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ON SEIFFERT–LIKE MEANS

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Abstract. We investigate the representation of homogeneous, symmetric means in the form

$$M(x,y) = \frac{x-y}{2f\left(\frac{x-y}{x+y}\right)}.$$

This allows for a new approach to comparing means. As an example, we provide optimal estimate of the form

$$(1-\mu)\min(x,y) + \mu\max(x,y) \leqslant M(x,y) \leqslant (1-\nu)\min(x,y) + \nu\max(x,y)$$

and

$$M\left(\tfrac{x+y}{2} - \mu\tfrac{x-y}{2}, \tfrac{x+y}{2} + \mu\tfrac{x-y}{2}\right) \leqslant N(x,y) \leqslant M\left(\tfrac{x+y}{2} - \nu\tfrac{x-y}{2}, \tfrac{x+y}{2} + \nu\tfrac{x-y}{2}\right)$$

for some known means.

We also introduce an integral operator on the set of means and investigate its properties.

1. Introduction, definitions and notation

In [20] and [21] H.-J. Seiffert introduced two means

$$\mathsf{P}(x,y) = \begin{cases} \dfrac{x-y}{2\arcsin\frac{x-y}{x+y}} & x \neq y \\ x & x = y \end{cases}, \tag{1}$$

$$\mathsf{T}(x,y) = \begin{cases} \dfrac{x-y}{2\arctan\frac{x-y}{x+y}} & x \neq y \\ x & x = y \end{cases}. \tag{2}$$

The papers of Sándor and Neuman [14, 16, 17, 19] show that both of them can be obtained using the same iterative procedure developed by Borchardt. Sándor also discovered that the first mean was known already to Pfaff (and suggested the letter P for its name) and that T can be obtained from P by a simple transformation ([18]). As in often happens in life, even if the means were in fact rediscovered by Seiffert, they are now referred to as the first and second Seiffert means.

Applying the Borchardt procedure with different starting points Sándor and Neuman discovered a new mean (called now Neuman-Sándor mean [14])

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$$\mathsf{NS}(x,y)=\begin{cases}\dfrac{x-y}{2\operatorname{arsinh}\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}, \tag{3}$$

and established a new formula for the well known logarithmic mean

$$\mathsf{L}(x,y)=\begin{cases}\dfrac{x-y}{2\operatorname{artanh}\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}. \tag{4}$$

The eye-catching similarity between all four means begets the following questions:

QUESTION 1.1. What about $\sin, \sinh, \tan, \tanh$? Will they also form means?

QUESTION 1.2. For positive x,y and a positive function $f:(0,1)\to\mathbb{R}$ let

$$F(x,y)=\begin{cases}\dfrac{|x-y|}{2f\left(\frac{|x-y|}{x+y}\right)} & x\neq y\\ x & x=y\end{cases}. \tag{5}$$

Under what assumptions on f the function F is a mean?

QUESTION 1.3. What means M can be represented in the form (5)?

The means of the form (5) will be called *Seiffert-like means.*

The aim of this paper is to answer the three question stated above and explore the subject. In particular, we introduce a metric and an algebraic structure on the set of means. Then we show how the representation of a mean in the form (5) can be used to investigate its properties. In particular, we offer a simple criterion for Schur convexity of means and criteria for finding optimal bounds of the form

$$(1-\mu)K(x,y)+\mu N(x,y)\leqslant M(x,y)\leqslant(1-\nu)K(x,y)+\nu N(x,y),$$
$$M\left(\tfrac{x+y}{2}-\mu\tfrac{x-y}{2},\tfrac{x+y}{2}+\mu\tfrac{x-y}{2}\right)\leqslant N(x,y)\leqslant M\left(\tfrac{x+y}{2}-\nu\tfrac{x-y}{2},\tfrac{x+y}{2}+\nu\tfrac{x-y}{2}\right),$$

where $K<M<N$ are homogeneous means.

Let us begin with some definitions and notation convention.

DEFINITION 1.1. A *mean* is a function $M:\mathbb{R}_+^2\to\mathbb{R}$ satisfying the following conditions:

(A) M is *symmetric*, i.e. $M(x,y)=M(y,x)$ for all $x,y\in\mathbb{R}_+$.

(B) M is *positively homogeneous* of order 1, i.e. for all $\lambda>0$ holds $M(\lambda x,\lambda y)=\lambda M(x,y)$.

(C) M is *internal*, i.e. $\min(x,y)\leqslant M(x,y)\leqslant\max(x,y)$.

The set of means will be denoted by $\mathscr{M}$.
The set of *strict means*, i.e. means satisfying the condition

(D) $\min(x,y) < M(x,y) < \max(x,y)$ whenever $x \neq y$,

will be denoted by $\mathscr{M}^\circ$.

DEFINITION 1.2. A function $f:(0,1) \to \mathbb{R}$ satisfying

$$\frac{z}{1+z} \leqslant f(z) \leqslant \frac{z}{1-z} \tag{6}$$

will be called *Seiffert function*. The set of Seiffert functions will be denoted by $\mathscr{S}$. By $\mathscr{S}^\circ$ we shall denote the set of *strict Seiffert functions* for which both inequalities in (6) are strict.

Note two important properties of Seiffert functions:

$$\lim_{z\to 0} f(z) = 0, \quad \text{and} \quad \lim_{z\to 0} \frac{f(z)}{z} = 1.$$

This means that a Seiffert function can be extended to a function defined on $[0,1)$ by setting $f(0) = 0$ and this extension is continuous and differentiable at $x = 0$. Sometimes it would be convenient to think about a Seiffert function as an odd function defined on the interval $(-1,1)$.

On the set of means we define two partial orders as follows:

$$\begin{aligned} M \leqslant N \quad &\text{if and only if } M(x,y) \leqslant N(x,y) \text{ for all } x \neq y,\\ M < N \quad &\text{if and only if } M(x,y) < N(x,y) \text{ for all } x \neq y. \end{aligned}$$

Similar partial orders are defined for Seiffert functions (we use the same symbol, as there is no chance for ambiguity)

$$\begin{aligned} m \leqslant n \quad &\text{if and only if } m(z) \leqslant n(z) \text{ for all } 0 < z < 1,\\ m < n \quad &\text{if and only if } m(z) < n(z) \text{ for all } 0 < z < 1. \end{aligned}$$

In what follows we shall denote the means by uppercase letters and the Seiffert functions by lowercase. We shall also use a special notation for well-known means (additionally to the four means defined above)

$\mathsf{A}(x,y) = \dfrac{x+y}{2}$	arithmetic mean
$\mathsf{G}(x,y) = \sqrt{xy}$	geometric mean
$\mathsf{H}(x,y) = \dfrac{2xy}{x+y}$	harmonic mean
$\mathsf{C}(x,y) = \dfrac{x^2+y^2}{x+y}$	contraharmonic mean

$$
\begin{aligned}
\mathsf{RMS}(x,y) &= \sqrt{\frac{x^2+y^2}{2}} && \text{root mean square}\\
\mathsf{M}_r(x,y) &= \left(\frac{x^r+y^r}{2}\right)^{1/r} && \text{power mean of order } r \ (\mathsf{M}_0=\mathsf{G})\\
\mathsf{AGM}(x,y) &= \frac{\pi}{4}\frac{x+y}{K\left(\frac{x-y}{x+y}\right)} && \text{arithmetic-geometric mean}\\
\mathsf{GINI}_{r,s}(x,y) &= \left(\frac{x^r+y^r}{x^r+y^r}\right)^{1/(r-s)} && \text{Gini means} \qquad (7)\\
\mathsf{STO}_{r,s}(x,y) &= \left(\frac{s}{r}\frac{x^r-y^r}{x^s-y^s}\right)^{1/(r-s)} && \text{Stolarsky means .}
\end{aligned}
$$

Here K denotes the complete elliptic function of the first kind.

