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The Buckling of an Elastic Layer Bonded to an Elastic Substrate in Plane Strain

The solution for buckling of a stiff elastic layer bonded to an elastic half-space under a transverse compressive plane strain is presented. The results are compared to an approximate solution that models the layer using beam theory. This comparison shows that the beam theory model is adequate until the buckling strain exceeds three percent, which occurs for modulus ratios less than 100. In these cases the beam theory predicts a larger buckling strain than the exact solution. In all cases the wavelength of the buckled shape is accurately predicted by the beam model. A buckling experiment is described and a discussion of buckling-induced delamination is given.

Introduction

Buckling of layers bonded to elastic substrates has recently gained importance in the semiconductor industry. Increasing the level of circuit integration leads to large numbers of dissimilar layers bonded together. These layered structures are assembled at elevated temperatures and then cooled. The different coefficients of thermal expansion can cause a layer to buckle as shown in Fig. 1. Buckling of a layer can lead to delamination. This will be discussed later.

Historically, interest in this problem was motivated by the buckling of the sandwich panels in aircraft. Previous works have all used beam theory (plate theory with a single axis of bending) models for the layer. Here we present a two-dimensional solution for the stresses in both the layer and the substrate. The results obtained are compared to those obtained using beam theory for the layer. Original work on this type of problem was done by Gough, Elam, and De Bruyne (1940). They considered eight geometrical cases experimentally and theoretically. Their model ignores the compression in the substrate and assumes that the tangential interface displacements are zero. This approximation agrees with the plate theory model for stiff layers (compared to the substrate) and small loads. Their approximations do not allow them to recover the Euler buckling load for a sandwich beam. Their experiments considered metals sandwiched between Onazote (expanded ebonite) foam. By sandwiching the stiff layer between the foam they avoided the possibility of delamination that is discussed below. However, the foam was permanently deformed by the metal during buckling and because no bond was made, the foam only provided compressive stresses across the interface.

Fig. 1 Copper film of 10 μm thickness in polymide substrate, buckling probably occurred during manufacture before top substrate was added. Here $\lambda/h \approx 4$ and $E_l/E \approx 35$. (Specimen courtesy of IBM).

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The problem of buckling of sandwich panels was approached using energy methods by Hoff and Maunter (1945). They took Poisson's ratio of the substrate to be zero and their calculations provide an upper bound on the buckling load. However, their calculations predict the energy of a skew mode of deformation that is greater than that in a symmetric mode of deformation, which is not correct. Furthermore, they do not recover the Euler buckling load as a limiting case. Their experiments also used a cellular substrate with a relative layer stiffness of about 2000. Chung and Testa (1969) considered fibers in a composite using beam theory for the fibers and ignored the contribution of the energy in the core to correctly predict the lower energy of the symmetric mode of deformation compared to the skew mode.

Goodier (1946) considered cylindrical buckling of sandwich plates using a form of Biot's equations (1965) for the initially stressed core and beam theory for the layer. This method leads to a reasonable approximate answer. However, his assumption that the tangential displacement is zero on the interface is too restrictive to recover the Euler buckling mode. Biot's equations are not derivable from a strain energy, but the terms omitted are of the same order as those ignored elsewhere in the analysis.

Stability of a Layered on a Half-Space: Two-Dimensional Solution

In order to treat the stability of a body under compression, we first consider an elastic body which in the reference state is in a state of initial stress S_{ij} referred to rectangular Cartesian axes x_i. For equilibrium of the reference state,

$$S_{ij,j}=0 \text{ and } S_{ij}=S_{ji}$$

throughout the body. The Lagrangian stress tensor T_{ij} in the deformed state is such that the stress vector T_i, measured per unit area of the reference state, is given by

$$T_i=T_{ji}n_j,$$

where n_i is the unit normal to the surface element in the reference state. For an elastic material we have

$$T_{ji}=\partial W/\partial u_{i,j},$$

where the strain energy W per unit volume of the reference state is a function of the Cauchy strains e_{ij} given by

$$e_{ij}=\frac{1}{2}\,(u_{i,j}+u_{j,i}+u_{r,i}\,u_{r,j})$$

in terms of the displacements u_i. Since T_{ij} is equal to S_{ij} in the reference state, we take

$$W=S_{ij}\,e_{ij}+W_0(\mathbf{e}),$$

and for linear behavior W_0 is taken to be given by

$$W_0=\frac{1}{2}\,c_{ijkl}\,e_{ij}\,e_{kl},$$

with the coefficients c_{ijkl} satisfying the usual symmetry relations. The stresses T_{ij} then become

$$T_{ji}=S_{ij}+S_{jr}\,u_{i,r}+c_{ijkl}\,u_{k,l} \tag{1}$$

if we ignore second-order terms in $u_{i,j}$.

