Realizing Mahlo set theory in type theory

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Abstract

After introducing the large set notion of Mahloness, this paper shows that constructive set theory with an axiom asserting the existence of a Mahlo set has a realizability interpretation in an extension of Martin-Löf type theory developed by A. Setzer.

1 Introduction

In a talk at a conference in Italy in October 1998, Martin-Löf addressed the *Problem of Impredicativity.* He expounded that the strongest version of type theory for which there exists a constructive justification (in terms of his meaning explanation) is a system of Martin-Löf type theory with a Mahlo universe, **MLM**, introduced by Setzer (cf. [17]). As was shown by Setzer [17], this type theory embodies considerable proof-theoretic strength. The consistency proof for a strong system of classical set theory, called **KPM**, which was introduced in [14], can be carried out in **MLM** by utilizing the ordinal analysis of **KPM** of [14] via the ordinal representation system of [13].

Another way of describing the strength of systems of type theory is by way of interpreting more familiar systems of set theory in them. Aczel has given an interpretation of constructive Zermelo-Fraenkel set theory, **CZF**, in Martin-Löf type theory, and for several large set notions this interpretation has been extended to incorporate **CZF** plus large set axioms (cf.[15], [16]). The objective of this paper is to show that **CZF** plus an axiom asserting the existence of a Mahlo set has a canonical interpretation in Setzer's type theory.

The paper is organized as follows: After recalling the axioms of **CZF**, we introduce the notion of set-inaccessibility. As the latter notion is still of a rather syntactic flavour, a more 'algebraic' characterization is sought in Section 3. Section 4 deals with the notion of Mahloness and explores to some extent its properties on the basis of Constructive Zermelo-Fraenkel Set Theory. The last Section 4 provides the interpretation of **CZF**+$\exists M \;\; M$ *Mahlo* in Setzer's extension of Martin-Löf type theory.

2 Large sets in constructive set theory

Constructive set theory grew out of Myhill's endeavours (cf.[12]) to discover a simple formalism that relates to Bishop's constructive mathematics as **ZFC** relates

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to classical Cantorian mathematics. Later on Aczel modified Myhill's set theory to a system which he called *Constructive Zermelo-Fraenkel Set Theory*, **CZF**, and corroborated its constructiveness by interpreting it in Martin-Löf's type theory (cf. [1]). The interpretation was in many ways canonical and can be seen as providing **CZF** with a standard model in type theory.

Definition 2.1 (Axioms of **CZF**) The language of **CZF** is the first order language of Zermelo Fraenkel set theory, LST, with the non logical primitive symbol $\in$. **CZF** is based on intuitionistic predicate logic with equality. The set theoretic axioms of axioms of **CZF** are the following:

1. **Extensionality** $\forall a\, \forall b\, (\forall y\, (y \in a \leftrightarrow y \in b) \rightarrow a = b)$.
2. **Pair** $\forall a\, \forall b\, \exists x\, \forall y\, (y \in x \leftrightarrow y = a \vee y = b)$.
3. **Union** $\forall a\, \exists x\, \forall y\, (y \in x \leftrightarrow \exists z \in a\, y \in z)$.
4. **Restricted Separation scheme** $\forall a\, \exists x\, \forall y\, (y \in x \leftrightarrow y \in a \wedge \varphi(y))$,

   for every *bounded* formula $\varphi(y)$, where a formula $\varphi(x)$ is bounded, or Δ_0, if all the quantifiers occurring in it are bounded, i.e. of the form $\forall x \in b$ or $\exists x \in b$.
5. **Subset Collection scheme**

   $$\begin{array}{l}\forall a\, \forall b\, \exists c\, \forall u\, \big(\forall x \in a\, \exists y \in b\, \varphi(x, y, u) \rightarrow \\ \exists d \in c\, (\forall x \in a\, \exists y \in d\, \varphi(x, y, u) \wedge \forall y \in d\, \exists x \in a\, \varphi(x, y, u))\big)\end{array}$$

   for every formula $\varphi(x, y, u)$.
6. **Strong Collection scheme**

   $$\begin{array}{l}\forall a\, \big(\forall x \in a\, \exists y\, \varphi(x, y) \rightarrow \\ \exists b\, (\forall x \in a\, \exists y \in b\, \varphi(x, y) \wedge \forall y \in b\, \exists x \in a\, \varphi(x, y))\big)\end{array}$$

   for every formula $\varphi(x, y)$.
7. **Infinity**

   $$\exists x \forall u \big[u \in x \leftrightarrow \big(0 = u \vee \exists v \in x (u = v \cup \{v\})\big)\big]$$

   where $y + 1$ is $y \cup \{y\}$, and 0 is the empty set, defined in the obvious way.
8. **Set Induction scheme**

   $$(IND_{\in}) \qquad \forall a\, (\forall x \in a\, \varphi(x) \rightarrow \varphi(a)) \rightarrow \forall a\, \varphi(a),$$

   for every formula $\varphi(a)$.

Let **CZF**$^-$ denote the theory **CZF** without Set Induction.

From Infinity, Set Induction, and Extensionality one can deduce that there exists exactly one set x such that $\forall u \big[u \in x \leftrightarrow \big(0 = u \vee \exists v \in x (u = v \cup \{v\})\big)\big]$; this set will be denoted by ω.