2. Answers to Questions 1.2 and 1.3

The next theorem gives complete (and rather surprising) answer to questions 1.2 and 1.3 stated in the previous section.

THEOREM 2.1. *The mapping* $\mathfrak{F}\colon \mathscr{S} \to \mathscr{S}$ *given by the formula*

$$
\mathfrak{F}(f)(x,y) = \begin{cases} \dfrac{|x-y|}{2f\left(\frac{|x-y|}{x+y}\right)} & x \neq y \\ x & x = y \end{cases}, \tag{8}
$$

establishes a one-to-one correspondence between $\mathscr{S}$ *and* $\mathscr{M}$ *that transforms* $\mathscr{S}^\circ$ *onto* $\mathscr{M}^\circ$*. Its inverse is given by the formula*

$$
\mathfrak{F}^{-1}(M)(z) = \frac{z}{M(1-z,1+z)}.
$$

Proof. It is obvious that $\mathfrak{F}(f)$ is symmetric and positively homogeneous. $\mathfrak{F}(f)$ is defined for all positive $x \neq y$, since $0 < \frac{|x-y|}{x+y} < 1$. Suppose that $x < y$ and let $z = \frac{y-x}{x+y}$. Then, the inequalities (6) read

$$
\frac{y-x}{2y} \leqslant f\left(\frac{y-x}{x+y}\right) \leqslant \frac{y-x}{2x}.
$$

This is equivalent to $x \leqslant \mathfrak{F}(f)(x,y) \leqslant y$ and shows that $\mathfrak{F}(f)$ satisfies (C). Also note that for $f \in \mathscr{S}^\circ$ the inequalities above are strict, which means that $\mathfrak{F}(f)$ belongs to $\mathscr{M}^\circ$.

Conversely, for $M \in \mathscr{M}$ we have

$$
M(x,y) = \frac{x+y}{2}M\left(\frac{x+y-(y-x)}{x+y}, \frac{x+y+(y-x)}{x+y}\right)
$$

$$= \frac{y-x}{2\dfrac{z}{M(1+z,1-z)}}, \tag{9}$$

and the function

$$m(z) = \frac{z}{M(1+z,1-z)}$$

in the denominator of the right-hand side is a Seiffert function, because $1-z \leqslant M(1+z,1-z) \leqslant 1+z$. Again, if M is strict, then so is m.

It is obvious from (9) and (8) that $m(z) = \mathfrak{F}^{-1}(M)$.

NOTE 1. In what follows we shall use the convention that the Seiffert function corresponding to a mean is denoted by the same lowercase letter as a mean and vice versa. Thus $\mathfrak{F}(m)$ will be denoted by M and $\mathrm{g}(z) = \frac{z}{\sqrt{1-z^2}}$.

A very important property of the mappings between the set of means and Seiffert functions is their anti-monotonicity.

COROLLARY 2.1. *The following conditions are equivalent*

(i) $M \leqslant (<) N$,

(ii) $m \geqslant (>) n$.

In many cases comparing the Seiffert functions is much easier than comparing the means, so the anti-monotonicity provides a new tool for proving inequalities between means.

The following corollary is a trivial consequence of Theorem 2.1.

COROLLARY 2.2. *For the means* min *and* max *hold*

$$\mathfrak{F}^{-1}(\min)(z) = \frac{z}{1-z}, \qquad \mathfrak{F}^{-1}(\max)(z) = \frac{z}{1+z}.$$

Let us transform (6) as follows:

$$\frac{1-z}{z} \leqslant \frac{1}{f(z)} \leqslant \frac{1+z}{z}$$
$$-1 \leqslant \frac{1}{f(z)} - \frac{1}{z} \leqslant 1. \tag{10}$$

This shows that the set of Seiffert means is bounded in some sense, and enables us to define a metric in $\mathscr{S}$ by

$$d_{\mathscr{S}}(f,g) = \sup_{0<z<1} \left| \frac{1}{f(z)} - \frac{1}{g(z)} \right|.$$

LEMMA 2.1. *The space* $(\mathscr{S}, d_{\mathscr{S}})$ *has the following properties*

1. $(\mathscr{S}, d_{\mathscr{S}})$ *is a complete metric space and* $\operatorname{diam}\mathscr{S} = 2$,
2. $(\mathscr{S}, d_{\mathscr{S}})$ *is a unit ball centered at the identity function* $id(z) = z$.

Proof. Completeness follows, since the convergence in $d_{\mathscr{S}}$ implies the point-wise convergence, hence the limit function satisfies (6). The inequality (10) implies the second property.

Clearly, the metric on $\mathscr{S}$ induces the metric on $\mathscr{M}$ by

$$d_{\mathscr{M}}(M,N) = d_{\mathscr{S}}(m,n).$$

Thus, by Lemma 2.1 the space of means is a unit ball centered at the arithmetic mean.

To obtain the explicit formula for $d_{\mathscr{M}}(M,N)$, take $z = \frac{|x-y|}{x+y}$ and write

$$\begin{aligned} d_{\mathscr{M}}(M,N) = d_{\mathscr{S}}(m,n) &= \sup_{0<z<1}\left|\frac{1}{m(z)} - \frac{1}{n(z)}\right| \\ &= \sup_{x\neq y}\frac{2}{|x-y|}\left|\frac{|x-y|}{2m(z)} - \frac{|x-y|}{2n(z)}\right| = 2\sup_{x\neq y}\left|\frac{M(x,y)-N(x,y)}{x-y}\right|. \end{aligned}$$

For more properties of this metric, see [8].

As an application, let us prove the following result

THEOREM 2.2. *For* $M,N \in \mathscr{M}$ *satisfying* $d_{\mathscr{M}}(M,N) < 2$, *there exists a unique mean* $K \in \mathscr{M}$ *such that for all* x,y

$$K(x,y) = K(M(x,y),N(x,y)).$$

NOTE 2. This result is known in case M,N are strict means. and also in case one of them is not strict, but continuous. There exist means that satisfy $d_{\mathscr{M}}(M,N) < 2$ and are not strict, and there are strict means with $d_{\mathscr{M}}(M,N) = 2$.

Proof. Define the mapping $\Phi : \mathscr{S} \to \mathscr{S}$ by $\Phi(P)(x,y) = P(M(x,y),N(x,y))$.

$$\begin{aligned} \frac{1}{\mathfrak{F}^{-1}(\Phi(P))(z)} &= \frac{P(M(1+z,1-z),N(1+z,1-z))}{z} = P\left(\frac{1}{m(z)},\frac{1}{n(z)}\right) \\ &= \frac{\left|\frac{1}{m(z)} - \frac{1}{n(z)}\right|}{2p\left(\frac{|m^{-1}(z)-n^{-1}(z)|}{m^{-1}(z)+n^{-1}(z)}\right)} = \frac{1}{2}\left(\left|\frac{1}{m(z)} - \frac{1}{n(z)}\right|\right)\frac{1}{p(u)}, \end{aligned}$$

where $u = \frac{|n(z)-m(z)|}{m(z)+n(z)}$. Thus

$$\left|\frac{1}{\mathfrak{F}^{-1}(\Phi(P))(z)} - \frac{1}{\mathfrak{F}^{-1}(\Phi(Q))(z)}\right| = \frac{1}{2}\left(\left|\frac{1}{m(z)} - \frac{1}{n(z)}\right|\right)\left(\left|\frac{1}{p(u)} - \frac{1}{q(u)}\right|\right),$$

which yields

$$d_{\mathscr{M}}(\Phi(P),\Phi(Q)) \leqslant \frac{1}{2}d_{\mathscr{M}}(M,N)d_{\mathscr{M}}(P,Q).$$

Applying the Banach fixed point theorem, we complete the proof.

