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Supersymmetric vertex models with domain wall boundary conditions

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By means of the Drinfeld twists, we derive the determinant representations of the partition functions for the $gl(1|1)$ and $gl(2|1)$ supersymmetric vertex models with domain wall boundary conditions. In the homogeneous limit, these determinants degenerate to simple functions. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2436986]

I. INTRODUCTION

The six-vertex model on a finite square lattice with the so-called domain wall (DW) boundary condition was first proposed by Korepin in Ref. 1, where recursion relations of the partition functions of the DW model were derived. In Refs. 2 and 3, it was found that the partition functions of the DW model can be represented as determinants. Taking the homogeneous limit of the spectral parameters, Sogo found that the partition function satisfies the Toda differential equations.[4] Then, by using the equations Korepin and Zinn-Justin obtained the bulk free energy of the system.[5] The determinant representations of DW partition functions are useful in solving some pure mathematical problems, such as the problem of alternating sign matrices.[6] By using the fusion method, Caradoc *et al.* obtained the determinant expressions of the partition functions for the spin-$k/2$ vertex models with the DW boundary conditions.[7] In Ref. 8, Bleher and Fokin obtained the large N asymptotics of the DW six-vertex model in the disordered phase.

Supersymmetric integrable models based on superalgebras are important for describing strongly correlated electronic systems of high T_c superconductivity.[9–15] In this paper, we investigate the $gl(1|1)$ and $gl(2|1)$ supersymmetric vertex models on an $N\times N$ square lattice with DW boundary conditions. By using the approach of Drinfeld twists,[16,18,19] we derive the determinant representations of the DW partition functions for the two systems. We find that the partition functions degenerate to simple functions in the homogeneous limit.

The framework of the paper is as follows. In Sec. II, we obtain the determinant representation of the partition function of the $gl(1|1)$ vertex model with DW boundary condition. Then, we derive the homogeneous limit of the partition function. In Sec. III, we solve the $gl(2|1)$ supersymmetric vertex model with the DW boundary conditions by deriving its DW partition functions exactly. In Sec. IV, we present some discussions.

II. $gl(1|1)$ VERTEX MODEL WITH DW BOUNDARY CONDITION

In this section, we study the DW boundary condition for the $gl(1|1)$ vertex model on an $N\times N$ square lattice.

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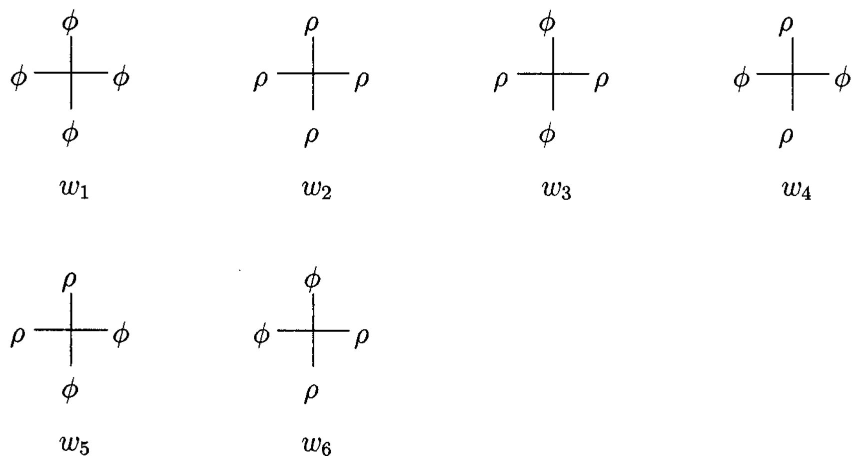

FIG. 1. Vertex configurations and their Boltzmann weights for the $gl(1|1)$ vertex model.

A. Description of the model

Let V be the two-dimensional irreducible $gl(1|1)$ module and $R \in End(V \otimes V)$ the R matrix associated with this representation. In this paper, we choose the FB (fermion-Boson) grading for V, i.e., $[1]=1,[2]=0$. The R matrix satisfies the graded Yang-Baxter equation (GYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{2.1}$$

where $R_{ij} \equiv R_{ij}(\lambda_i, \lambda_j)$ with spectral parameters specified by λ_i. Explicitly,

$$R_{12}(\lambda_1,\lambda_2) = \begin{pmatrix} c_{12} & 0 & 0 & 0 \\ 0 & a_{12} & b_{12} & 0 \\ 0 & b_{12} & a_{12} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.2}$$

where

$$a_{12} = a(\lambda_1,\lambda_2) \equiv \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2 + \eta}, \quad b_{12} = b(\lambda_1,\lambda_2) \equiv \frac{\eta}{\lambda_1 - \lambda_2 + \eta},$$

$$c_{12} = c(\lambda_1,\lambda_2) \equiv \frac{\lambda_1 - \lambda_2 - \eta}{\lambda_1 - \lambda_2 + \eta}, \tag{2.3}$$

and η is the crossing parameter which could be normalized to 1 in the rational cases we are considering.

