left(\frac{z}{D}\right)^2 + ... + a_6 \left(\frac{z}{D}\right)^5$$

and all $a_i$ constants determined by models and/or experiment, or any function which decays sufficiently rapidly. Maxwell's equations (1) imply

$$\Delta_{y,z} B_y = \Delta_{y,z} B_z = 0$$

where $\Delta_{y,z} = \partial_y^2 + \partial_z^2$. Both wave equations (for $B_y$ and $B_z$) can be easily solved to give

$$B_y = e(z+iy) + f(z-iy), \quad B_z = g(z+iy) + h(z-iy).$$

Hence, if we ask that equations (1) be solved as well, we end up with

$$B_y = e(z+iy) + f(z-iy), \quad B_z = -ie(z+iy) + if(z-iy)$$

If we further restrict ourselves to real magnetic fields, we obtain

$$B_y = e(z+iy) + \bar{e}(z-iy) \tag{3}$$
$$B_z = -ie(z+iy) + i\bar{e}(z-iy) \tag{4}$$

so $B_y$ and $B_z$ are given by twice the real and imaginary parts of the function $e(z+iy)$ respectively. A possibility for having a magnetic field whose $B_y$ component fall off on axis is given by the six parameter Enge function [1] as

$$B_y = \frac{1}{2(1+e^{E(z+iy)})} + \frac{1}{2(1+e^{E(z-iy)})} \tag{5}$$

which would force $B_z$ to have the form

$$B_z = \frac{-i}{2(1+e^{E(z+iy)})} + \frac{i}{2(1+e^{E(z-iy)})} \tag{6}$$

for some complex function $E(z+iy)$. If we consider the simple case $E(z+iy) = z+iy$ then equations (5) and (6) simplify to

$$B_y = \frac{(1+e^z\cos(y))}{1+2e^z\cos(y)+e^{2z}}, B_z = \frac{-e^z\sin(y)}{1+2e^z\cos(y)+e^{2z}}.$$

This may be extended to include as many parameters of the Enge function as desired, the only restriction being that $E = E(z+iy)$.

## EXTENSION TO HIGHER ORDER MULTIPOLES

In order to extend the fringe fields to higher order multipoles, it is instructive to rewrite a few of the already known higher order multipoles in a way which can be seen to explicitly solve Maxwell's equations - that is to express them in the form ((3),(4)) - only, this time coordinates $x$ and $y$ are used rather than $y$ and $z$. This is done by introducing the complex coordinates $u = \frac{1}{\sqrt{2}}(x+iy)$ and $v = \frac{1}{\sqrt{2}}(x-iy)$ and by defining the transformation / rescaling of Maxwell's equations: $B_u = \frac{1}{\sqrt{2}}(B_x + iB_y)$, $B_v = \frac{1}{\sqrt{2}}(B_x - iB_y)$ and $B_z^{'} = \frac{1}{\sqrt{2}}B_z$, $z^{'} = \frac{1}{\sqrt{2}}z$ and dropping primes, we have:

$$\partial_u B_u + \partial_z B_z = 0 \quad (7)$$
$$\partial_v B_v + \partial_z B_z = 0 \quad (8)$$
$$\partial_z B_u - \partial_v B_z = 0 \quad (9)$$
$$\partial_z B_v - \partial_u B_z = 0 \quad (10)$$

From (7) and (8), one can see immediately that, in the absence of any fringe fields, the general solution of Maxwell's equations for any magnet, acting transversely only and without fringe ($B_z = 0$) is given by $B_u = f(v)$ and $B_v = h(u)$ for some functions $f$ and $h$. The case of an $n$-pole multipole is given by $B_u = iv^{\frac{n-2}{2}}$, $B_v = -iu^{\frac{n-2}{2}}$ and $B_z = 0$, so a quadrupole is $B_u = iv$, $B_v = -iu$ and $B_z = 0$.