3. New Seiffert-like means

Seiffert introduced two means corresponding to arcsin and arctan. Two other means mentioned in the introduction come from their hyperbolic companions. In this section we extend this quartet by showing that also sin, tan, sinh and tanh are Seiffert functions, thus providing answer to the Question 1.1.

The two lemmas that follow establish inequalities between corresponding Seiffert functions. Some inequalities presented here are known (see e.g. [12]), some are rather trivial. We give here their straightforward proofs just for completeness. Consider first functions with graph lying below the main diagonal.

LEMMA 3.1. *The inequalities*

$$t > \operatorname{arsinh} t > \arctan t > \tanh t > \frac{t}{1+t} \tag{11}$$

hold for all $t > 0$. *Moreover,*

$$\operatorname{arsinh} t > \sin t \tag{12}$$

holds for $0 < t < \pi/2$ *and*

$$\sin t > \arctan t \tag{13}$$

is valid for $0 < t < 1$.

Proof. Since $\cosh^2 t > \left(1+\frac{t^2}{2}\right)^2 > 1+t^2$, integrating form 0 to t the inequalities

$$1 > \frac{1}{\sqrt{1+t^2}} > \frac{1}{1+t^2} > \frac{1}{\cosh^2 t},$$

we obtain first three inequalities in (11). To prove the last one, observe that the graph of the convex function $\cosh t$ and the straight line $1+t$ intersect at two points: $t=0$ and $t=t_0>0$. Thus $\frac{1}{\cosh^2 t} - \frac{1}{(1+t)^2}$ is positive for $0<t<t_0$ and negative for $t>t_0$. Therefore, the function

$$h(t) = \tanh t - \frac{t}{1+t} = \int_0^t \frac{1}{\cosh^2 t} - \frac{1}{(1+t)^2}dt$$

increases from $h(0)=0$ to $h(t_0)$ and then decreases to $h(\infty)=0$, hence is nonnegative. This completes the proof of the rightmost inequality in (11).

To prove (12), note that for $0<t<\pi/2$ the inequalities

$$\begin{aligned}(1+t^2)\cos^2 t &= (1+t^2)\frac{\cos 2t+1}{2} < (1+t^2)\frac{1-2t^2+2t^4/3+1}{2}\\ &= 1-\tfrac{1}{3}t^4(2-t^2) < 1\end{aligned}$$

hold. Thus $\frac{1}{\sqrt{1+t^2}} > \cos t$ and we obtain (12) by integration.

To prove (13), observe that for $0<t<1$

$$(1+t^2)\cos t > (1+t^2)(1-t^2/2) = 1+\frac{t^2(1-t^2)}{2} > 1,$$

and apply the same argument as above.
Lemma 3.1 shows that the following inequalities hold

$$\mathsf{A} < \mathsf{NS} < \mathfrak{F}(\sin) < \mathsf{T} < \mathfrak{F}(\tanh) < \max .$$

And now consider the functions with graph above the diagonal.

LEMMA 3.2. *The inequalities*

$$t < \sinh t < \tan t < \operatorname{artanh} t < \frac{t}{1-t} \tag{14}$$

hold for all $0 < t < 1$. *Moreover,*

$$\sinh t < \arcsin t < \operatorname{artanh} t \tag{15}$$

hold for $0 < t < 1$. *The functions* arcsin *and* tan *are not comparable in* $(0,1)$.

Proof. The inequalities (14) follow from (11) and the fact, that the graph of an inverse function is symmetric with respect to the main diagonal. The same argument applied to (12) implies the first inequality in (15), while the second inequality can be obtained by integration of $\frac{1}{\sqrt{1-t^2}} < \frac{1}{1-t^2}$.
It follows from (13) and the remark about the graph of an inverse function that $\arcsin t < \tan t$ for $t < \sin 1$, while $\arcsin 1 > \tan 1$.
Lemma 3.2 implies the following chains of inequalities between means

$$\mathsf{A} > \mathfrak{F}(\sinh) > \begin{matrix} \mathsf{P} \\ \mathfrak{F}(\tan) \end{matrix} > \mathsf{L} > \min .$$

So all hyperbolic and inverse offspring of sine and tangent functions form Seiffert-like means. We shall see in a while that both of them are much more fertile.

4. Integral transformation

In this section we introduce a operator acting on a subset of Seiffert means and investigate its properties.

THEOREM 4.1. *If* $f \in \mathscr{M}$ *is concave and*

$$I(f)(z) = \int_0^z \frac{f(t)}{t} dt,$$

then $I(f)$ *is also concave and* $f(z) \leqslant I(f)(z) \leqslant z$ *for all* $z \in (0,1)$. *If* f *is strictly convex these inequalities are sharp.*
Similarly, if f *is convex, then* $I(f)$ *is also convex and* $f(z) \geqslant I(f)(z) \geqslant z$.

Proof. First, note that setting $f(0)=0$ we extend f to a concave function on $[0,1)$. Inequalities (6) imply $f'(0)=1$ so $f(z)\leqslant z$ by concavity. Moreover, the divided difference $f(t)/t$ decreases which yields concavity of $I(f)$, and implies

$$f(z)=\int_0^z \frac{f(z)}{z}dt \leqslant \int_0^z \frac{f(t)}{t}dt \leqslant \int_0^z dt = z.$$

The proof in case of convex function is similar.

The operator I is monotone on the set of functions where it exists, and because $\max$ is concave and $\min$ is convex, we obtain the following corollary.

COROLLARY 4.1. *If f is a Seiffert function such that $I(f)$ exists, then $I(f)$ is also a Seiffert function.*

In particular we conclude that for $n=0,1,2,\dots$ the following functions are Seiffert functions

$$\begin{gathered}\mathfrak{F}(I^n(\sin)),\ \ \mathfrak{F}(I^n(\tan)),\ \ \mathfrak{F}(I^n(\arcsin)),\ \mathfrak{F}(I^n(\arctan)),\\ \mathfrak{F}(I^n(\sinh)),\ \mathfrak{F}(I^n(\tanh)),\ \mathfrak{F}(I^n(\operatorname{arsinh})),\ \mathfrak{F}(I^n(\operatorname{artanh})).\end{gathered}$$

Since all functions mentioned in Lemma 3.1 are concave, we have the following grid of inequalities between means:

$$\mathsf{A} < \begin{array}{ccccccc} \dots < \mathfrak{F}(I^2(\operatorname{arsinh})) & < & \mathfrak{F}(I(\operatorname{arsinh})) & < & \mathfrak{F}(\operatorname{arsinh}) & = \mathsf{NS} \\ \wedge & & \wedge & & \wedge & \\ \dots < \ \ \mathfrak{F}(I^2(\sin)) & < & \mathfrak{F}(I(\sin)) & < & \mathfrak{F}(\sin) & \\ \wedge & & \wedge & & \wedge & \\ \dots < \mathfrak{F}(I^2(\arctan)) & < & \mathfrak{F}(I(\arctan)) & < & \mathfrak{F}(\arctan) & = \mathsf{T} \\ \wedge & & \wedge & & \wedge & \\ \dots < \ \ \mathfrak{F}(I^2(\tanh)) & < & \mathfrak{F}(I(\tanh)) & < & \mathfrak{F}(\tanh) & \end{array}.$$

The horizontal lines are granted by Theorem 4.1 and vertical ones follow from Lemma 3.1.