In the following, we study the $gl(1|1)$ six-vertex model on a two-dimensional (2D) $N \times N$ square lattice. At any site of the lattice, there may be a vacuum ϕ or a single fermion ρ. A vertex configuration in the lattice is constructed by two nearest particle states in a horizontal line and two nearest particle states in a vertical line. For the present model, there are altogether six possible weights corresponding to a vertex configuration. The possible configurations and their corresponding Boltzmann weights w_i are given in Fig. 1. One may check that these weights preserve the fermion numbers. If we assign horizontal lines parameters $\{\lambda_j\}$ and vertical lines parameters $\{z_k\}$, and let

$$w_1 = 1, \quad w_2 = c(\lambda,z), \quad w_3 = w_4 = a(\lambda,z), \quad w_5 = w_6 = b(\lambda,z), \tag{2.4}$$

then the Boltzmann weights correspond to the elements of the $gl(1|1)$ R matrix (2.2).

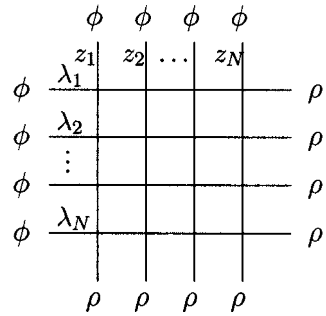

FIG. 2. DW boundary condition for the $gl(1|1)$ vertex model.

We now propose the $gl(1|1)$ vertex model with DW boundary condition. Similar to Refs. 1 and 3, if the ends of the square lattice satisfy the special boundary condition, that is, the states of all the left and top ends are vacuum ϕ while the states of all the right and bottom ends are single fermion ρ, we then call the boundary condition the DW boundary condition.

A configuration with the DW boundary condition is shown in Fig. 2. In this figure, λ_i and z_j are parameters associated with the horizontal and vertical lines, respectively. The partition function for this system is then given by

$$Z_N = \sum \prod_{i=1}^{6} w_i^{n_i}, \tag{2.5}$$

where the summation is over all possible configurations and n_i is the number of configurations with the Boltzmann weight w_i. By means of the R matrix, the partition function may be rewritten as

$$Z_N = \prod_{i=0}^{N-1} \rho_{(N-i)} \prod_{j=0}^{N-1} \phi_{(N-j)} \prod_{k=1}^{N} \prod_{l=1}^{N} R_{kl}(\lambda_k, z_l) \prod_{j=1}^{N} \rho_{(j)} \prod_{i=1}^{N} \phi_{(i)}. \tag{2.6}$$

Here $\rho_{(i)}$ ($\phi_{(i)}$) stands for the state ρ (ϕ) in the ith horizontal (vertical) line.

Following Korepin, we use the transfer matrix to arrange the partition function. Define the monodromy matrix $T_k(\lambda_k)$ along the horizontal lines,

$$T_k(\lambda_k) = R_{k,N}(\lambda_k, z_N) R_{k,N-1}(\lambda_k, z_{N-1}) \cdots R_{k,1}(\lambda_k, z_1) \equiv \begin{pmatrix} A(\lambda_k) & B(\lambda_k) \\ C(\lambda_k) & D(\lambda_k) \end{pmatrix}_{(k)}. \tag{2.7}$$

Therefore, in terms of $T_k(\lambda_k)$ the partition function is equal to

$$\begin{aligned} Z_N &= \prod_{i=0}^{N-1} \langle 1|_{(N-i)} \prod_{j=0}^{N-1} \langle 0|_{(N-j)} \prod_{k=1}^{N} T_k(\lambda_k) \prod_{j=1}^{N} |1\rangle_{(j)} \prod_{i=1}^{N} |0\rangle_{(i)} \\ &= \prod_{i=0}^{N-1} \langle 1|_{(N-i)} C(\lambda_1) C(\lambda_2) \cdots C(\lambda_N) \prod_{i=1}^{N} |0\rangle_{(i)}. \end{aligned} \tag{2.8}$$

Here we have introduced the notation

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.9}$$

to denote the particle states ϕ and ρ, respectively.

B. Exact solution of the partition function

In this section, we will compute the partition function (2.8) for the $gl(1|1)$ vertex model with DW boundary conditions by using the Drinfeld twist approach.