We now consider an elastic half-space of homogeneous isotropic material with a stiffer layer of isotropic material of uniform thickness h bonded to its surface (Fig. 2(*a*)). We assume conditions of plane strain and consider stability under compression parallel to the layer. The compression is assumed to induce the same uniform compressive strain ϵ in the layer and in the half-space and we take the compressed state to be the reference state. We use x and y for x_1 and x_2 with the x-axis along the interface as in Fig. 2(*a*). The stresses S_{yy} and S_{xy} are zero while S_{xx} has the values $-P_L$ and $-P$ in the layer and in the half-space, respectively, given by

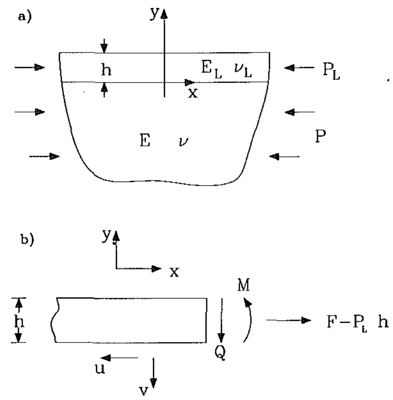

Fig. 2 (*a*) **The geometry of the layer buckling problem; (*b*) the sign conventions and notation used for the plate model for the layer**

$$P=\epsilon\,E/(1-\nu^2),\ P_L=\epsilon\,E_L/(1-\nu_L^2),$$

where E and ν denote Young's modulus and Poisson's ratio for the materials. The stiffness ratio K is defined as

$$K=\frac{E_L(1-\nu^2)}{E(1-\nu_L^2)}=\frac{\mu_L(1-\nu)}{\mu(1-\nu_L)}=\frac{P_L}{P},$$

where μ is the shear modulus. We look for the lowest value of ϵ for which an adjacent position of equilibrium is possible with the upper surface of the layer traction-free and with no body force.

For infinitesimal deformations from the compressed state we use the stress-strain relation (1) together with the approximation that W_0 has the isotropic form of the strain energy for the materials in the unstressed state. This assumes that the effect of the compression on the elastic moduli can be neglected and it is called the "engineering approximation" by Pearson (1956). For equilibrium of a perturbed state $T_{ji,j}=0$ and for the half-space this leads to

$$\mu\nabla^2u+(\lambda+\mu)\,(u_x+v_y)_x-Pu_{xx}=0$$
$$\mu\nabla^2v+(\lambda+\mu)\,(u_x+v_y)_y-Pv_{xx}=0, \tag{2}$$

where u_x, etc., and derivatives of the displacements u, v and λ is Lamé's constant. If we look for solutions of (2) for which u and v are proportional to sin x and cos x, respectively, we obtain the solution

$$u=(-Aae^{ay}+B\omega e^{by})\sin\omega x,$$
$$v=(A\omega e^{ay}-Bbe^{by})\cos\omega x, \tag{3}$$

where A and B are arbitrary constants and a and b are given by

$$a=\omega(1-P/\mu)^{1/2},\ b=\omega[1-P(1-2\nu)/\mu(1-\nu)]^{1/2}. \tag{4}$$

The solution (3) is similar in form to the solution for a Rayleigh wave on the surface of a half-space with P replacing ρc_R^2, where ρ is the density and c_R is the Rayleigh wave speed. The displacements go to zero as y goes to negative infinity. The tractions on the interface $y=0$ are T_{yx} and T_{yy} and if we adjust the constants in the solution (3) so that

$$u=hU\sin\omega x,\ v=hV\cos\omega x,\ \text{on } y=0, \tag{5}$$

then we find that in terms of the amplitudes U, V we have on $y=0$

$$T_{xy}=\mu(\gamma U+\delta V)\sin\omega x,\ T_{yy}=\mu(\alpha U+\beta V)\cos\omega x, \tag{6}$$

where

$$\alpha=\delta=-\Omega(\Omega^2+\xi^2-2\xi\eta)/\Delta,\ \beta=\xi(\Omega^2-\xi^2)/\Delta,$$

$$\gamma=b\beta/a,\ \Delta=\Omega^2-\xi\eta,\ \Omega=\omega h,\ \xi=ah,\ \eta=bh. \tag{7}$$

Equality of the coefficients α and δ is related to the fact that Eqs. (2) are self-adjoint and a Reciprocal Theorem holds.