Before providing a similar picture for the other four functions, recall that there is no comparison between arcsine and tangent functions. Nevertheless, the operator I quickly rectifies this irregularity.

LEMMA 4.1. *For $0<z<1$ we have*

$$I(\arcsin)(z)=\int_0^z \frac{\arcsin t}{t}dt < \int_0^z \frac{\tan t}{t}dt = I(\tan)(z).$$

Proof. Let $q(t)=\arcsin t-\tan t$. As shown in the proof of Lemma 3.2, for $t<\sin 1\approx 0.841$ the inequality $q(t)<0$ holds. We have $q'(t)=\frac{\cos^2 t-\sqrt{1-t^2}}{\sqrt{1-t^2}\cos^2 t}$. In the interval $(\pi/4,1)$ the numerator of q' is convex (being a sum of two convex functions), $q'(\pi/4)<0$ and $\lim_{t\to 1^-} q'(t)=\infty$, which implies that q' changes sign exactly once. As $\pi/4<\sin 1$, we conclude that $q(t)$ changes sign exactly once in the interval $(0,1)$.

This implies that the function $u(z) = \int_0^z q(t)/t dt$ has exactly one local minimum, and since $u(0) = 0$ and $u(1) \approx -0.06$, it is negative, which completes the proof.
Now Theorem 4.1 together with Lemmas 3.2 and 4.1 yield

$$\begin{array}{ccccccccc}
 & & \mathfrak{F}(\sinh) & < & \mathfrak{F}(I(\sinh)) & < & \mathfrak{F}(I^2(\sinh)) & < \dots & \\
 & & \vee & & \vee & & \vee & & \\
\mathsf{P} & = & \mathfrak{F}(\arcsin) & < & \mathfrak{F}(I(\arcsin)) & < & \mathfrak{F}(I^2(\arcsin)) & < \dots & \\
 & & & & \vee & & \vee & & < \mathsf{A}. \\
\vee & & \mathfrak{F}(\tan) & < & \mathfrak{F}(I(\tan)) & < & \mathfrak{F}(I^2(\tan)) & < \dots & \\
 & & \vee & & \vee & & \vee & & \\
\mathsf{L} & = & \mathfrak{F}(\operatorname{artanh}) & < & \mathfrak{F}(I(\operatorname{artanh})) & < & \mathfrak{F}(I^2(\operatorname{artanh})) & < \dots &
\end{array}$$

5. Schur convexity

Given a symmetric, convex set $D \subset \mathbb{R}^2$, a partial order in D is defined by

$$(x_1, y_1) \prec (x_2, y_2) \Leftrightarrow x_1 + y_1 = x_2 + y_2 \text{ and } \max(x_1, y_1) \leqslant \max(x_2, y_2).$$

A symmetric function $h : D \to \mathbb{R}$ is called Schur-convex if it preserves this partial order, i.e. if

$$(x_1, y_1) \prec (x_2, y_2) \Rightarrow h(x_1, y_1) \leqslant h(x_2, y_2), \tag{16}$$

and Schur-concave if the partial order gets reversed.
Setting $c = (x_1 + y_1)/2$, $t_1 = |x_1 - y_1|/2$, $t_2 = |x_2 - y_2|/2$, we see that $(x_1, y_1) \prec (x_2, y_2)$ is equivalent to $t_1 \leqslant t_2$, and thus we can say that h is Schur-convex (resp. concave) if and only if for all c the function $h(c+t, c-t)$ increases (resp. decreases) for $t > 0$, cf. [11, I.3.A.2.b]. It will be useful to introduce the strict Schur-convexity: it is when the inequality in (16) is strict whenever (x_1, y_1) is not a permutation of (x_2, y_2). In this case all reasoning in this section remains valid with the adverb 'strictly' added to all mentioned properties.

Schur convexity of means is an interesting subject investigated by many mathematicians (see e.g. [2, 22, 7] and the references therein).

In case of a homogeneous symmetric mean, the Schur-convexity condition may be written in a very simple form: $M(x, y)$ is Schur-convex (resp. concave) if and only if the function $s(t) = M(1+t, 1-t)$ increases in the unit interval. Let us see how this condition translates into the language of Seiffert functions. We have $s(t) = M(1+t, 1-t) = \frac{t}{m(t)}$, so the following result is valid.

THEOREM 5.1. *A mean M is Schur-convex (resp. concave) if and only if the function $m(z)/z$ decreases (res. increases).*

Note that if a Seiffert function f is concave (resp. convex), then its divided difference $f(z)/z$ decreases (resp. increases), so we obtain

COROLLARY 5.1. *If $f \in \mathscr{S}$ is concave (resp. convex), then the mean $\mathfrak{F}(f)$ is Schur-convex (resp. Schur-concave).*

COROLLARY 5.2. *For $n=0,1,2,\dots$ the means*

$$\mathfrak{F}(I^n(\sin)),\quad \mathfrak{F}(I^n(\operatorname{arsinh})),\quad \mathfrak{F}(I^n(\arctan)),\quad \mathfrak{F}(I^n(\tanh))$$

are strictly Schur-convex, while the means

$$\mathfrak{F}(I^n(\sinh)),\quad \mathfrak{F}(I^n(\arcsin)),\quad \mathfrak{F}(I^n(\operatorname{artanh})),\quad \mathfrak{F}(I^n(\tan))$$

are Schur-concave.

6. Means with varying arguments

For $0<t<1, 0<z<1$ and $f\in\mathscr{S}$ the inequalities

$$\frac{z}{1+z}<\frac{1}{t}\frac{tz}{1+tz}\leqslant\frac{f(tz)}{t}\leqslant\frac{1}{t}\frac{tz}{1-tz}<\frac{z}{1-z},$$

are valid, which shows that $f^{\{t\}}(z)=f(tz)/t$ are also Seiffert functions. Note that $\lim_{t\to 0} f^{\{t\}}(z)=z$, thus this process defines a homotopy between f and id. It is a matter of simple transformation to verify, that

$$F^{\{t\}}(x,y)=\mathfrak{F}(f^{\{t\}})(x,y)=\mathfrak{F}(f)\left(\frac{x+y}{2}-t\frac{x-y}{2},\frac{x+y}{2}+t\frac{x-y}{2}\right).$$

There are numerous papers on comparison between means of the form $M^{\{t\}}$ and other classical means (see e.g. [3, 4, 5, 6, 9, 13, 23]).
The most popular problem is formulated as follows: given two means M,N satisfying $\mathsf{A}<M<N$, find optimal p,q such that $N^{\{p\}}\leqslant M\leqslant N^{\{q\}}$. By Corollary 2.1 the inequalities $N^{\{p\}}<M<N^{\{q\}}$ are equivalent to $n^{\{p\}}>m>n^{\{q\}}$. The inequality $\mathsf{A}<N$ implies $n(z)<z$. Assume additionally that the function $\widehat{n}(z)=n(z)/z$ is strictly decreasing, (in case of classical means, their Seiffert functions are usually concave, so this condition is satisfied). Then the following inequalities are equivalent:

$$n^{\{p\}}(z)>m(z)>n^{\{q\}}(z)\equiv\frac{n(pz)}{pz}>\frac{m(z)}{z}>\frac{n(qz)}{qz}\equiv p<\frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}<q.$$

Similar reasoning can be applied in case $N<M<\mathsf{A}$. Thus we have proven the following theorem.