Let us remark that even though we believe that Drinfeld twists do exist for all algebras, so far only those related to A-type (super) algebras and to XYZ model have been explictly constructed.[17–21]

Let σ be any element of the permutation group $\mathcal{S}_N$. We then define the following lower-triangular matrix:[19,20]

$$F_{1\cdots N}=\sum_{\sigma\in\mathcal{S}_N}\sum^{*}_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(N)}}\prod_{j=1}^{N}P^{\alpha_{\sigma(j)}}_{(\sigma(j))}S(c,\sigma,\alpha_\sigma)R^{\sigma}_{1\cdots N}, \tag{2.10}$$

where $P^{\alpha}_{(k)}$ has the elements $(P^{\alpha}_{(k)})_{mn}=\delta_{\alpha m}\delta_{\alpha n}$ at the kth space with root indices $\alpha=1,2$, the sum Σ^{*} is taken over all nondecreasing sequences of the labels $\alpha_{\sigma(i)}$,

$$\begin{aligned}&\alpha_{\sigma(i+1)}\geqslant\alpha_{\sigma(i)}\quad\text{if }\sigma(i+1)>\sigma(i),\\&\alpha_{\sigma(i+1)}>\alpha_{\sigma(i)}\quad\text{if }\sigma(i+1)<\sigma(i),\end{aligned} \tag{2.11}$$

and the c-number function $S(c,\sigma,\alpha_\sigma)$ is given by

$$S(c,\sigma,\alpha_\sigma)\equiv\exp\left\{\frac{1}{2}\sum_{l>k=1}^{N}(1-(-1)^{[\alpha_{\sigma(k)}]})\delta_{\alpha_{\sigma(k)},\alpha_{\sigma(l)}}\ln(1+c_{\sigma(k)\sigma(l)})\right\}. \tag{2.12}$$

In Eq. (2.10), $R^{\sigma}_{12\cdots N}$ is the N-fold R matrix which can be decomposed in terms of elementary R matrices [Eq. (2.2)] by the decomposition law

$$R^{\sigma'\sigma}_{1\cdots N}=R^{\sigma}_{\sigma'(1\cdots N)}R^{\sigma'}_{1\cdots N}. \tag{2.13}$$

We showed in Ref. 18 that the F matrix is nondegenerate and satisfies the relation

$$F_{\sigma(1\cdots N)}(z_{\sigma(1)},\ldots,z_{\sigma(N)})R^{\sigma}_{1\cdots N}(z_1,\ldots,z_N)=F_{1\cdots N}(z_1,\ldots,z_N). \tag{2.14}$$

The nondegeneracy of the F matrix means that its column vectors form a complete basis, which is called the F basis. The nondegeneracy also ensures that the F matrix is invertible. The inverse is given by Ref. 19,

$$F^{-1}_{1\cdots N}=F^{*}_{1\cdots N}\prod_{i<j}\Delta^{-1}_{ij}, \tag{2.15}$$

with

$$\Delta_{ij}=\mathrm{diag}((1+c_{ij})(1+c_{ji}),a_{ji},a_{ij},1) \tag{2.16}$$

and

$$F^{*}_{1\cdots N}=\sum_{\sigma\in\mathcal{S}_N}\sum^{**}_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(N)}}S(c,\sigma,\alpha_\sigma)R^{\sigma^{-1}}_{\sigma(1\cdots N)}\prod_{j=1}^{N}P^{\alpha_{\sigma(j)}}_{(\sigma(j))}. \tag{2.17}$$

Here the sum Σ^{**} is taken over all possible α_i, which satisfies the following nonincreasing constraints:

$$\begin{aligned}&\alpha_{\sigma(i+1)}\leqslant\alpha_{\sigma(i)}\quad\text{if }\sigma(i+1)<\sigma(i),\\&\alpha_{\sigma(i+1)}<\alpha_{\sigma(i)}\quad\text{if }\sigma(i+1)>\sigma(i).\end{aligned} \tag{2.18}$$

Working in the F basis, the entries of the monodromy matrix (2.7) can be simplified to symmetric forms; e.g., the lower entry $C(\lambda)$ becomes[19]

$$\tilde{C}(\lambda) \equiv F_{12\cdots N} C(\lambda) F^{-1}_{12\cdots N} = \sum_{i=1}^{N} b(\lambda, z_i) E^{12}_{(i)} \otimes_{j \neq i} \mathrm{diag}\,(2a(\lambda, z_j), 1)_{(j)}, \tag{2.19}$$

where $E^{\alpha\beta}_{(l)}$ (α, $\beta = 1,2$; $l = 1,2,\ldots,N$) are generators of the superalgebra $gl(1|1)$ at the site l. From Eq. (2.19), one can see that in the F basis all compensating terms (polarization clouds) in the original expression of $C(\lambda)$ in terms of local generators disappear from $\tilde{C}(\lambda)$.