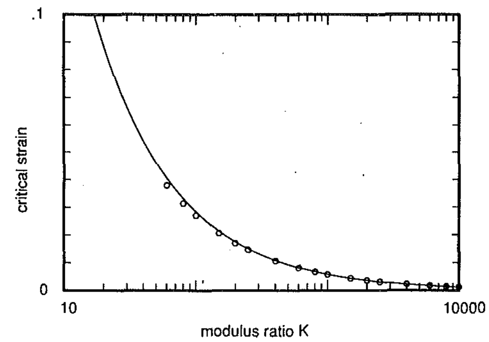

Fig. 3 The buckling strain as a function of the layer modulus ratio. Comparison of the full elasticity results Eq. (9) (circles) and the approximate plate theory results of (16) (solid curves) for $\nu = 0.25$

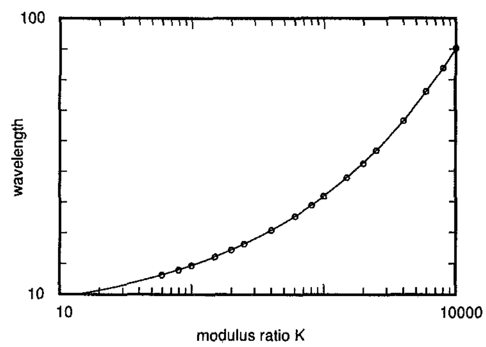

Fig. 4 The wavelength of the buckled shape as a function of the layer modulus ratio. Comparison of the full elasticity results Eq. (9) (circles) and the approximate plate theory results of (17) (solid curves) for $\nu = 0.25$.

Equations similar to (2) apply in the layer. We can change the signs of a and b in (3) and still have a solution of Eqs. (2). Combining the solutions for the layer we can choose the four constants in the solutions so that the upper surface of the layer is free from traction and so that the displacements on the lower surface have the values (5). We then find the stresses T^L_{yx} and T^L_{yy} in the layer at the interface $y=0$ are given by (6) with μ replaced by μ_L and α, β, γ, δ replaced by α_L, β_L, γ_L, δ_L where

$$\begin{aligned}
\alpha_L = \delta_L = \Omega\{ &[(\xi_L^2+\Omega^2)^2 - 4\xi_L\eta_L\Omega^2] \\
&\times (\eta_L^2+\Omega^2-2\xi_L\eta_L)\sinh \xi_L \sinh \eta_L \\
&- 2\xi_L\eta_L(\xi_L^2+\Omega^2)(\xi_L^2+3\Omega^2)[\cosh(\xi_L-\eta_L)-1]\}/\Delta_L \\
\beta_L = \xi_L(\xi_L^2-\Omega^2)&[4\xi_L\eta_L\Omega^2 \sinh \xi_L \cosh \eta_L \\
&- (\xi_L^2+\Omega^2)^2 \sinh \eta_L \cosh \xi_L]/\Delta_L \\
\gamma_L = \eta_L(\xi_L^2-\Omega^2)&[4\xi_L\eta_L\Omega^2 \sinh \eta_L \cosh \xi_L \\
&- (\xi_L^2+\Omega^2)^2 \sinh \xi_L \cosh \eta_L)]/\Delta_L \\
\Delta_L = \xi_L\eta_L(\xi_L^2-\Omega^2)& \cosh \xi_L \cosh \eta_L \\
&- \Omega^2(\xi_L^2+\Omega^2-2\xi_L\eta_L) \sinh \xi_L \sinh \eta_L \\
&+ 4\xi_L\eta_L\Omega^2(\xi_L^2+\Omega^2)[\cosh(\xi_L-\eta_L)-1]. \quad (8)
\end{aligned}$$

In these equations ξ_L and η_L have the values $a_L h$, $b_L h$ where a_L and b_L are given by expressions of the same form as (4) but evaluated using P_L and the material constants for the layer.