THEOREM 6.1. *Let M and N be two means with Seiffert functions m and n, respectively. Suppose that $\widehat{n}(z)$ is strictly monotone and let $p_0=\inf_z\frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$ and $q_0=\sup_z\frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$.*
If $\mathsf{A}(x,y)<M(x,y)<N(x,y)$ for all $x\neq y$ then the inequalities

$$N^{\{p\}}(x,y)\leqslant M(x,y)\leqslant N^{\{q\}}(x,y)$$

hold if and only if $p \leqslant p_0$ *and* $q \geqslant q_0$.
If $N(x,y) < M(x,y) < \mathsf{A}(x,y)$ *for all* $x \neq y$ *then the inequalities*

$$N^{\{q\}}(x,y) \leqslant M(x,y) \leqslant N^{\{p\}}(x,y)$$

hold if and only if $p \leqslant p_0$ *and* $q \geqslant q_0$.

To illustrate this theorem, let us consider two examples featuring the contraharmonic mean $\mathsf{C}(x,y) = \frac{x^2+y^2}{x+y}$.

EXAMPLE 6.1. Let $M(x,y) = \frac{x^2+xy+y^2}{x+\sqrt{xy}+y}$. It is known (cf. [24]) that for $x \neq y$

$$\mathsf{A}(x,y) < M(x,y) < \mathsf{C}(x,y)$$

hold. We have

$$\widehat{m}(z) = \frac{2+\sqrt{1-z^2}}{3+z^2}, \ \widehat{\mathsf{c}}(z) = \frac{1}{1+z^2}, \ \frac{\widehat{\mathsf{c}}^{-1}(\widehat{m}(z))}{z} = \sqrt{\frac{1-\sqrt{1-z^2}}{z^2}}.$$

The function $1-\sqrt{1-u}$ is convex, so its divided difference increases from $1/2$ to 1, therefore we obtain the optimal inequalities

$$\mathsf{C}^{\{\sqrt{2}/2\}}(x,y) \leqslant M(x,y) \leqslant \mathsf{C}(x,y).$$

EXAMPLE 6.2. The contraharmonic mean belongs to the family of Gini means defined in general case by (7). They are increasing with respect to parameters r,s, thus for $0 < \alpha < 2$ we have

$$\mathsf{A}(x,y) = \mathsf{GINI}_{1,0}(x,y) < \mathsf{GINI}_{1,\alpha}(x,y) < \mathsf{GINI}_{1,2}(x,y) = \mathsf{C}(x,y).$$

Fix α and let $M = \mathsf{GINI}_{1,\alpha}$. An easy calculation shows that

$$\frac{\widehat{\mathsf{c}}^{-1}(\widehat{m}(z))}{z} = 2^{-\frac{1}{2(\alpha-1)}} \sqrt{\frac{[(1+z)^\alpha + (1-z)^\alpha]^{1/(\alpha-1)} - 2^{1/(\alpha-1)}}{z^2}}.$$

We shall show that the function under the square root increases. To this end, we shall use the following version of de l'Hospital's rule ([15]): if $f,g : I \to \mathbf{R}$ are differentiable in the interval I such that g' does not vanish and $(f'/g')(x)$ increases (decreases), then for any $a \in I$ so does the divided difference $\frac{f(x)-f(a)}{g(x)-g(a)}$.
Let $f(x) = [(1+z)^\alpha + (1-z)^\alpha]^{1/(\alpha-1)}$ and $g(x) = z^2$.
We intend to show that $(f'/g')(z)$ increases in $(0,1)$. We have

$$\begin{aligned}\frac{f'}{g'}(z) &= [(1+z)^\alpha + (1-z)^\alpha]^{(2-\alpha)/(\alpha-1)} \times \frac{\alpha}{2(\alpha-1)} \frac{(1+z)^{\alpha-1} - (1-z)^{\alpha-1}}{z} \\ &=: h_1(z) \times \frac{\alpha}{2(\alpha-1)} \frac{h_2(z)}{z}.\end{aligned}$$

Case 1: $1<\alpha<2$. The power function z^α is convex, so $(1+z)^\alpha+(1-z)^\alpha$ increases and so does $h_1(z)$. The function $h_2(z)$ is positive and convex, so its divided difference increases, thus f'/g' increases being a product of positive increasing functions.
Case 2: $0<\alpha<1$. The power function z^α is concave, so $(1+z)^\alpha+(1-z)^\alpha$ decreases, thus $h_1(z)$ increases. The function $h_2(z)$ is negative and concave, so its divided difference decreases, and the negative factor $\frac{\alpha}{2(\alpha-1)}$ turns it into positive increasing. Thus again f'/g' increases.

Therefore

$$\sqrt{\frac{\alpha}{2}}=\lim_{z\to 0}\frac{\widehat{\mathsf{c}}^{-1}(\widehat{m}(z))}{z}\leqslant\frac{\widehat{\mathsf{c}}^{-1}(\widehat{m}(z))}{z}\leqslant\frac{\widehat{\mathsf{c}}^{-1}(\widehat{m}(1))}{1}=1$$

and we obtain the optimal inequalities

$$\mathsf{C}\left(\frac{x+y}{2}+\sqrt{\frac{\alpha}{2}}\frac{x-y}{2},\frac{x+y}{2}-\sqrt{\frac{\alpha}{2}}\frac{x-y}{2}\right)\leqslant\mathsf{GINI}_{1,\alpha}(x,y)\leqslant\mathsf{C}(x,y).$$

Using the same technique as in Example 6.2 we obtain the following results for power means $\mathsf{GINI}_{0,\alpha}(x,y)$:

EXAMPLE 6.3. Note that $\mathsf{RMS}(x,y)=\mathsf{GINI}_{0,2}(x,y)$. For $1<\alpha<2$ the following inequalities

$$\mathsf{RMS}\left(\tfrac{x+y}{2}+p\tfrac{x-y}{2},\tfrac{x+y}{2}-p\tfrac{x-y}{2}\right)\leqslant\mathsf{GINI}_{0,\alpha}(x,y)\leqslant\mathsf{RMS}\left(\tfrac{x+y}{2}+q\tfrac{x-y}{2},\tfrac{x+y}{2}-q\tfrac{x-y}{2}\right)$$

hold if and only if $p\leqslant\sqrt{\alpha-1}$ and $q\geqslant\sqrt{4^{1-1/\alpha}-1}$.

EXAMPLE 6.4. For $-1<\alpha<1$ the following inequalities

$$\mathsf{H}(x,y)\leqslant\mathsf{GINI}_{0,\alpha}(x,y)\leqslant\mathsf{H}\left(\frac{x+y}{2}+q\frac{x-y}{2},\frac{x+y}{2}-q\frac{x-y}{2}\right)$$

hold if and only if $q\leqslant\sqrt{\frac{1-\alpha}{2}}$.

7. Aproximation by convex combination of means

Suppose for all positive distinct x,y three means K,M,N satisfy the inequalities $K(x,y)<M(x,y)<N(x,y)$. Our goal is to determine the best possible constants μ and ν such that the inequalities

$$(1-\mu)K(x,y)+\mu N(x,y)\leqslant M(x,y)\leqslant(1-\nu)K(x,y)+\nu N(x,y) \tag{17}$$

are valid for all x,y. In terms of Seiffert functions, the inequalities (17) have the form

$$\frac{1-\mu}{k(z)}+\frac{\mu}{n(z)}\leqslant\frac{1}{m(z)}\leqslant\frac{1-\nu}{k(z)}+\frac{\nu}{n(z)},$$

which is equivalent to

$$\mu \leqslant \frac{\frac{1}{m(z)} - \frac{1}{k(z)}}{\frac{1}{n(z)} - \frac{1}{k(z)}} \leqslant \nu. \tag{18}$$

Thus we have the following result.