Applying the F matrix and its inverse to the states $|0\rangle_{(1)} \otimes \cdots \otimes |0\rangle_{(N)}$ and $\langle 1|_{(N)} \otimes \cdots \otimes \langle 1|_{(1)}$, we have

$$\begin{aligned} F_{1\cdots N}|0\rangle_{(1)} \otimes \cdots \otimes |0\rangle_{(N)} &= |0\rangle_{(1)} \otimes \cdots \otimes |0\rangle_{(N)}, \\ \langle 1|_{(N)} \otimes \cdots \otimes \langle 1|_{(1)} F^{-1}_{1\cdots N} &= \langle 1|_{(N)} \otimes \cdots \otimes \langle 1|_{(1)} \prod_{i<j} (2a(z_i, z_j))^{-1}. \end{aligned} \tag{2.20}$$

Substituting Eqs. (2.19) and (2.20) into the partition function (2.8), we obtain

$$\begin{aligned} Z_N &= \prod_{i=0}^{N-1} \langle 1|_{(N-i)} C(\lambda_1) C(\lambda_2) \cdots C(\lambda_N) \prod_{i=1}^{N} |0\rangle_{(i)} \\ &= \prod_{i=0}^{N-1} \langle 1|_{(N-i)} F^{-1}_{1\cdots N} F_{1\cdots N} C(\lambda_1) \cdots C(\lambda_N) F^{-1}_{1\cdots N} F_{1\cdots N} \prod_{i=1}^{N} |0\rangle_{(i)} \\ &= \prod_{i<j} (2a(z_i, z_j))^{-1} \prod_{i=0}^{N-1} \langle 1|_{(N-i)} \tilde{C}(\lambda_1) \cdots \tilde{C}(\lambda_N) \prod_{i=1}^{N} |0\rangle_{(i)} \\ &= \prod_{i<j} (2a(z_i, z_j))^{-1} \prod_{i=0}^{N-1} \langle 1|_{(N-i)} \times 2^{N(N-1)/2} \sum_{i_1 < \cdots < i_N} B_N(\lambda_1, \ldots, \lambda_N | z_{i_1}, \ldots, z_{i_N}) E^{12}_{(i_1)} \cdots E^{12}_{(i_n)} \prod_{i=1}^{N} |0\rangle_{(i)} \\ &= \prod_{i<j} a^{-1}(z_i, z_j) B_N(\lambda_1, \ldots, \lambda_N | z_1, \ldots, z_N), \end{aligned} \tag{2.21}$$

where

$$B_N(\lambda_1, \ldots, \lambda_N | z_1, \ldots, z_N) = \sum_{\sigma \in \mathcal{S}_N} \mathrm{sign}(\sigma) \prod_{k=1}^{N} b(\lambda_k, z_{\sigma(k)}) \prod_{l=k+1}^{N} a(\lambda_k, z_{\sigma(l)}) = \det \mathcal{B}(\{\underline{\lambda}\}, \{\underline{z}\}), \tag{2.22}$$

with $\mathcal{B}(\{\underline{\lambda}\}, \{\underline{z}\})$ being an $N \times N$ matrix with elements

$$(\mathcal{B}(\{\underline{\lambda}\}, \{\underline{z}\}))_{\alpha\beta} = b(\lambda_\alpha, z_\beta) \prod_{\gamma=1}^{\alpha-1} a(\lambda_\gamma, z_\beta). \tag{2.23}$$

Therefore, the partition function of the $gl(1|1)$ vertex model is given by the following determinant of the $N \times N$ matrix:

$$Z_N = \prod_{i<j} a^{-1}(z_i, z_j) \det \mathcal{B}(\{\underline{\lambda}\}, \{\underline{z}\}). \tag{2.24}$$

C. Homogeneous limit of the partition function

In this section, we discuss the homogeneous limit, i.e., when $\lambda_1=\lambda_2=\cdots=\lambda_N$ and $z_1=z_2=\cdots=z_N$, of the partition function (2.24).

For later convenience, we rewrite the inhomogeneous partition function more explicitly as

$$Z_N=\prod_{i<j}\frac{z_i-z_j+\eta}{z_i-z_j}\times\begin{vmatrix} b(\lambda_1,z_1) & \cdots & b(\lambda_1,z_k) & \cdots & b(\lambda_1,z_N)\\ b(\lambda_2,z_1)a(\lambda_1,z_1) & \cdots & b(\lambda_2,z_k)a(\lambda_1,z_k) & \cdots & b(\lambda_2,z_N)a(\lambda_1,z_N)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ b(\lambda_N,z_1)\prod_{\gamma=1}^{N-1}a(\lambda_\gamma,z_1) & \cdots & b(\lambda_N,z_k)\prod_{\gamma=1}^{N-1}a(\lambda_\gamma,z_k) & \cdots & b(\lambda_N,z_N)\prod_{\gamma=1}^{N-1}a(\lambda_\gamma,z_N)\end{vmatrix}. \tag{2.25}$$