The first large set axioms proposed in the context of constructive set theory was the *Regular Extension Axiom*, **REA**, which Aczel introduced to accommodate inductive definitions in **CZF** (cf. [1], [3]).

Definition 2.2 A set c is said to be *regular* if it is transitive, inhabited (i.e. $\exists u\, u \in c$) and for any $u \in c$ and set $R \subseteq u \times c$ if $\forall x \in u\, \exists y\ \langle x, y\rangle \in R$ then there is a set $v \in c$ such that

$$\forall x \in u\, \exists y \in v\ \langle x, y\rangle \in R \ \wedge\ \forall y \in v\, \exists x \in u\ \langle x, y\rangle \in R.$$

We write $Reg(a)$ for 'a is regular'.

REA is the principle

$$\forall x\, \exists y\ (x \in y\ \wedge\ Reg(y)).$$

Definition 2.3 Let **INAC** be the principle

$$\forall x\, \exists y\ (x \in y\ \wedge\ Reg(y) \text{ and } y \text{ is a model of } \mathbf{CZF}^-),$$

i.e. the structure $\langle y, \in\restriction (y \times y)\rangle$ is a model of $\mathbf{CZF}^-$.

We say that a set is *set-inaccessible* if it is regular and a model of $\mathbf{CZF}^-$ and write $\mathbf{INAC}(y)$ for 'y is set-inaccessible'.

Remark 2.4 As it makes perfect sense to study notions of largeness in set theories without or with restricted Set Induction, we have formalized set-inaccessibility by requiring that y is a model of $\mathbf{CZF}^-$ rather than **CZF**. On the other hand, if one assumes Set Induction in the background theory than $\mathbf{INAC}(y)$ readily implies that y is a model of Set Induction as well, and hence $y \models \mathbf{CZF}$.

The formalization of the notion of inaccessibility in Definition 2.3 is somewhat awkward as it is very syntactic in that it requires a satisfaction predicate for formulae interpreted over a set. An alternative and more 'algebraic' characterization will be given in the next section.

Viewed classically inaccessible sets are closely related to inaccessible cardinals. Let V_α denote the αth level of the von Neumann hierarchy.

Proposition 2.5 (**ZFC**) *I is set-inaccessible if and only if $I = V_\kappa$ for some strongly inaccessible cardinal κ.*

Proof: This is a consequence of the proof of [15], Corollary 2.7. □

Proposition 2.6 *Let* **EM** *denote the principle of excluded middle. The theories* $\mathbf{CZF}^- + \mathbf{INAC} + \mathbf{EM}$ *and*

$$\mathbf{ZFC} + \forall\alpha\, \exists\kappa\, (\alpha < \kappa\ \wedge\ \kappa \textit{ is a strongly inaccessible cardinal})$$

have the same proof theoretic strength.

Proof: [5], Lemma 2.10. □

3 A nicer rendering of set-inaccessibility

Definition 3.1 Let $\Omega := \{x : x \subseteq \{0\}\}$. Ω is the class of truth values with 0 representing falsity and $1 = \{0\}$ representing truth. Classically one has $\Omega = \{0, 1\}$ but intuitionistically one cannot conclude that those are the only truth values.

For $a \subseteq \Omega$ define

$$\begin{aligned}\bigwedge a &= \{x \in 1 : (\forall u \in a) x \in u\} \\ \bigvee a &= \{x \in 1 : (\exists u \in a) x \in u\} \ \ (= \bigcup a).\end{aligned}$$

A class B is $\bigwedge$-*closed* if for all $a \in B$, whenever $a \subseteq \Omega$, then $\bigwedge a \in B$.

Definition 3.2 For sets a, b let ${}^a b$ be the class of all functions with domain a and with range contained in b. Let $\mathbf{mv}({}^a b)$ be the class of all sets $r \subseteq a \times b$ satisfying $\forall u \in a\, \exists v \in b\, \langle u, v\rangle \in r$. A set c is said to be *full in* $\mathbf{mv}({}^a b)$ if $c \subseteq \mathbf{mv}({}^a b)$ and

$$\forall r \in \mathbf{mv}({}^a b)\, \exists s \in c\, s \subseteq r.$$

The expression $\mathbf{mv}({}^a b)$ should be read as the collection of *multi-valued functions* from a to b.

The *Fullness* axiom is the assertion

$$\forall a \forall b\, \exists c \subseteq \mathbf{mv}({}^a b)\, \big[\forall r \in \mathbf{mv}({}^a b)\, \exists s \in c\, s \subseteq r\big].$$

Let $\mathbf{CZF}^*$ be the theory $\mathbf{CZF}^-$ bereft of the Subset Collection scheme.

Proposition 3.3 *On the basis of* $\mathbf{CZF}^*$*, the Subset Collection scheme and the Fullness axiom are equivalent.*

Proof: The proof of [15], Proposition 2.3 does not use Set Induction. □

Proposition 3.4 ($\mathbf{CZF}^-$) *I is set-inaccessible if and only if the following are satisfied:*

1. *I is a regular set,*
2. *$\omega \in I$,*
3. $(\forall a \in I) \bigcup a \in I$,
4. *I is $\bigwedge$-closed,*
5. $(\forall a, b \in I)\big[\{x \in 1 : a = b\} \in I \wedge \{x \in 1 : a \in b\} \in I\big]$.
6. $(\forall a, b \in I)(\exists c \in I)\, \big[c \textit{ is full in } \mathbf{mv}({}^a b)\big]$.