In order that the solutions for the layer and the half-space represents an adjacent position of equilibrium, we must impose continuity of the tractions T_{yx}, T_{yy} across the interface $y=0$. This leads to two homogeneous linear equations for U, V and in order for a solution for U, V to exist the determinant of the coefficients must be zero. This gives

$$(\mu_L\alpha_L - \mu\alpha)^2 - (\mu_L\beta_L - \mu\beta)(\mu_L\gamma_L - \mu\gamma) = 0. \quad (9)$$

For given values of the material constants this equation is a relation between $\Omega = \omega h$ and the compressive strain ϵ for which adjacent equilibrium solutions exist. The critical value ϵ_c at which the layer will buckle is found from (9) by determining the value of Ω for which ϵ is a minimum.

The circles in Fig. 3 show the buckling strain ϵ_c versus the stiffness ratio K for the case when $\nu = \nu_L = 0.25$. The nondimensional buckling wavelength $\lambda_c/h = 2\pi/\Omega$ is shown by the circles in Fig. 4.

Beam Approximation for the Layer

When the buckling wavelength is large compared to the thickness of the layer, the behavior of the layer can be described approximately by using beam theory (plate theory with a single axis of bending). We assume that after deflection of the compressed layer, an axial force $F - P_L h$, a moment M and a shear force Q act on a section of the layer, as indicated in Fig. 2(*b*), and the lower surface of the layer has tractions T_{yx}, T_{yy}. The displacements at the lower surface are denoted by u, v. With the assumption that plane sections remain plane, the vertical displacement at the central line will be v and the horizontal displacement will be $u - hv_x/2$. The moment and additional axial force are then given by

$$M = \frac{E_L h^3}{12(1-\nu_L^2)}\, v_{xx},\ F = \frac{E_L h}{(1-\nu_L^2)} \left(u_x - \frac{h}{2}\, v_{xx}\right). \quad (10)$$

For equilibrium of the layer

$$\frac{dM}{dx} + P_L h v_x - Q - \frac{h}{2}\, T_{yx} = 0$$

$$\frac{dQ}{dx} + T_{yy} = 0,\ \frac{dF}{dx} - T_{yx} = 0, \quad (11)$$

where the effect of the geometry change on the action of the force $P_L h$ has been taken into account. If we differentiate the first of Eqs. (11) to eliminate Q and use the values (10) for M and F we find that

$$\frac{E_L h^2}{(1-\nu_L^2)} \left(\frac{h}{3}\, v_{xxxx} - \frac{1}{2}\, u_{xxx}\right) + P_L\, h v_{xx} + T_{yy} = 0$$

$$\frac{E_L h}{(1-\nu_L^2)} \left(u_{xx} - \frac{h}{2}\, v_{xxx}\right) - T_{yx} = 0. \quad (12)$$

If we assume that u and v are given by (5) and use expressions (6) for T_{yx} and T_{yy} in Eqs. (12), we get two homogeneous linear equations for U, V, and in order for a solution to exist we must have

$$\left[\alpha + \frac{K\Omega^3}{(1-\nu)}\right]^2 - \left[\beta + \frac{2K\Omega^2}{(1-\nu)}\,(\Omega^2/3 - \epsilon)\right]\left[\gamma + \frac{2K\Omega^2}{(1-\nu)}\right] = 0. \quad (13)$$

The limiting values of the coefficients $\alpha, \ldots$ as P/μ goes to zero are

$$\alpha = \delta = -2\Omega\,\frac{(1-2\nu)}{(3-4\nu)},\ \beta = \gamma = 4\Omega\,\frac{(1-\nu)}{(3-4\nu)} \quad (14)$$

and they are the response coefficients for an isotropic half-space with surface displacements (5). If we use these values as approximations for $\alpha, \ldots$ in (13) we find that the strain ϵ for which an adjacent equilibrium solution exists is given approximatly by

$$\epsilon = \left\{\frac{(3-4\nu)K}{24(1-\nu)^2}\,\Omega^2 + \frac{1}{\Omega} + \frac{(1-2\nu)}{2(1-\nu)} + \frac{\Omega}{3} + \frac{1}{2K\Omega^2}\right\} \times \left\{\frac{(3-4\nu)K}{2(1-\nu)^2} + \frac{1}{\Omega}\right\}^{-1}. \quad (15)$$