It is easy to check that in the limit $\lambda_1\to\lambda,\lambda_2\to\lambda,\ldots,\lambda_N\to\lambda$, we have

$$Z_N=\prod_{i<j}\frac{z_i-z_j+\eta}{z_i-z_j}\prod_{i=1}^{N}\frac{\eta}{\lambda-z_i+\eta}\times\begin{vmatrix} 1 & \cdots & 1 & \cdots & 1\\ a(\lambda,z_1) & \cdots & a(\lambda,z_k) & \cdots & a(\lambda,z_N)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ a^{N-1}(\lambda,z_1) & \cdots & a^{N-1}(\lambda,z_k) & \cdots & a^{N-1}(\lambda,z_N)\end{vmatrix}, \tag{2.26}$$

where $a^k(\lambda,z_i)$ stands for the kth power of $a(\lambda,z_i)$. For the homogeneous limit of the parameters z_k, we first compute the limit $z_2\to z_1\equiv z$. Taylor expanding the second column as $z_2\to z$,

$$a^k(\lambda,z_2)=a^k(\lambda,z)+(a^k(\lambda,z))'(z_2-z)+\mathcal{O}((z_2-z)^2),$$

where X', X'', and $X^{(n)}$ stand for the 1st, 2nd and nth order derivatives of X with respect to the parameter z, respectively, and subtracting the first column from the second, Eq. (2.26) becomes

$$Z_N=(-\eta)\prod_{j=3}^{N}\left(\frac{z-z_j+\eta}{z-z_j}\right)^2\prod_{j>i=3}\frac{z_i-z_j+\eta}{z_i-z_j}\left(\frac{\eta}{\lambda-z+\eta}\right)^2\prod_{i=3}^{N}\frac{\eta}{\lambda-z_i+\eta}\times\begin{vmatrix} 1 & 0 & 1 & \cdots & 1\\ a(\lambda,z) & (a(\lambda,z))' & a(\lambda,z_3) & \cdots & a(\lambda,z_N)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ a^{N-1}(\lambda,z) & (a^{N-1}(\lambda,z))' & a^{N-1}(\lambda,z_3) & \cdots & a^{N-1}(\lambda,z_N)\end{vmatrix}. \tag{2.27}$$

Then, Taylor expanding the third column as $z_3\to z$,

$$a^k(\lambda,z_3)=a^k(\lambda,z)+(a^k(\lambda,z))'(z_3-z)+\frac{1}{2}(a^k(\lambda,z))''(z_3-z)^2+\mathcal{O}((z_3-z)^3),$$

and subtracting multiples of previous columns, we obtain

$$Z_N = \frac{(-\eta)^3}{2}\prod_{j=4}^{N}\left(\frac{z-z_j+\eta}{z-z_j}\right)^3 \prod_{j>i=4}\frac{z_i-z_j+\eta}{z_i-z_j}\left(\frac{\eta}{\lambda-z+\eta}\right)^3\prod_{i=4}^{N}\frac{\eta}{\lambda-z_i+\eta}$$
$$\times\begin{vmatrix} 1 & 0 & 0 & 1 & \cdots & 1 \\ a(\lambda,z) & (a(\lambda,z))' & (a(\lambda,z))'' & a(\lambda,z_4) & \cdots & a(\lambda,z_N) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ a^{N-1}(\lambda,z) & (a^{N-1}(\lambda,z))' & (a^{N-1}(\lambda,z))'' & a^{N-1}(\lambda,z_4) & \cdots & a^{N-1}(\lambda,z_N) \end{vmatrix}. \quad (2.28)$$

Continuing with such process, we obtain, instead of the determinant of the $N\times N$ matrix, the following determinant of the $(N-1)\times(N-1)$ matrix for the partition function:

$$Z_N = \frac{(-\eta)^{\Sigma_{i=1}^{N-1} i} b^N(\lambda,z)}{\prod_{i=2}^{N-1} i!}\begin{vmatrix} (a(\lambda,z))' & (a(\lambda,z))'' & \cdots & (a(\lambda,z))^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ (a^{N-1}(\lambda,z))' & (a^{N-1}(\lambda,z))'' & \cdots & (a^{N-1}(\lambda,z_N))^{(N-1)} \end{vmatrix}. \quad (2.29)$$