Proof: Firstly, suppose that I is set-inaccessible. Then (1)–(5) are obvious. (6) follows from the proof of [15], Lemma 2.6 which only requires $\mathbf{CZF}^-$.

Now assume that (1)-(6) hold. The regularity of I implies that I is a model of Strong Collection and (6) implies that I is a model of Subset Collection by Proposition 3.3 providing that I is a model of the remaining axioms of $\mathbf{CZF}^-$.

By (2), I is a model of Infinity. From $\omega \in I$ and the transitivity of I we obtain $2 = \{0, 1\} \in I$. If $a, b \in I$ let $f : 2 \to I$ be the function defined by $f(0) = a$ and $f(1) = b$. The range of f is $\{a, b\}$. Regularity of I implies that the range of f is in I and thus $\{a, b\} \in I$. The latter shows that I is a model of Pairing. By (3), I is a model of Union. It remains to verify that I is a model of Restricted Separation. Firstly, we will show that for every restricted formula θ which contains only parameters from I there exists a set $c \in \Omega \cap I$ such that

$$\theta \leftrightarrow 0 \in c. \tag{1}$$

The proof of (1) follows the proof of [4], Proposition 3.7. We proceed by induction on the construction of θ. Note that, by Extensionality, c is unique.

If θ is of the form $a = b$ or $a \in b$, then the claim follows from (5).

Next we address the propositional connectives. Let $c_1, c_2 \in \Omega \cap I$ such that

$$\phi_i \leftrightarrow 0 \in c_i.$$

Then $c_\wedge := \bigwedge\{c_1, c_2\} \in \Omega \cap I$ (by (4)) and

$$[\phi_1 \wedge \phi_2] \leftrightarrow 0 \in c_\wedge.$$

Similarly $c_\vee = \bigvee\{c_1, c_2\} \in \Omega \cap I$ (by (3)) and $[\phi_1 \vee \phi_2] \leftrightarrow 0 \in c_\vee$. Let $c_\to := \bigwedge\{c_2 : x \in c_1\} \in \Omega$. As $c_\to$ is the range of the function $x \mapsto c_2$ with domain c_1, regularity of I implies $c_\to \in I$. Moreover,

$$[\phi_1 \to \phi_2] \leftrightarrow 0 \in c_\to.$$

Set $c_\neg := \bigwedge\{0 : x \in c_1\}$. As $0 \in \Omega \cap I$, the above shows $c_\neg \in \Omega \cap I$. As $0 = 1 \leftrightarrow 0 \in 0$ and $\neg\phi_1 \leftrightarrow [\phi_1 \to 0 = 1]$, it follows that $\neg\phi_1 \leftrightarrow 0 \in c_\neg$.

Finally, we address the bounded quantifiers. Suppose that $a \in I$ and that for all $x \in a$ there exists a $c_x \in \Omega \cap I$ such that $\phi(x) \leftrightarrow x \in c_x$. Let f be the function with domain a such that $f(x) = c_x$. Let b be the range of f. As $f : a \to I$, the regularity of I implies $b \in I$. By (3) and (4) we get $\bigwedge b, \bigvee b \in \Omega \cap I$. Moreover,

$$\begin{aligned} (\forall x \in a)\phi(x) &\leftrightarrow 0 \in \bigwedge b, \\ (\exists x \in a)\phi(x) &\leftrightarrow 0 \in \bigvee b, \end{aligned}$$

concluding the proof of (1).

Now let $a \in I$ and let $\phi(x)$ be a restricted formula with all parameters in I. Then for every $x \in a$ there exists exactly one set $c_x \in I$ such that $c_x \in \Omega$ and $\phi(x) \leftrightarrow 0 \in c_x$. For each set $x \in a$ let d_x be the function with domain c_x such that $d_x(u) = x$ for $u \in c_x$. Regularity of I implies that $\mathbf{ran}(d_x)$ (the range of d_x) is in I. Let g be the function with domain a satisfying $g(x) = \mathbf{ran}(d_x)$. Then $g : a \to I$ and, by the regularity of I, we get $\mathbf{ran}(g) \in I$. As

$$\{x \in a : \phi(x)\} = \{x \in a : 0 \in c_x\} = \{x \in a : x \in \mathbf{ran}(d_x)\} = \bigcup \mathbf{ran}(g)$$

we get $\{x \in a : \phi(x)\} \in I$. □

Corollary 3.5 (**CZF**) *I is set-inaccessible if and only if the following are satisfied:*

1. *I is a regular set,*
2. *$\omega \in I$,*
3. *$(\forall a \in I) \bigcup a \in I$,*
4. *I is $\bigwedge$-closed,*
5. *$(\forall a, b \in I)(\exists c \in I)\,\big[c$ is full in $\mathbf{mv}({}^{a}b)\big]$.*

Proof: In the presence of Set Induction for restricted formulas, clause (5) of Proposition is not needed in the proof of (1). If θ is the formula $a = b$, one uses a double Set Induction on a, b and the equivalence

$$a = b \;\leftrightarrow\; \big((\forall x \in a)(\exists y \in b)[x = y] \;\wedge\; (\forall x \in b)(\exists y \in a)[x = y]\big)$$

to show (1). If θ is the formula $a \in b$ one uses the equivalence $a \in b \;\leftrightarrow\; (\exists y \in b) a = y$. □

4 Mahloness in constructive set theory

This section introduces the notion of a Mahlo set and explores some of its **CZF** provable properties.