THEOREM 7.1. *For three means satisfying $K < M < N$ the inequalities (17) hold if and only if*

$$\mu \leqslant \inf_{0<z<1} \frac{\frac{1}{m(z)} - \frac{1}{k(z)}}{\frac{1}{n(z)} - \frac{1}{k(z)}} \text{ and } \sup_{0<z<1} \frac{\frac{1}{m(z)} - \frac{1}{k(z)}}{\frac{1}{n(z)} - \frac{1}{k(z)}} \leqslant \nu.$$

Let us illustrate this result by finding the optimal bounds of eight Seiffert-like means discussed in Section 3 by convex combination of min and max. In this case the inequalities (18) read

$$\mu \leqslant \frac{1}{2}\left(\frac{1}{f(z)} - \frac{1}{z} + 1\right) \leqslant \nu,$$

thus we have to find the upper and lower bound of the function $\frac{1}{f(z)} - \frac{1}{z}$. In all eight cases we can write $f(z) = z + cz^3 + \ldots$, so $\lim_{z\to 0} \frac{1}{f(z)} - \frac{1}{z} = 0$.

EXAMPLE 7.1. Let $f(z) = \sin z$. The function $\cos z$ is concave in $(0, \pi/2)$ thus so are $\cos(tz)$ and $\int_0^1 \cos(tz)dt = \frac{\sin z}{z}$. Since the reciprocal of a positive concave function is convex, $\frac{z}{\sin z}$ is convex and consequetly, its divided difference

$$\frac{1}{z}\left(\frac{z}{\sin z} - 1\right) = \frac{1}{\sin z} - \frac{1}{z}$$

increases from 0 to $1/\sin(1) - 1$. Thus

$$\frac{x+y}{2} \leqslant \mathfrak{F}(\sin)(x,y) \leqslant \left(1 - \frac{1}{2\sin 1}\right)\min(x,y) + \frac{1}{2\sin 1}\max(x,y).$$

EXAMPLE 7.2. Let $f(z) = \arcsin z$. To investigate monotonicity of $\frac{1}{\arcsin z} - \frac{1}{z}$, set $z = \sin t$ and use the result of the previous example to see, that the range of this function is $(2/\pi - 1, 0)$. Thus,

$$\frac{1}{\pi}\min(x,y) + \left(1 - \frac{1}{\pi}\right)\max(x,y) \leqslant \mathsf{P}(x,y) \leqslant \frac{x+y}{2}.$$

EXAMPLE 7.3. Let $f(z) = \tan z$. Then,

$$\left(\frac{1}{\tan z} - \frac{1}{z}\right)' = \frac{1}{z^2} - \frac{\tan'(z)}{\tan^2(z)} = \frac{\sin^2 z - z^2}{z^2 \sin^2 z} < 0,$$

so $\frac{1}{\tan z} - \frac{1}{z}$ decreases from 0 to $1/\tan(1) - 1$ and consequently

$$\left(1 - \frac{1}{2\tan 1}\right)\min(x,y) + \frac{1}{2\tan 1}\max(x,y) \leqslant \mathfrak{F}(\tan)(x,y) \leqslant \frac{x+y}{2}.$$

EXAMPLE 7.4. In case $f(z) = \arctan z$, we set $z = \tan t$ and use the previous example to get

$$\frac{x+y}{2} \leqslant \mathsf{T}(x,y) \leqslant \left(1 - \frac{2}{\pi}\right) \min(x,y) + \frac{2}{\pi} \max(x,y).$$

EXAMPLE 7.5. The story of hyperbolic tangent is quite similar to that of tangent, the only difference is that $\sinh z \geqslant z$.

$$\frac{x+y}{2} \leqslant \mathfrak{F}(\tanh)(x,y) \leqslant \left(1 - \frac{1}{2\tanh 1}\right) \min(x,y) + \frac{1}{2\tanh 1} \max(x,y).$$

EXAMPLE 7.6. As shall be expected, for the inverse hyperbolic tangent we shall use Example 7.5 and the substitution $t = \tanh z$. Nevertheless, it is wise to note that $\lim_{z\to 1} \operatorname{artanh} z = \infty$, so in this case there is only trivial lower bound

$$\min(x,y) \leqslant \mathsf{L}(x,y) \leqslant \frac{x+y}{2}.$$

EXAMPLE 7.7. An attempt to use the same approach as in Example 7.1 fails for $f(z) = \sinh z$, because $\sinh z/z$ is convex and the reciprocal of a convex function may not be convex. So we use a bit more complicated approach.

$$\begin{aligned} \left(\frac{1}{\sinh z} - \frac{1}{z}\right)' &= \frac{\sinh^2 z - z^2 \cosh z}{z^2 \sinh^2 z} \\ &= \frac{\cosh z}{\sinh^2 z} \left(\frac{\sinh z \cosh^{-1/2} z}{z} + 1\right) \left(\frac{\sinh z \cosh^{-1/2} z}{z} - 1\right). \end{aligned} \tag{19}$$

Let $q(z) = \sinh z \cosh^{-1/2} z$. Then $q''(z) = \frac{1}{4} \cosh^{-3/5} z \sinh z(\cosh^2 z - 3)$, hence q is concave in $(0, \operatorname{arcosh}\sqrt{3})$. Since $\operatorname{arcosh}\sqrt{3} \approx 1.14$, we conclude that $q(z)/z$ decreases in the interval $(0,1)$, and thus the expression (19) is negative. Therefore we have

$$\left(1 - \frac{1}{2\sinh 1}\right) \min(x,y) + \frac{1}{2\sinh 1} \max(x,y) \leqslant \mathfrak{F}(\sinh)(x,y) \leqslant \frac{x+y}{2}.$$

And finally

EXAMPLE 7.8. For $f(z) = \operatorname{arsinh} z$, the substitution $z = \sinh t$ converts $\frac{1}{\operatorname{arsinh} z} - \frac{1}{z}$ into $\frac{1}{t} - \frac{1}{\sinh t}$ and monotonicity follows from the previous example, since $1 > z > t$. The optimal inequalities then are

$$\frac{x+y}{2} \leqslant \mathfrak{F}(\operatorname{arsinh})(x,y) \leqslant \left(1 - \frac{1}{2\operatorname{arsinh} 1}\right) \min(x,y) + \frac{1}{2\operatorname{arsinh} 1} \max(x,y).$$

8. Harmonic representation of means

Now it is time to combine the results presented in sections 4 and 6 to obtain some interesting identities for mean.

Initially we shall establish the relationship between a mean M and its integral representation $\mathfrak{F}\circ I\circ\mathfrak{F}^{-1}(M)$.

Changing variable in the integral defining the operator I we obtain

$$I(m)(z)=\int_0^z\frac{m(t)}{t}dt=\int_0^1\frac{m(uz)}{u}du=\int_0^1 m^{\{u\}}(z)du,$$

and we deduce

$$\frac{1}{\mathfrak{F}(I(m))(x,y)}=\frac{2I(m)\left(\frac{|x-y|}{x+y}\right)}{|x-y|}=\frac{2\int_0^1 m^{\{u\}}\left(\frac{|x-y|}{x+y}\right)du}{|x-y|}=\int_0^1\frac{du}{M^{\{u\}}(x,y)}.$$

We shall call M the harmonic representation of $\mathfrak{F}(I(m))$. Our goal is to characterize means which allow for harmonic representation.