To simplify the determinant in Eq. (2.29) further, we investigate its elements. Computing $(a^j(\lambda,z))^{(n)}$, we obtain

$$(a^j(\lambda,z))^{(n)} = \sum_{k=1}^{n} f_k \prod_{l=0}^{k-1}(j-l)a^{j-k}(\lambda,z), \quad (2.30)$$

where f_k are functions of $\{(a(\lambda,z))^{(l)}\}$ $(l=1,\ldots,n)$ and are independent of j in Eq. (2.30). One can easily obtain f_k for the first and last terms,

$$f_1 = (a(\lambda,z))^{(n)}, \quad f_n = ((a(\lambda,z))')^n. \quad (2.31)$$

Thus, by means of the properties of determinants, the partition function of the $gl(1|1)$ vertex model is simplified to the following function:

$$Z_N = \frac{(-\eta)^{\Sigma_{i=1}^{N-1} i}(b(\lambda,z))^N}{\prod_{i=2}^{N-1} i!}\prod_{i=2}^{N-1} i![(a(\lambda,z))']^{\Sigma_{i=1}^{N-1} i} = (b(\lambda,z))^{N^2}. \quad (2.32)$$

III. *gl*(2|1) VERTEX MODEL WITH DW BOUNDARY CONDITION

A. Description of the model

Let $R\in End(V\otimes V)$ be the R matrix associated with the three-dimensional (3D) irreducible $gl(2|1)$ module. Choosing the FFB (fermion-fermion-Boson) grading for V, i.e., $[1]=[2]=1$, $[3]=0$, then the R matrix reads

$$R_{12}(\lambda_1,\lambda_2) = \begin{pmatrix} c_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{12} & 0 & -b_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{12} & 0 & 0 & 0 & b_{12} & 0 & 0 \\ 0 & -b_{12} & 0 & a_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{12} & 0 & b_{12} & 0 \\ 0 & 0 & b_{12} & 0 & 0 & 0 & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{12} & 0 & a_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (3.1)$$

which satisfies the GYBE [Eq. (2.1)]. Here a_{12}, b_{12}, and c_{12} are the same as those given in the previous section. The basis vectors $|0\rangle$, $|1\rangle$, and $|2\rangle$ of the 3D $gl(2|1)$ representation are given by

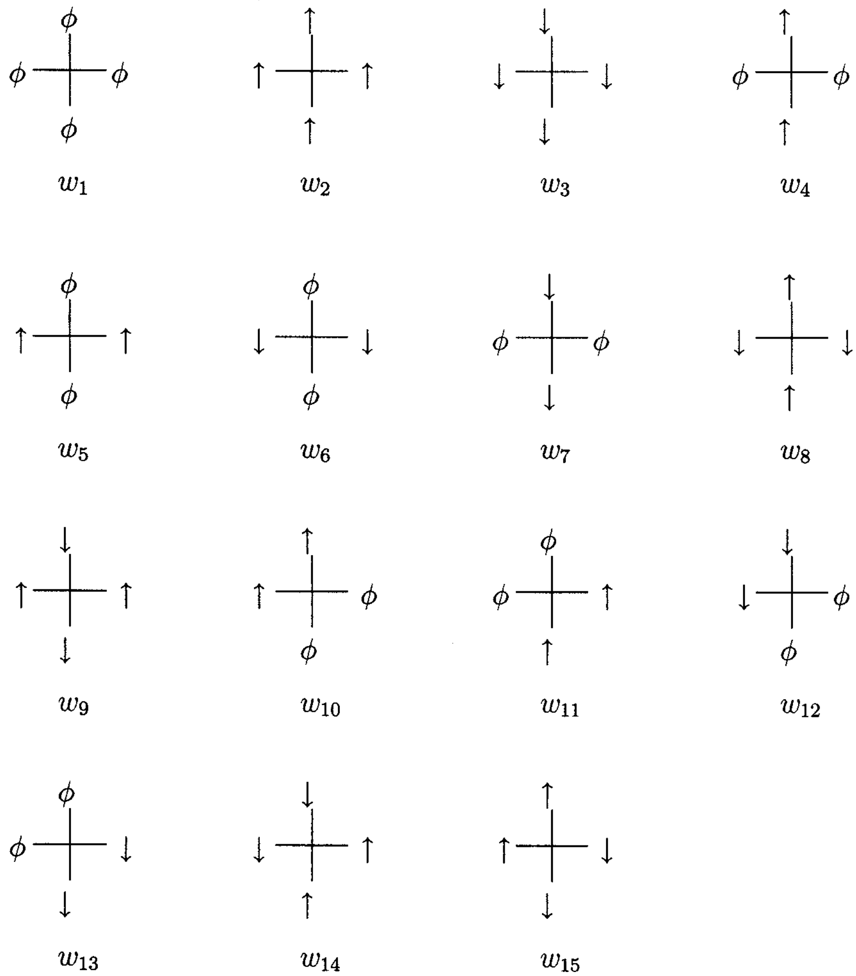

FIG. 3. Vertex configurations of the $gl(2|1)$ vertex model and their Boltzmann weights.

$$|0\rangle=\begin{pmatrix}0\\0\\1\end{pmatrix},\quad |1\rangle=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad |2\rangle=\begin{pmatrix}0\\1\\0\end{pmatrix}. \tag{3.2}$$