Recall that in classical set theory a cardinal κ is said to be *weakly Mahlo* if the set $\{\rho < \kappa : \rho \textit{ is regular}\}$ is stationary in κ. A cardinal μ is *strongly Mahlo* if the set $\{\rho < \kappa : \rho \textit{ is a strongly inaccessible cardinal}\}$ is stationary in μ.

Definition 4.1 A set M is said to be *Mahlo* if M is set-inaccessible and for every $R \in \mathbf{mv}({}^{M}M)$ there exists a set-inaccessible $I \in M$ such that

$$\forall x \in I\, \exists y \in I\, \langle x, y\rangle \in R.$$

Proposition 4.2 (**ZFC**) *A set M is Mahlo if and only if $M = V_\mu$ for some strongly Mahlo cardinal μ.*

Proof: This is an immediate consequence of Proposition 2.5. □

Lemma 4.3 (**CZF$^-$**) *If M is Mahlo and $R \in \mathbf{mv}({}^{M}M)$, then for every $a \in M$ there exists a set-inaccessible $I \in M$ such that $a \in I$ and*

$$\forall x \in I\, \exists y \in I\, \langle x, y\rangle \in R.$$

Proof: Set $S := \{\langle x, \langle a, y\rangle\rangle : \langle x, y\rangle \in R\}$. Then $S \in \mathbf{mv}({}^{M}M)$ too. Hence there exists $I \in M$ such that $\forall x \in I\, \exists y \in I\, \langle x, y\rangle \in S$. Now pick $c \in I$. Then $\langle c, d\rangle \in S$ for some $d \in I$. Moreover, $d = \langle a, y\rangle$ for some y. In particular, $a \in I$.

Further, for each $x \in I$ there exists $u \in I$ such that $\langle x, u\rangle \in S$. As a result, $u = \langle a, y\rangle$ and $\langle x, y\rangle \in R$ for some y. Since $u \in I$ implies $y \in I$, the latter shows that $\forall x \in I\, \exists y \in I\, \langle x, y\rangle \in R$. □

Lemma 4.4 (**CZF**$^-$) *Let M be Mahlo. If $\forall x \in M\, \exists y \in M\, \phi(x,y)$, then there exists $S \in \mathbf{mv}({}^M M)$ such that*

$$\forall xy\, [\langle x,y\rangle \in S \to \phi(x,y)].$$

Proof: The assumption yields $\forall x \in M\, \exists z \in M\, \psi(x,z)$, where

$$\psi(x,z) \quad := \quad \exists y \in M\, (z = \langle x,y\rangle \,\wedge\, \phi(x,y)).$$

By Strong Collection there exists a set S such that $\forall x \in M\, \exists z \in S\, \psi(x,z)$ and $\forall z \in S\, \exists x \in M\, \psi(x,z)$. As a result, $S \in \mathbf{mv}({}^M M)$ and $\forall x \in M\, \exists y \in M\, \langle x,y\rangle \in S$. Moreover, if $\langle x,y\rangle \in S$, then $y \in M$ and $\phi(x,y)$ holds. □

Corollary 4.5 (**CZF**$^-$) *Let M be Mahlo. If $\forall x \in M\, \exists y \in M\, \phi(x,y)$, then for every $a \in M$ there exists a set-inaccessible $I \in M$ such that $a \in I$ and*

$$\forall x \in I\, \exists y \in I\, \phi(x,y).$$

Proof: This follows from Lemma 4.4 and Lemma 4.3. □

In a paper from 1911 Mahlo [7] investigated two hierarchies of regular cardinals. In view of its early appearance this work is astounding for its refinement and its audacity in venturing into the higher infinite. Mahlo called the cardinals considered in the first hierarchy π_α-*numbers*. In modern terminology they are spelled out as follows:

κ is 0-*weakly inaccessible*	iff	κ is regular;
κ is $(\alpha+1)$-*weakly inaccessible*	iff	κ is a regular limit of α-weakly inaccessibles
κ is λ-*weakly inaccessible*	iff	κ is α-weakly inaccessible for every $\alpha < \lambda$

for limit ordinals λ. Mahlo also discerned a second hierarchy which is generated by a principle superior to taking regular fixed-points. Its starting point is the class of ρ_0-numbers which later came to be called weakly Mahlo cardinals.

A hierarchy of em strongly α-inaccessible cardinals is analogously defined, except that the strongly 0-inaccessibles are the strongly inaccessible cardinals.