The main point is that, for dipole fringe fields, one needs to go from a magnetic field which is one dimensional to one that is two dimensional whereas for multipoles one has to go from two dimensional field to a three dimensional one. This presents the problem that the complete solution to the three dimensional Laplace equation is not really known. A formal solution due to Whittaker is known and may be given by

$$\varphi(x, y, z) = \int_0^{2\pi} f(z + ix\cos\vartheta + iy\sin\vartheta)\mathrm{d}\vartheta$$

where $\Delta_{x,y,z}\varphi = \partial_x^2\varphi + \partial_y^2\varphi + \partial_z^2\varphi = 0$. However, it is extremely difficult to translate this into a real solution and the only well-known one is $\varphi = (z + ix\cos\vartheta + iy\sin\vartheta)^{-1}$ which gives the standard solution $2\pi/r$ with $r = \sqrt{x^2 + y^2 + z^2}$. Therefore, we try a different approach, and, rather than solving Laplace and then further restricting the general solution by substituting it into the Maxwell equations, we assume a general form the multipole fringe fields should have and then we solve the resulting constraints. In full, the equations to be solved are ((7), ..., (10)) and we assume that the fringe fields have the following form

$$\tilde{B}_u = \frac{f_1(u,v,z) + f_2(u,v,z)e^z}{(1 + 2f_3(u,v)e^z + e^{2z})}$$
$$\tilde{B}_v = \frac{f_4(u,v,z) + f_5(u,v,z)e^z}{(1 + 2f_3(u,v)e^z + e^{2z})}$$
$$\tilde{B}_z = \frac{f_6(u,v,z) + f_7(u,v,z)e^z}{(1 + 2f_3(u,v)e^z + e^{2z})}.$$

This is based on a generalisation of the form the fringe fields take for the dipole case. Essentially, there are only two types of differentials that we have to look at and these are

$$\partial_u B_u = \frac{\partial_u f_1 + \partial_u f_2 e^z}{A} - \frac{2(f_1 + f_2 e^z)e^z\partial_u f_3}{A^2}$$
$$\partial_z B_u = \frac{\partial_z f_1 + \partial_z f_2 e^z}{A} - \frac{2(f_1 + f_2 e^z)(e^z f_3 + e^{2z})}{A^2}$$

where $A = 1 + 2f_3e^z + e^{2z}$. For the remaining differentials, we simply implement the following changes sequentially

$$\partial_v B_u = \partial_u B_u \quad (u \leftrightarrow v)$$
$$\partial_u B_v = \partial_u B_u, \quad (f_1 \to f_4,\ f_2 \to f_5)$$
$$\partial_v B_v = \partial_u B_v \quad (u \leftrightarrow v)$$
$$\partial_z B_v = \partial_z B_u \quad (f_1 \to f_4,\ f_2 \to f_5)$$
$$\partial_u B_z = \partial_u B_u \quad (f_1 \to f_6,\ f_2 \to f_7)$$
$$\partial_v B_z = \partial_u B_z \quad (u \leftrightarrow v)$$
$$\partial_z B_z = \partial_z B_u \quad (f_1 \to f_6,\ f_2 \to f_7).$$

As all equations ((7), ..., (10))are equal to zero, we take out a factor of $A^2$ and we can now equate all coefficients of $e^z$ giving:

$$e^{3z}: \quad \partial_u f_2 + \partial_z f_7 - f_7 = 0 \quad (11)$$
$$\partial_v f_5 + \partial_z f_z - f_7 = 0 \quad (12)$$
$$\partial_u f_7 - \partial_z f_5 + f_5 = 0 \quad (13)$$
$$\partial_v f_7 - \partial_z f_2 + f_2 = 0 \quad (14)$$
$$e^{2z}: \quad f_2\partial_u f_3 + f_6 - f_3 f_7 = 0 \quad (15)$$
$$f_5\partial_v f_3 + f_6 - f_3 f_7 = 0 \quad (16)$$
$$f_7\partial_u f_3 - f_4 + f_3 f_5 = 0 \quad (17)$$
$$f_7\partial_v f_3 - f_1 + f_3 f_2 = 0 \quad (18)$$
$$e^z: \quad f_1\partial_u f_3 + f_3 f_6 - f_7 = 0 \quad (19)$$
$$f_4\partial_v f_3 + f_3 f_6 - f_7 = 0 \quad (20)$$
$$f_6\partial_u f_3 + f_5 - f_3 f_4 = 0 \quad (21)$$
$$f_6\partial_v f_3 + f_2 - f_3 f_1 = 0 \quad (22)$$
$$e^0: \quad \partial_u f_1 + \partial_z f_6 = 0 \quad (23)$$
$$\partial_v f_4 + \partial_z f_6 = 0 \quad (24)$$
$$\partial_u f_6 - \partial_z f_4 = 0 \quad (25)$$
$$\partial_v f_6 - \partial_z f_1 = 0. \quad (26)$$

Note that we have not included all the steps and the above equations represent the original set with all possible simplifications, taking into account the set itself. Note that, equations ((11), ..., (14)) and ((23), ..., (26)) may be solved independently of the rest and they can therefore be dealt with later. From equation (18), using (15) and (19), we see

$$f_7(\partial_v f_3\partial_u f_3 + f_3^2 - 1) = 0.$$

Had we looked at equations (17) and (21) instead, using (16) and (20), we would have had

$$f_6(\partial_v f_3\partial_u f_3 + f_3^2 - 1) = 0$$

with the same result from equation (22). Now, $f_6$ and $f_7$ cannot both be zero as this would mean $B_z = 0$, therefore we must have

$$\partial_v f_3\partial_u f_3 + f_3^2 - 1 = 0$$

whose general solution is given by $f_3 = \sin h(u, v)$ with $h(u, v) = \frac{1}{b}u + bv + c$, and $b$ and $c$ constant. Substituting this back into ((15), ..., (22)) gives the relations $f_2 = b^2 f_5$ and $f_1 = b^2 f_4$ and the equations reduce to just two independent ones which may be written as

$$\frac{1}{b} f_2 \cos h + f_6 - f_7 \sin h = 0 \quad (27)$$

$$f_6 \cos h + \frac{1}{b} f_2 - \frac{1}{b} f_1 \sin h = 0. \quad (28)$$

Using $f_1 = b^2 f_4$ and equations (23) and (24) we see that we require

$$b^2 \partial_u f_4 = \partial_v f_4$$

which can be solved by the method of characteristics to give $f_4 = f_4(h, z)$. Using this with equations (25) and (26) we see that $f_6 = f_6(h, z)$. Similarly, $f_2 = b^2 f_5$ applied to (11) and (12) and, subsequently (13) and (14) gives $f_5 = f_5(h, z)$ and $f_7 = f_7(h, z)$. This leaves six equations to be satisfied from the original system ((11), ..., (26)), namely (27) and (28) together with