Now we consider the $gl(2|1)$ 15-vertex model on an $N\times N$ square lattice. Excluding the double occupancy, there are three possible electronic states, i.e., up spin $\uparrow$, down spin $\downarrow$, and vacuum ϕ, at each site of the lattice. For this model, the configuration of the vertex is decided by the electronic states around it. Corresponding to the $gl(2|1)$ invariance, there are altogether 15 possible configurations, which preserve the fermion numbers and spins, with nonzero Boltzmann weights. They are shown in Fig. 3. The Boltzmann weights associated with the vertices are given by the elements of the R matrix (3.1),

$$\begin{gathered} w_1=1,\quad w_2=w_3=c(\lambda,z),\quad w_4=\cdots=w_9=a(\lambda,z),\\ w_{10}=\cdots=w_{13}=b(\lambda,z),\quad w_{14}=w_{15}=-b(\lambda,z). \end{gathered} \tag{3.3}$$

The $gl(2|1)$ supersymmetric vertex model with DW boundary condition is defined as follows. At the left and top ends, all electrons are in vacuum states, while at the right and bottom ends,

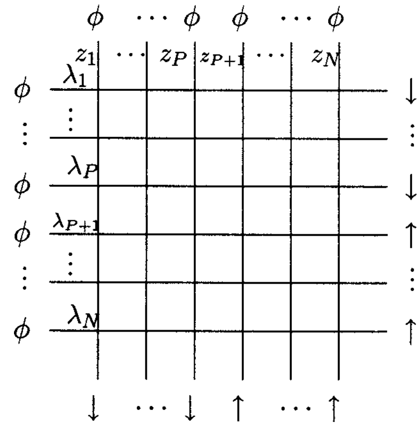

FIG. 4. DW boundary condition for the $gl(2|1)$ vertex model.

there are P down spin states (corresponding to the 1st$-P$th lines) and $N-P$ up spin states [corresponding to the $(P+1)$th$-N$th lines]. The boundary condition is shown in Fig. 4. The DW partition function of the $gl(2|1)$ vertex model is given by

$$Z_N = \sum \prod_{i=1}^{15} w_i^{n_i} = \prod_{i=0}^{P-1} \uparrow_{(N-i)} \prod_{j=0}^{P-1} \downarrow_{(P-j)} \prod_{i=0}^{N-1} \phi_{(N-i)} \prod_{k=1}^{N} \prod_{l=1}^{N} R_{kl}(\lambda_k, z_l) \prod_{j=1}^{P} \downarrow_{(j)} \prod_{j=P+1}^{N} \uparrow_{(j)} \prod_{i=1}^{N} \phi_{(i)}$$
$$= \prod_{i=0}^{P-1} \uparrow_{(N-i)} \prod_{j=0}^{P-1} \downarrow_{(P-j)} \prod_{i=0}^{N-1} \phi_{(N-i)} \prod_{k=1}^{N} T_k(\lambda_k) \prod_{j=1}^{P} \downarrow_{(j)} \prod_{j=P+1}^{N} \uparrow_{(j)} \prod_{i=1}^{N} \phi_{(i)}$$
$$= \prod_{i=0}^{P-1} \uparrow_{(N-i)} \prod_{j=0}^{P-1} \downarrow_{(P-j)} C_2(\lambda_1) \ldots C_2(\lambda_P) C_1(\lambda_{P+1}) \ldots C_1(\lambda_N) \prod_{i=1}^{N} \phi_{(i)}, \tag{3.4}$$

where n_i is the number of configurations with the weights w_i and $T_k(\lambda_k)$ is the monodromy matrix along the horizontal lines and is defined by

$$T_k(\lambda_k) = R_{k,N}(\lambda_k, z_N) R_{k,N-1}(\lambda_k, z_{N-1}) \cdots R_{k,1}(\lambda_k, z_1)$$
$$\equiv \begin{pmatrix} A_{11}(\lambda_k) & A_{12}(\lambda_k) & B_1(\lambda_k) \\ A_{21}(\lambda_k) & A_{22}(\lambda_k) & B_2(\lambda_k) \\ C_1(\lambda_k) & C_2(\lambda_k) & D(\lambda_k) \end{pmatrix}_{(k)} . \tag{3.5}$$

Similar to the $gl(1|1)$ vertex model case, the partition function (3.4) can be computed by using the approach of Drinfeld twists, as can be seen in the next section.