In classical set theory the notion of a strongly Mahlo cardinal is much stronger than that of a strongly inaccessible cardinal. This is e.g. reflected by the fact that for every strongly Mahlo cardinal μ and $\alpha < \mu$ the set of strongly α-inaccessible cardinals below μ is closed an unbounded in μ (cf.[6], Ch.I,Proposition 1.1). In the following we show that similar relations hold true in the context of constructive set theory as well.

Definition 4.6 An *ordinal* is a transitive set whose elements are transitive too. We use letters $\alpha, \beta, \gamma, \delta$ to range over ordinals.

Let A, B be classes. A is said to be *unbounded in B* if

$$\forall x \in B\, \exists y \in A\, (x \in y \,\wedge\, y \in B).$$

Let Z be set. Z is said to be *α-set-inaccessible* if Z is set-inaccessible and there exists a family $(X_\beta)_{\beta\in\alpha}$ of sets such that for all $\beta \in \alpha$ the following hold:

- X_β is unbounded in Z.
- X_β consists of set-inaccessible sets.
- $\forall y \in X_\beta \, \forall \gamma \in \beta \; X_\gamma$ is unbounded in y.

The function F with domain α satisfying $F(\beta) = X_\beta$ will be called a *witnessing function for the α-set-inaccessibility of Z.*

Corollary 4.7 (**CZF**) *If Z is α-set-inaccessible and $\beta \in \alpha$, then Z is β-set-inaccessible.*

Lemma 4.8 (**CZF**) *If Z is set-inaccessible, then Z is α-set-inaccessible iff for all $\beta \in \alpha$ the β-set-inaccessibles are unbounded in Z.*

Proof: One direction is obvious. So suppose that for all $\beta \in \alpha$ the β-set-inaccessibles are unbounded in Z; thus

$$\forall \beta \in \alpha \forall x \in Z \exists u \in Z \big(x \in u \wedge u \text{ is } \beta\text{-set-inaccessible} \big).$$

Using Strong Collection, there is a set S such that S consists of triples $\langle \beta, u, x \rangle$, where $\beta \in \alpha$, $x \in u \in Z$ and u is β-set-inaccessible, and for each $\beta \in \alpha$ and $x \in Z$ there is a triple $\langle \beta, u, x \rangle \in S$. Put

$$S_\beta = \{ u : \exists x \in Z \, \langle \beta, u, x \rangle \in S \}.$$

Again by Strong Collection there exists a set $\mathcal{F}$ of functions such that for $\beta \in \alpha$ and any $u \in S_\beta$ there is a function $f \in \mathcal{F}$ witnessing the β-set-inaccessibility of u, and, conversely, any $f \in \mathcal{F}$ is a witnessing function for some $u \in S_\beta$ for some $\beta \in \alpha$. Now define a function F with domain α via

$$F(\beta) = S_\beta \cup \bigcup \{ f(\beta) : \; f \in \mathcal{F} \; \wedge \; \beta \in \mathbf{dom}(f) \}.$$

As S_β is unbounded in Z, so is $F(\beta)$. Let $y \in F(\beta)$ and suppose $\gamma \in \beta$. If $y \in S_\beta$, then there is an $f \in \mathcal{F}$ witnessing the β-set-inaccessibility of y, thus $f(\gamma)$ is unbounded in y and a fortiori $F(\gamma)$ is unbounded in y.

Now assume that $y \in f(\beta)$ for some $f \in \mathcal{F}$. As $f \restriction \beta$ witnesses the β-set-inaccessibility of y, $f(\gamma)$ is unbounded in y, thus $F(\gamma)$ is unbounded in y.

As a result, F is a witnessing function for the α-set-inaccessibility of Z. □

The preceding lemma shows that the notion of being α-set-inaccessible is closely related to Mahlo's π_α-numbers. To state this precisely, we recall the notion of κ being *α-strongly inaccessible* (for ordinals α and cardinals κ) which is defined as α-weak inaccessibility except that κ is also required to be a strong limit, i.e. $\forall \rho < \kappa \, (2^\rho < \kappa)$.

Corollary 4.9 (**ZFC**) *Let $Z = V_\kappa$ be set-inaccessible. Then κ is α-strongly inaccessible iff V_κ is α-set-inaccessible.*

Theorem 4.10 (**CZF**) *Let M be Mahlo. Then for every $\alpha \in M$, the set of α-set-inaccessibles is unbounded in M.*

Proof: We will prove this by induction on α. Suppose this is true for all $\beta \in \alpha$. By the regularity of M we get

$$\forall x \in M\, \exists y \in M\, \big[x \in y \ \wedge\ \forall \beta \in \alpha\, \exists z \in y\ z \text{ is } \beta\text{-set-inaccessible}\big]. \tag{2}$$

Using Lemma 4.4 and Lemma 4.3, we conclude that for every $a \in M$ there exists a set-inaccessible $I \in M$ such that $a \in I$ and

$$\forall x \in I\, \exists y \in I\, \big(x \in y \ \wedge\ \forall \beta \in \alpha\, \exists z \in y\ z \text{ is } \beta\text{-set-inaccessible}\big).$$

Hence the β-set-inaccessibles are unbounded in I and, by Lemma 4.8, I is α-set-inaccessible. As a result, the α-set-inaccessibles are unbounded in M. □