$$\partial_u f_2 + \partial_z f_7 - f_7 = 0 \quad (29)$$

$$\partial_v f_7 - \partial_z f_2 + f_2 = 0 \quad (30)$$

$$\partial_u f_1 + \partial_z f_6 = 0 \quad (31)$$

$$\partial_v f_6 - \partial_z f_1 = 0. \quad (32)$$

After cross-differentiation, equations (31) and (32) give

$$\partial^2_{uv} f_6 + \partial^2_z f_6 = 0$$

$$\partial^2_{uv} f_1 + \partial^2_z f_1 = 0.$$

Now, we can re-express the partial derivatives in $u$ and $v$ in terms of $h$ only and the equations simplify to $\triangle f_1 = \triangle f_6 = 0$ with $\triangle = \partial^2_h + \partial^2_z$ and we can introduce the co-ordinates $w = h + iz$, $\tilde{w} = h - iz$. Note that this operation is equivalent to complex conjugation in the $z$ co-ordinate only and the function $h$ is untouched. Therefore we have the solutions $f_1 = p_1(w) + q_1(\tilde{w})$ and $f_6 = p_6(w) + q_6(\tilde{w})$. Substituting this back into (31) and (32), we see that the solutions are further constrained to $f_1 = -ibp_6 + ibq_6 + k$ from which we can get $f_4$ via $f_4 = \frac{1}{b^2} f_1$. Subsequently, we can get $f_2$ from (28) and hence $f_5$ via $f_5 = \frac{1}{b^2} f_2$ and $f_7$ from 27. The general result, in terms of $p_6$ and $q_6$ may be summarised as follows (with $h = \frac{1}{b}u + bv + c$):

$$f_1 = -ibp_6 + ibq_6 + k \quad (33)$$

$$f_2 = (-ibp_6 + ibq_6 + k) \sin h - (bp_6 + bq_6) \cos h \quad (34)$$

$$f_3 = \sin h \quad (35)$$

$$f_4 = \frac{1}{b}(-ip_6 + iq_6 + \frac{k}{b}) \quad (36)$$

$$f_5 = \frac{1}{b}(-ip_6 + iq_6 + \frac{k}{b}) \sin h - \frac{1}{b}(p_6 + q_6) \cos h \quad (37)$$

$$f_6 = p_6 + q_6 \quad (38)$$

$$f_7 = (p_6 + q_6) \sin h + (-ip_6 + iq_6 + \frac{k}{b}) \cos h. \quad (39)$$

So we are left with equations (29) and (30) to be solved. Upon substitution of ((33), ..., (39)), this is actually seen to be trivially satisfied with no further constraints on any of the $f$'s. In fact, the results can be seen to imply the following solution to Maxwell's equations:

$$B_u = -ibf(h + iz) + ibg((h - iz) \quad (40)$$

$$B_v = -\frac{i}{b} f(h + iz) + \frac{i}{b} g(h - iz) \quad (41)$$

$$B_z = f(h + iz) + g(h - iz) \quad (42)$$

with $h$ being the same as defined earlier. Note that, the relations $f_2 = b^2 f_5$ and $f_1 = b^2 f_4$ found earlier imply that $B_u \propto B_v$ which means that no physical magnetic fields can be represented this way. However, because of the linearity of Maxwell's equations, it is possible to add, together with multiplicative constants, as many of these solutions as required. When we do this, we must also make sure that the field decays as $z \to \infty$ and that the field is equivalent to the hard edge model when we are inside the magnet. The full details of the result will be published elsewhere [3] and we only go through the quadrupole case below. Let $B_z$ be given by

$$B_z = \sum_{j=1,2} c_j[(h_j + iz)F_j(h_j + iz) + (h_j - iz)G_j(h_j - iz)]$$

with similar expresssion for $B_u$ and $B_v$, according to the format (40), (41) and (42) and where $h_j = \frac{1}{b_j}u + b_j v$. The functions $F_i$ and $G_i$ are chosen to give the desired decays. Therefore, inside the magnet, we are left with the following constraints on the $b_i$'s and $c_i$'s:

$$b_1 = \pm\frac{1}{b_2}, \quad c_1 = -c_2 = \frac{1}{2(b_2^2 - \frac{1}{b_2^2})}.$$

The results are extendible to higher order multipoles in a straightforward way, however, the higher the order, the more three dimensional solutions discussed above need to be included.

## CONCLUSIONS

A closed form analytic expression was presented for multipole fringe fields extendible to any order. The complete derivation and details will be made given in [3]. It is hoped that the results summarised in this paper will be facilitate the design of machines like ns-FFAGs to a higher degree of accuracy at an early stage in a given project.

## REFERENCES

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