The minimum value of ϵ is the buckling strain. For K greater

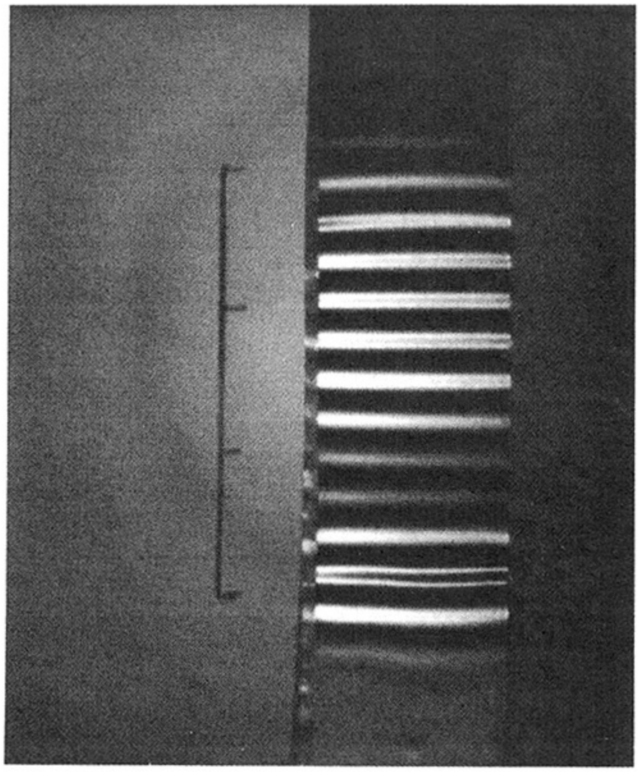

Fig. 5 A top view of the buckled configuration of a mylar film on a rubber substrate, $K=2140$ and $h=44\ \mu$m, centimeter scale shown

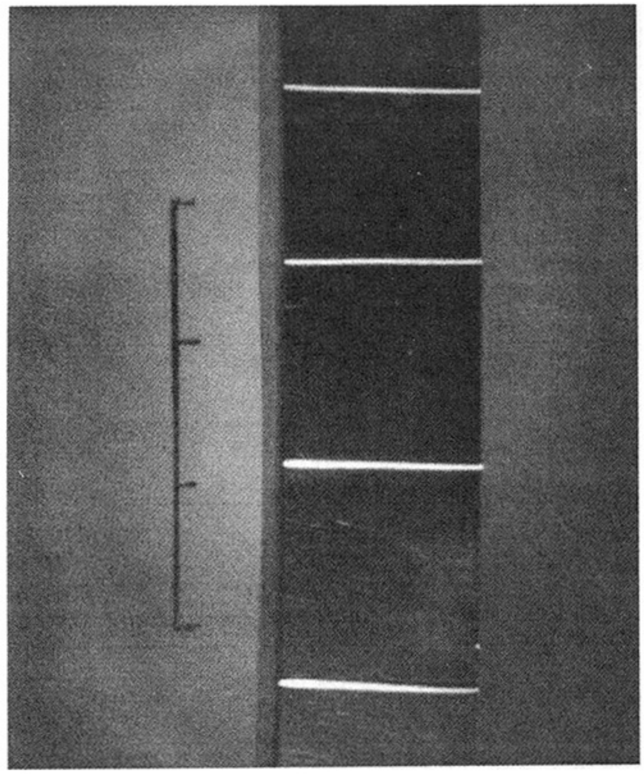

Fig. 6 Post-buckling delamination pattern of a mylar film bonded to a rubber substrate, $K=465$, centimeter scale shown

than $(3-4\nu)/(1-\nu)(1-2\nu)$ it can be shown that the equation $d\epsilon/d\Omega=0$ has at most one positive root. For large values of the stiffness ratio K, ϵ is a minimum when