Corollary 4.11 (**CZF**) *Let M be Mahlo. If $\alpha \in M$, then M is α-set-inaccessible.*

Proof: Follows from Theorem 4.10 and Lemma 4.8. □

5 Realizing set theory in Mahlo type theory

This section assumes familiarity with [17]. To commence we give a brief description of Setzer's Mahlo universe (cf. [17]) which will be denoted by $(\mathbf{M}, \mathbf{T})$.[1]

Definition 5.1 $(\mathbf{M}, \mathbf{T})$ is a universe closed under the usual type constructors $\mathbb{N}, \mathbb{N}_0, \mathbb{N}_1, \Pi, \Sigma, +, I, W$, and for every function

$$f^* : \big(\Sigma x : \mathbf{M}.\mathbf{T}(x) \to \mathbf{M}\big) \ \to\ \big(\Sigma x : \mathbf{M}.\mathbf{T}(x) \to \mathbf{M}\big)$$

there exists an element $\widehat{\mathbf{U}}_{f^*} : \mathbf{M}$ together with a decoding function

$$\mathbf{s}_{f^*} : \mathbf{T}(\widehat{\mathbf{U}}_{f^*}) \ \to\ \mathbf{M}$$

such that - letting $\mathbf{U}_{f^*} := \mathbf{T}(\widehat{\mathbf{U}}_{f^*})$ and $\mathbf{T}_{f^*}(x) := \mathbf{T}(\mathbf{s}_{f^*}(x))$ -

$$(\mathbf{U}_{f^*}, \mathbf{T}_{f^*})$$

is a universe satisfying the following properties:

(A) $(\mathbf{U}_{f^*}, \mathbf{T}_{f^*})$ is closed under the type constructors $\mathbb{N}, \mathbb{N}_0, \mathbb{N}_1, \Pi, \Sigma, +, I, W$.

(B) f^* possesses a restriction $\mathbf{Res}_{f^*}$ to $\mathbf{U}_{f^*}$ such that

$$\mathbf{Res}_{f^*}\ :\ \big(\Sigma x : \mathbf{U}_{f^*}.\, \mathbf{s}_{f^*}(x) \to \mathbf{U}_{f^*}\big) \ \to\ \big(\Sigma x : \mathbf{U}_{f^*}.\, \mathbf{s}_{f^*}(x) \to \mathbf{U}_{f^*}\big)$$

and

$$\iota_{f^*} \circ \mathbf{Res}_{f^*} \ =\ f^* \circ \iota_{f^*},$$

where

$$\iota_{f^*}\ :\ \big(\Sigma x : \mathbf{U}_{f^*}.\, \mathbf{s}_{f^*}(x) \to \mathbf{U}_{f^*}\big) \ \to\ \big(\Sigma x : \mathbf{M}.\, \mathbf{T}(x) \to \mathbf{M}\big)$$

is defined by[2]

$$\iota_{f^*}(\langle r, t\rangle) \ :=\ \langle \mathbf{s}_{f^*}(r), \lambda x.\mathbf{s}_{f^*}(tx)\rangle.$$

[1] It is denoted (V,T) in [17].

[2] We denote the pairing function of the Σ-type by $\langle , \rangle$.

Furthermore, let $(\mathbb{U}, \mathbb{T})$ be a universe above $(\mathbf{M}, \mathbf{T})$, i.e. there is an element $\widehat{\mathbf{M}} : \mathbb{U}$ such that $\mathbb{T}(\widehat{\mathbf{M}}) = \mathbf{M}$ (so $\widehat{\mathbf{M}}$ is a code for $\mathbf{M}$ in $\mathbb{U}$) and there is an embedding function $\jmath : \mathbf{M} \to \mathbb{U}$ satisfying $\mathbb{T}(\jmath(x)) = \mathbf{T}(x)$ for $x : \mathbf{M}$. Set

$$\mathbb{V} := W(\mathbb{U}, \mathbb{T}).$$

By Aczel's proof (cf.[1]), $\langle \mathbb{V}, \doteq, \dot{\in} \rangle$ is a realizability model for **CZF**; if one assumes closure of $(\mathbf{M}, \mathbf{T})$ under taking W-types, it also realizes **REA** as was shown by Aczel in [3]. To be precise, the realizability interpretation in $\langle \mathbb{V}, \doteq, \dot{\in} \rangle$ proceeds as follows. Each theorem ϕ of $\mathbf{CZF} + \mathbf{REA}$ is translated into a proposition ϕ^* of **MLM** such that $\mathbf{MLM} \vdash t \in \phi^*$ for a suitable term t. The latter will be shortened into

$$\langle \mathbb{V}, \doteq, \dot{\in} \rangle \models \phi.$$

Here we want to show that $\langle \mathbb{V}, \doteq, \dot{\in} \rangle$ realizes the axiom

$$\exists x \;\; x \texttt{ is Mahlo}$$

as well. Set

$$\mathbf{V}_M := W(\mathbf{M}, \mathbf{T}).$$

Lemma 5.2 *There are one-place functions assigning $\bar{\alpha} : \mathbf{M}$ and $\tilde{\alpha} : \mathbf{T}(\bar{\alpha}) \to \mathbf{V}_M$ to $\alpha : \mathbf{V}_M$ such that if $\alpha = \sup(a, b)$ where $a \in \mathbf{M}$ and $b : \mathbf{T}(a) \to \mathbf{V}_M$ then $\bar{\alpha} = a : \mathbf{M}$ and $\tilde{\alpha} = b : \mathbf{T}(a) \to \mathbf{V}_M$. Moreover, $\alpha = \sup(\bar{\alpha}, \tilde{\alpha}) : \mathbf{V}_M$ for $\alpha : \mathbf{V}$.*