Observe that the integral operator I defined on the set of Riemann integrable Seiffert functions is not a bijection. Indeed, two functions that differ on a finite set produce the same outcome. Due to this smoothing properties of the integral, the harmonic representation (if exists) is not unique. Fortunately, we can cope with this inconvenience by restricting the range of the operator I to the set of continuous Seiffert functions. Note that in this case the $I(\cdot)$ is differentiable. Thus our task, in terms of Seiffert means can be formulated as follows

QUESTION 8.1. Characterize the Seiffert function which are of the form $I(m)$ for some continuous Seiffert function m.

Differentiating the formula $n(z)=I(m)(z)=\int_0^x\frac{m(t)}{t}dt$ we obtain $n'(x)=\frac{m(x)}{x}$. If we assume m is a Seiffert mean, the inequalities (6) yields

$$\frac{1}{1+z}\leqslant n'(z)\leqslant\frac{1}{1-z},$$

which implies that $\mathrm{id}\cdot n'$ is a Seiffert function. One can easily verify $I(\mathrm{id}\cdot n')=n$. This leads us to the following theorem:

THEOREM 8.1. *Let g be a real function defined on the interval $(0,1)$. The following conditions are equivalent*

- $\lim_{z\to0}g(z)=0$, *g is continuously differentiable, and for all $0<z<1$*

$$\frac{1}{1+z}\leqslant g'(z)\leqslant\frac{1}{1-z},\tag{20}$$

- *there exists a continuous Seiffert function f such that $g=I(f)$.*

Proof. Multiplying (20) by z we see that $f(z) = zg'(z)$ is a continuous Seiffert function and clearly $I(f) = g$.
Conversely, if f is continuous, then $g = I(f)$ is differentiable. Since $\lim_{z\to 0} f(z)/z = 1$ we claim $\lim_{z\to 0} g(z) = 0$. Differentiating g we obtain $g'(z) = f(z)/z$, which yields (20) because f fulfills (6).

COROLLARY 8.1. *If N is a continuous harmonic representation of M, then $n(z) = zm'(z)$.*

EXAMPLE 8.1. For the first Seiffert mean P (see (1)) we have $z\mathsf{p}'(z) = \frac{z}{\sqrt{1-z^2}}$. This is the Seiffert function of the geometric mean G. Thus we obtain the identity

$$\mathsf{P}(x,y) = \left(\int_0^1 \frac{dt}{\mathsf{G}^{\{t\}}(x,y)}\right)^{-1}.$$

EXAMPLE 8.2. Similarly, for the second Seiffert mean T (see (2)) we have $z\mathsf{t}'(z) = \frac{z}{1+z^2}$. This is the Seiffert function of the contra-harmonic mean C. Thus we obtain the identity

$$\mathsf{T}(x,y) = \left(\int_0^1 \frac{dt}{\mathsf{C}^{\{t\}}(x,y)}\right)^{-1}.$$

EXAMPLE 8.3. For the logarithmic and the harmonic means are linked together by the formula $\mathsf{h}(z) = z\mathsf{l}'(z)$, so we have

$$\mathsf{L}(x,y) = \left(\int_0^1 \frac{dt}{\mathsf{H}^{\{t\}}(x,y)}\right)^{-1}.$$

EXAMPLE 8.4. The Neuman-Sándor mean (3) and the root-mean square form another good pair

$$\mathsf{NS}(x,y) = \left(\int_0^1 \frac{dt}{\mathsf{RMS}^{\{t\}}(x,y)}\right)^{-1}.$$

Now we shall check the harmonic representations of the four new means introduced in Section 3.

EXAMPLE 8.5. For $g(z) = \sin z$ we want to show that g' satisfies (20). Obviously $\cos z < 1 < 1/(1-z)$. To prove the other part observe that

$$(1+z)\cos z > (1+z)(1-z^2/2) > 1 + z(1-z/2) > 1,$$

thus (20) holds, and one easily verifies that $z\cos z$ is the Seiffert function of the mean $M(x,y) = \mathsf{A}(x,y)/\cos\frac{|x-y|}{x+y}$, which implies

$$\frac{x-y}{2\sin\frac{x-y}{x+y}} = \left(\int_0^1 \frac{dt}{M^{\{t\}}(x,y)}\right)^{-1}.$$

EXAMPLE 8.6. Now let $g(z) = \tan z$. We have

$$\frac{1}{1+z} < 1 < \frac{1}{\cos^2 z} = \frac{1}{(1+\sin z)(1-\sin z)} < \frac{1}{1-z},$$

so $z/\cos^2 z$ is the Seiffert function. It corresponds to the mean $M(x,y) = \mathsf{A}(x,y)\cos^2 \frac{|x-y|}{x+y}$ and

$$\frac{x-y}{2\tan\frac{x-y}{x+y}} = \left(\int_0^1 \frac{dt}{M^{\{t\}}(x,y)}\right)^{-1}.$$

EXAMPLE 8.7. With the hyperbolic sine the situation is simple. We have

$$1 < \cosh z = \sum_{m=0}^{\infty} \frac{z^{2m}}{(2m)!} < \sum_{m=0}^{\infty} z^m = \frac{1}{1-z},$$

thus $z\cosh z$ is the Seiffert function, and its mean $M(x,y) = \mathsf{A}(x,y)/\cosh\frac{|x-y|}{x+y}$ satisfies

$$\frac{x-y}{2\sinh\frac{x-y}{x+y}} = \left(\int_0^1 \frac{dt}{M^{\{t\}}(x,y)}\right)^{-1}.$$

EXAMPLE 8.8. The last function is the hyperbolic tangent. Its derivative is $\cosh^{-2} z$ and $\cosh^{-2}(1) \approx 0.41997 < \frac{1}{2}$, so the left inequality in (20) does not hold, and this yields the mean $\frac{x-y}{2\sinh\frac{x-y}{x+y}}$ does not have any harmonic representation.

9. Harmonic representation of the AGM mean.

This section is devoted to the arithmetic-geometric mean given by the formula

$$\mathsf{AGM}(x,y) = \left(\frac{2}{\pi}\int_0^{\pi/2} \frac{d\varphi}{\sqrt{x^2\cos^2\varphi + y^2\sin^2\varphi}}\right)^{-1}.$$

To find its Seiffert function let us recall the famous result of Gauss [10]

$$\mathsf{AGM}(1-z, 1+z) = \frac{\pi}{2K(z)}, \tag{21}$$

where K is the complete elliptic integral of the first kind

$$K(z) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-z^2\sin^2\varphi}} = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-z^2t^2}}.$$

Comparing (21) and (9) we see that $\mathsf{agm}(z) = \frac{2}{\pi} zK(z)$. We shall show that AGM admits the harmonic representation. By Theorem 8.1 it is enough to show that agm' satisfies (20). To this end let us recall the power series expansion of K ([1, 900.00])

$$K(z) = \frac{\pi}{2}\left(1 + \sum_{m=1}^{\infty}\left[\frac{(2m-1)!!}{(2m)!!}\right]^2 z^{2m}\right).$$

We have

$$\mathsf{agm}'(z) = \frac{2}{\pi}\left(K(z) + z\frac{dK}{dz}\right) = 1 + \sum_{m=1}^{\infty}(2m+1)\left[\frac{(2m-1)!!}{(2m)!!}\right]^2 z^{2m}. \tag{22}$$