$$\Omega_c^3=\frac{12(1-\nu)^2}{K(3-4\nu)} \tag{16}$$

and with this value of Ω, (15) gives

$$\epsilon_c=\frac{\Omega_c}{12}\left\{3+\frac{(1-2\nu)}{(1-\nu)}\,\Omega_c+\frac{2}{3}\,\Omega_c^2+\frac{(3-4\nu)}{12(1-\nu)^2}\,\Omega_c^2\right\}\left\{1+\frac{\Omega_c^2}{6}\right\}^{-1} \tag{17}$$

as an approximation to the buckling strain. For $\nu=\nu_L=0.25$ and $K\geq 100$, the value of Ω_c given by (16) differs by less than one percent from the value which minimizes ϵ determined from Eq. (9), and the approximation for the critical strain given by (16) and (17) overestimates ϵ_c determined from (9) by less than five percent. The theory assumes small strains so it is less reliable for smaller values of K when the values predicted for the critical strain are rather large (over 2.5 percent for $K=100$). For $K>1000$, ϵ_c from (16) and (17) is within one percent of ϵ_c determined from (9) so that the approximation (16), (17) is accurate enough. These results are shown by the solid curves in Figs. 3 and 4 $\nu=0.25$. The Euler buckling value of ϵ_c for a strip in plane strain with the same buckling wavelength is $\Omega^2/12$; for large K, Ω is small and (17) is close to three times the Euler value.

Similar approximations to those of this section are made by Gough, Elam, and de Bruyne (1940) but their analysis assumes that there is no horizontal displacement at the interface and the effect of the interfacial shear stress on the equilibrium of the layer is ignored. The plane stress value for the buckling strain given by Gough, Elam, and de Bruyne when converted to plane strain is $\Omega^2/4$ with Ω given by (16), in agreement with the approximation (17) for small values of Ω.

Discussion

Figure 5 shows the buckled configuration of a mylar film on a rubber substrate. The modulus ratio, K, is 2140 for these materials and the layer thickness is 44 μm. Poisson's ratio for the substrate is close to 0.5. The relative wavelength, λ/h, of the buckled shape is approximately 57, which compares well to the value of 54 predicted by (16). The buckling strain for this configuration was roughly the magnitude of that predicted by (17) but was not measured accurately enough to report here. The modulus of the substrate used in this case was very low. Attempts were made to use a stiffer rubber substrate with a modulus ratio $K=465$. The result of this experiment is shown in Fig. 6. Unlike Fig. 5, the configuration in Fig. 6 is in the unloaded state. The periodic bright lines correspond to narrow regions of delamination of the mylar from the rubber substrate and are kinks in the film. These kinks formed after buckling occurred in the film.

The process that formed the periodic delaminations in Fig. 6 is as follows. Due to local imperfections, buckling first occurs in a small region of the film. However, the interface cannot support the tensile normal stresses that occur during buckling. Thus the film delaminates in a small region, this immediately relieves the compressive strain in the system and the characteristics wavelength of the buckled film disappears. The kink in the film effectively cuts the film into two parts that must now be treated as film segments bonded to the substrate. Further loading will cause the axial strain to again increase in the film, however the strain in the film will not reach the value of the strain in the substrate near the delamination. There is a characteristic distance from the delamination for the strain in the film to build up to a large percentage of the strain in the substrate. This distance can be predicted by considering beam segments bonded to a half-space as in Shield and Kim (1992). At this distance from the first delamination, the film again buckles and caused another delamination kink. This process was repeated three times to arrive at the configuration shown in Fig. 6 with periodic delaminations separated by the characteristic distance defined above.

Buckling-induced delamination is characteristic of film/substrate combinations with low modulus ratios and hence large buckling strains. In is very difficult to construct such a system with an interface that can withstand the stresses in the buckled configuration. One solution this problem is to sandwich a layer between two substrates as in Gough, Elam, and de Bruyne (1940). The only change to the analysis for this case is that the quantity h defined above would be the half thickness of the layer. The delamination behavior discussed here shows the importance of considering the possibility of elastic buckling of thin films. Such buckling is highly probable in polymer-based electronic devices made up of conducting thin films that have high stiffness and polymeric substrates that may undergo a large shrinkage. The shrinkage strain may be caused by either polymerization or thermal contraction during cooling. Although the buckled configuration may never be observed, it still may be the original cause of delamination and failure of the film.

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