Proof: [2], Theorem 2.1. □

By recursion on $\mathbf{V}_M$, define $h : \mathbf{V}_M \to \mathbb{V}$ by

$$h(\alpha) \;\; = \;\; \sup_{\mathbb{V}}(\jmath(\bar{\alpha}), h \circ \tilde{\alpha}),$$

where $\alpha = \sup_{\mathbf{V}_M}(\bar{\alpha}, \tilde{\alpha})$. Finally define $\beta : \mathbb{V}$ by $\bar{\beta} := \widehat{W}(\widehat{\mathbf{M}}, \jmath)$ and $\tilde{\beta} := h$. Note that

$$\mathbb{T}(\bar{\beta}) = \mathbf{V}_M.$$

Lemma 5.3 $\langle \mathbb{V}, \doteq, \dot{\in} \rangle \models \beta$ `is a regular model of` **CZF**.

Proof: The same as in Corollary 4.8 of [15]. □

Definition 5.4 Let $\mathbf{Fam}(\mathbf{M}) := \Sigma x : \mathbf{M}. \mathbf{T}(x) \to \mathbf{M}$.

Definition 5.5 To each function $f : \mathbf{V}_M \to \mathbf{V}_M$ we are going to associate a lifting

$$f^* : \mathbf{Fam}(\mathbf{M}) \to \mathbf{Fam}(\mathbf{M}).$$

Let $z = \langle a, b \rangle : \mathbf{Fam}(\mathbf{M})$. Then $\widehat{W}(a, b) : \mathbf{M}$. Put $A := \mathbf{T}(a)$ and $B := (x)\mathbf{T}(b(x))$. By recursion on $W(A, B)$ define a function

$$e_{A,B} : W(A, B) \to \mathbf{V}_M$$

by letting

$$e_{A,B}(\sup(u,g)) := \sup_{\mathbf{V}_M}(B(u),(x)e_{A,B}(g(x))),$$

and define

$$\ell : \mathbf{Fam}(\mathbf{M}) \to \mathbf{V}_M$$

by

$$\ell(z) := \sup_{\mathbf{V}_M}(W(A,B), e_{A,B}).$$

A mapping

$$\wp : \mathbf{V}_M \to \mathbf{Fam}(\mathbf{M})$$

is defined by letting

$$\wp(\alpha) := \langle \bar{\alpha}, (x)\overline{\tilde{\alpha}(x)} \rangle$$

for $\alpha = \sup_{\mathbf{V}_M}(\bar{\alpha}, \tilde{\alpha})$.

Finally put

$$f^*(z) := \wp(f(\ell(z))).$$

Lemma 5.6 *Let* $\alpha : \mathbf{V}_M$ *and* $\hat{\alpha} := \ell(\wp(\alpha))$. *If* $\langle \mathbf{V}_M, \dot{\in}, \doteq \rangle \models \mathbf{Tran}(\alpha)$, *then*

$$\langle \mathbf{V}_M, \dot{\in}, \doteq \rangle \models \alpha \subseteq \hat{\alpha} \wedge \mathbf{Reg}(\hat{\alpha}).$$

Proof: See [3], Proof of A2.1. □

Theorem 5.7 $\langle \mathbb{V}, \doteq, \dot{\in} \rangle \models \beta$ `is Mahlo`.

Proof: Suppose

$$\langle \mathbb{V}, \doteq, \dot{\in} \rangle \models \forall x \dot{\in} \beta \, \exists y \dot{\in} \beta \, \phi(x,y),$$

where ϕ is a set-theoretic formula with parameters from $\mathbb{V}$. As β is a regular model of **CZF** by Lemma 5.3, we obtain

$$\langle \mathbb{V}, \doteq, \dot{\in} \rangle \models \forall x \dot{\in} \beta \, \exists z \dot{\in} \beta \left[\mathbf{Tran}(z) \wedge \forall u \dot{\in} x \cup \{x\} \, \exists y \dot{\in} z \, \phi(u,y) \right]. \qquad (3)$$

By the axiom of choice, which is valid in type theory, there is a function

$$f : \mathbf{V}_M \to \mathbf{V}_M$$

such that for all $i : \mathbf{V}_M$,

$$\langle \mathbb{V}, \doteq, \dot{\in} \rangle \models \mathbf{Tran}(\tilde{\beta}(f(i))) \wedge \left(\forall u \dot{\in} \tilde{\beta}(i) \cup \{\tilde{\beta}(i)\}\right) \exists y \dot{\in} \tilde{\beta}(f(i)) \, \phi(u,y). \qquad (4)$$