Denoting the m^{th} coefficient in (22) by c_m we see that

$$\frac{c_{m+1}}{c_m} = \frac{2m+3}{2m+1}\left[\frac{(2m+1)!!(2m)!!}{(2m+2)!!(2m-1)!!}\right]^2 = \frac{(2m+1)(2m+3)}{(2m+2)^2} < 1,$$

and since $c_1 = 3/4$ we conclude that $c_m < 1$ for all $m \geqslant 1$. Thus $1 < \mathsf{agm}'(z) < 1 + z + z^2 + \ldots = 1/(1-z)$.
Theorem 8.1 implies that the arithmetic-geometric mean admits the harmonic representation. To derive its explicit form, recall that the derivative of K is given by $K'(z) = \frac{E(z)}{z(1-z^2)} - \frac{K(z)}{z}$ (see e.g. [1, 710.00], $E(z) = \int_0^{\pi/2}\sqrt{1 - z^2\sin^2\varphi d\varphi}$ is the complete elliptic integral of the second kind), thus

$$z \cdot \mathsf{agm}'(z) = \frac{2}{\pi}\left(zK(z) + z^2K'(z)\right) = \frac{2}{\pi}\frac{z}{1-z^2}E(z).$$

As $\frac{z}{1-z^2}$ is the Seiffert function of the harmonic mean we obtain the formula

$$V(x,y) = \frac{\pi\mathsf{H}(x,y)}{2E\left(\frac{|x-y|}{x+y}\right)} = \frac{\pi\mathsf{H}(x,y)}{2E\left(\sqrt{1 - \frac{\mathsf{G}^2(x,y)}{\mathsf{A}^2(x,y)}}\right)}$$
$$= \frac{\pi\mathsf{G}^2(x,y)}{2\int_0^{\pi/2}\sqrt{\mathsf{A}^2(x,y)\cos^2\varphi + \mathsf{G}^2(x,y)\sin^2\varphi d\varphi}}.$$

This mean has a nice geometric interpretation: in the ellipse with semi-axes $\mathsf{G}(x,y)$ and $\mathsf{A}(x,y)$ it represents the ratio of the area of the inscribed disc to the semi-perimeter of the ellipse.

10. Miscellanea

In this section, we collect some facts about means and Seiffert functions, that might be useful for future investigations.

This surprising result follows from the results of Section 6.

THEOREM 10.1. *If M is a mean satisfying $M(x,y) \leqslant \mathsf{A}(x,y)$, and $M^{\{1/2\}}(x,y) = M(\frac{3x+y}{4}, \frac{x+3y}{4})$, then $(M^{\{1/2\}})^2/\mathsf{A}$ is a mean.*

Note that in general M^2/A is not a mean (take the harmonic mean as a counterexample).

Proof. Let f be the Seiffert mean of M. Then the function $2f(z/2)$ is the Seiffert function of $M^{\{1/2\}}$. Consider the function $g(z) = 4f^2(z/2)/z$. Since $z \leqslant f(z)$, we have $z \leqslant g(z)$ and

$$g(z) \leqslant \frac{4}{z}\frac{z^2/4}{(1-z/2)^2} < \frac{z}{1-z}.$$

Thus g is a Seiffert function and its corresponding mean is

$$\frac{|x-y|z}{8f^2(z/2)} = \left(\frac{|x-y|}{2\cdot 2f(z/2)}\right)^2 \frac{2}{x+y} = \frac{(M^{\{1/2\}})^2(x,y)}{\mathsf{A}(x,y)}.$$

This result may be generalized as follows

COROLLARY 10.1. *If M is a mean satisfying $M(x,y) \leqslant \mathsf{A}(x,y)$, then $(M^{\{t\}})^{\frac{1}{t}}\mathsf{A}^{\frac{t-1}{t}}$ is also a mean.*

To prove this, it is enough to show that $g(z) = z[g(tz)/(tz)]^{\frac{1}{t}}$ is a Seiffert function. We leave the details to the reader.

REMARK 10.1. It is easy to see that $z+az^3$ is a Seiffert function if and only if $-1/2 \leqslant a \leqslant 4$.

THEOREM 10.2. *The inequalities*

$$\mathsf{A}(x,y) \leqslant \frac{|x-y|}{2\sin\frac{x-y}{x+y}} \leqslant \mathsf{A}(x,y)\frac{6\mathsf{A}^2(x,y)}{5\mathsf{A}^2(x,y)+\mathsf{G}^2(x,y)} \tag{23}$$

hold.

Proof. The sine function satisfies $z > \sin z > z - z^3/6$. It follows from Remark 10.1 that the last function in this chain is a Seiffert function, and a simple calculation shows that its mean is the rightmost mean in (23).

For two Seiffert functions f_M and f_N and a homogeneous but necessarily symmetric mean K, the function $g(z) = K(f_M(z), f_N(z))$ is a Seiffert function corresponding to $MN/K(N,M)$. In particular, if K is M,N invariant (see Theorem 2.2), then MN/K is also a mean.

Now we shall show some facts about power series representation of Seiffert functions. The first one is trivial.

THEOREM 10.3. *If $0 \leqslant a_n \leqslant 1$ for $n > 1$, then the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is a Seiffert function.*

Proof. Clearly, the series converges for $0 < z < 1$ and

$$z \leqslant f(z) \leqslant \sum_{n=1}^{\infty} z^n = \frac{z}{1-z}.$$

The two following theorems concern alternating series.

THEOREM 10.4. *Let $1 = a_1 \geqslant a_2 \geqslant \ldots \geqslant 0$ be a convex sequence (i.e. satisfying $2a_k \leqslant a_{k-1} + a_{k+1}$ for $k = 2,3,\ldots$). Then the function $f(z) = z + \sum_{n=2}^{\infty}(-1)^{n+1} a_n z^n$ is a Seiffert function.*

Proof. Note first, that

$$f(z) = z - z^2(a_2 - a_3 z) - z^4(a_4 - a_5 z) - \ldots \leqslant z.$$

Let $b_n = a_n - a_{n+1}$. This sequence decreases monotonically to 0 and we have

$$\begin{aligned}(1+z)f(z) &= z + \sum_{n=2}^{\infty}(-1)^{n+1}a_n z^n + \sum_{n=1}^{\infty}(-1)^{n+1}a_n z^{n+1} \\ &= z + z^2(b_1 - b_2 z) + z^4(b_3 - b_4 z) + \ldots \geqslant z,\end{aligned}$$

thus $f(z) \geqslant \frac{z}{1+z}$.

COROLLARY 10.2. *The following functions are Seiffert:*

$$\log(1+z), \quad 2\frac{z - \log(1+z)}{z}, \quad 3\frac{\log(1+z) - z + z^2/2}{z^2}, \ \ldots.$$

THEOREM 10.5. *Let a_n, $n \geqslant 1$ be nonnegative numbers satisfying $a_1 \leqslant 1/2, 1 \geqslant a_2 \geqslant a_3 \geqslant \ldots$, then $f(z) = z - a_1 z^3 + \sum_{n=2}^{\infty}(-1)^n a_n z^{2n+1}$ is a Seiffert function.*

Proof. As we know from Remark 10.1 that $z - a_1 z^3 \geqslant \frac{z}{1+z}$, thus

$$f(z) \geqslant \frac{z}{1+z} + z^5(a_2 - a_3 z^2) + z^9(a_4 - a_5 z^2) + \ldots \geqslant \frac{z}{1+z}.$$

On the other hand

$$f(z) \leqslant \sum_{n=0}^{\infty} z^{2n+1} < \sum_{n=1}^{\infty} z^n = \frac{z}{1-z},$$

which completes the proof.

COROLLARY 10.3. *The following are Seiffert functions:*

$$\sin z, \quad 6\frac{z - \sin z}{z^2}, \quad 120\frac{\sin z - z + z^3/6}{z^4}, \ \ldots,$$

$$2\frac{1 - \cos z}{z}, \quad 24\frac{\cos z - 1 + z^2/z}{z^3}, \ \ldots.$$

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