Let $f^* : \mathbf{Fam}(\mathbf{M}) \to \mathbf{Fam}(\mathbf{M})$ be the lifting of f as defined in Definition 5.5. Then $(\mathbf{U}_{f^*}, \mathbf{T}_{f^*})$ is the subuniverse of $\mathbf{M}$ determined by f^*, and $\mathbf{s}_{f^*} : \mathbf{U}_{f^*} \to \mathbf{M}$ is the function which injects $\mathbf{U}_{f^*}$ into $\mathbf{M}$; in particular, $\mathbf{T}_{f^*}(v) = \mathbf{T}(\mathbf{s}_{f^*}(v))$ for $v : \mathbf{U}_{f^*}$. Set

$$\mu := \ell(\langle \widehat{\mathbf{U}}^{f^*}, \mathbf{s}_{f^*} \rangle).$$

Then $\mu : \mathbf{V}_M$ and

$$\langle \mathbf{V}_M, \dot{\in}, \doteq \rangle \models \mathbf{INAC}(\mu). \qquad (5)$$

Consequently,

$$\langle \mathbb{V}, \dot\in, \doteq \rangle \models \tilde\beta(\mu) \,\dot\in\, \beta \;\wedge\; \mathbf{INAC}(\tilde\beta(\mu)). \tag{6}$$

Suppose $\langle \mathbb{V}, \dot\in, \doteq \rangle \models \rho \,\dot\in\, \tilde\beta(\mu)$. Then $\langle \mathbb{V}, \dot\in, \doteq \rangle \models \rho \doteq \tilde\beta(\delta)$ for some $\delta : \mathbf{V}_M$. Then $\langle \mathbf{V}_M, \dot\in, \doteq \rangle \models \delta \,\dot\in\, \mu$. Pick $\delta_0 : \mathbf{V}_M$ such that

$$\langle \mathbf{V}_M, \dot\in, \doteq \rangle \models \delta \,\dot\in\, \delta_0 \;\wedge\; \delta_0 \,\dot\in\, \mu \;\wedge\; \mathbf{Tran}(\delta_0).$$

We may assume $\delta_0 = \tilde\mu(i)$ for some $i : \bar\mu$ as well. Set $\delta_1 := \ell(\wp(\delta_0))$. Then

$$\langle \mathbf{V}_M, \dot\in, \doteq \rangle \models \delta_1 \,\dot\in\, \mu \;\wedge\; \delta_0 \subseteq \delta_1 \;\wedge\; \mathbf{Tran}(\delta_1).$$

Let $\delta_2 := \ell(f^*(\wp(\delta_0)))$. As $\ell(f^*(\wp(\delta_0))) = \ell(\wp(f(\delta_1)))$, from Lemma 5.6 we obtain

$$\langle \mathbf{V}_M, \dot\in, \doteq \rangle \models f(\delta_1) \subseteq \delta_2. \tag{7}$$

Furthermore, since $\delta_0 = \tilde\mu(i)$ and $\mathbf{U}_{f^*}$ is closed under $\mathbf{Res}_{f^*}$ (in the sense of Definition 5.1,(B)) we also get

$$\langle \mathbf{V}_M, \dot\in, \doteq \rangle \models \delta_2 \,\dot\in\, \mu,$$

whence

$$\langle \mathbb{V}, \dot\in, \doteq \rangle \models \tilde\beta(\delta_2) \,\dot\in\, \tilde\beta(\mu). \tag{8}$$

Using (4), we obtain

$$\langle \mathbb{V}, \dot\in, \doteq \rangle \models \big(\forall u \,\dot\in\, \tilde\beta(\delta) \cup \{\tilde\beta(\delta)\}\big)\; \exists y \,\dot\in\, \tilde\beta(f(\delta_1))\; \phi(u,y),$$

and hence, using (7),

$$\langle \mathbb{V}, \dot\in, \doteq \rangle \models \big(\forall u \,\dot\in\, \tilde\beta(\delta) \cup \{\tilde\beta(\delta)\}\big)\; \exists y \,\dot\in\, \tilde\beta(\delta_2)\; \phi(u,y). \tag{9}$$

(8) and (9) imply $\langle \mathbb{V}, \dot\in, \doteq \rangle \models \exists y \,\dot\in\, \tilde\beta(\delta_2)\; \phi(\tilde\beta(\delta), y)$, and therefore

$$\langle \mathbb{V}, \dot\in, \doteq \rangle \models \exists y \,\dot\in\, \tilde\beta(\delta_2)\; \phi(\rho, y).$$

The upshot of the above is that

$$\langle \mathbb{V}, \dot\in, \doteq \rangle \models \mathbf{INAC}(\tilde\beta(\mu)) \;\wedge\; \forall x \,\dot\in\, \tilde\beta(\mu) \exists y \,\dot\in\, \tilde\beta(\mu)\; \phi(x,y),$$

verifying that

$$\langle \mathbb{V}, \dot\in, \doteq \rangle \models \exists u \,\dot\in\, \beta \,\big[\mathbf{INAC}(u) \;\wedge\; \forall x \,\dot\in\, u\, \exists y \,\dot\in\, u\; \phi(x,y)\big].$$

As a result, β `is Mahlo`. □

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