B. Exact solution of the partition function

We now compute the partition function (3.4) using the Drinfeld twist method. The F matrix for this case is still defined by Eq. (2.10), except that now $\alpha = 1, 2, 3$. The inverse of the F matrix is given by

$$F^{-1}_{1\cdots N} = F^{*}_{1\cdots N} \prod_{i<j} \Delta_{ij}^{-1}, \tag{3.6}$$

with

$$\Delta_{ij} = \mathrm{diag}(4a_{ij}a_{ji}, a_{ji}, a_{ji}, a_{ij}, 4a_{ij}a_{ji}, a_{ji}, a_{ij}, a_{ij}, 1). \tag{3.7}$$

Working in the F basis, the lower entries C_1 and C_2 of the monodromy matrix (3.5) are simplified to symmetry forms, that is, they can be written as[19]

$$\tilde{C}_2(\lambda) = \sum_{i=1}^{N} b(\lambda,z_i)E_{(i)}^{2,3} \otimes_{j\neq i} \mathrm{diag}(a(\lambda,z_j),2a(\lambda,z_j),1)_{(j)}, \tag{3.8}$$

$$\begin{aligned}\tilde{C}_1(\lambda) = &\sum_{i=1}^{N} b(\lambda,z_i)E_{(i)}^{1,3} \otimes_{j\neq i} \mathrm{diag}(2a(\lambda,z_j),a(\lambda,z_j)(a(z_i,z_j))^{-1},1)_{(j)} \\ &+ \sum_{i\neq j=1}^{N} \frac{a(\lambda,z_i)b(\lambda,z_j)b(z_i,z_j)}{a(z_i,z_j)} E_{(i)}^{1,2}E_{(j)}^{2,3} \otimes_{k\neq i,j}(2a(\lambda,z_k),a(\lambda,z_k)a^{-1}(z_i,z_k),1)_{(k)}.\end{aligned} \tag{3.9}$$

Here $E_{(l)}^{\alpha\beta}$ (α, β=1,2,3; l=1,2,$\cdots$,N) are the generators of $gl(2|1)$ on the lth vertical line.

In the following we will identify $|0\rangle$, $|1\rangle$, and $|2\rangle$ with the vacuum ϕ, spin-up $\uparrow$, and spin-down $\downarrow$ states, respectively. Note that under the action of F the state $|0\rangle_{(1)} \otimes \cdots \otimes |0\rangle_{(N)}$ is invariant, that is,

$$F_{1\cdots N}|0\rangle_{(1)} \otimes \cdots \otimes |0\rangle_{(N)} = |0\rangle_{(1)} \otimes \cdots \otimes |0\rangle_{(N)}. \tag{3.10}$$

Thus, substituting Eqs. (3.8)–(3.10) into the partition function (3.4), we obtain

$$\begin{aligned}Z_N &= \prod_{i=0}^{P-1} \langle 1|_{(N-i)} \prod_{j=0}^{P-1} \langle 2|_{(P-j)} C_2(\lambda_1)\cdots C_2(\lambda_P)C_1(\lambda_{P+1})\cdots C_1(\lambda_N)\prod_{i=1}^{N}|0\rangle_{(i)} \\ &= \prod_{i=0}^{P-1} \langle 1|_{(N-i)} \prod_{j=0}^{P-1} \langle 2|_{(P-j)} F_{1\cdots N}^{-1}\tilde{C}_2(\lambda_1)\cdots \tilde{C}_2(\lambda_P)\tilde{C}_1(\lambda_{P+1})\cdots \tilde{C}_1(\lambda_N)\prod_{i=1}^{N}|0\rangle_{(i)} \\ &= \sum_{i_1<\cdots<i_P}\ \sum_{i_{P+1}<\ldots<i_N}\ \prod_{l=0}^{P-1} \langle 1|_{(N-l)} \prod_{j=0}^{P-1} \langle 2|_{(P-j)} F_{1\cdots N}^{-1} \prod_{j=1}^{P} E_{(i_j)}^{23} \prod_{j=P+1}^{N} E_{(i_j)}^{13} \prod_{l=0}^{N} |0\rangle_{(l)} \\ &\quad \times 2^{P(P+1)/2+(N-P)(N-P+1)/2} \prod_{k=1}^{P}\prod_{l=P+1}^{N} a(\lambda_k,z_{i_l}) \det \mathcal{B}_P(\{\lambda_1,\ldots,\lambda_P\},\{z_{i_1},\ldots,z_{i_P}\}) \\ &\quad \times \det \mathcal{B}_{N-P}(\{\lambda_{P+1},\ldots,\lambda_N\},\{z_{i_{P+1}},\ldots,z_{i_N}\}) \\ &\equiv \sum_{i_1<\cdots<i_P}\ \sum_{i_{P+1}<\cdots<i_N}\ \prod_{l>k=1}^{P} (2a(z_{i_k},z_{i_l}))^{-1} \prod_{l>k=P+1}^{N} (2a(z_{i_k},z_{i_l}))^{-1} G(\{\underline{z}\}) \\ &\quad \times 2^{P(P+1)/2+(N-P)(N-P+1)/2} \prod_{k=1}^{P}\prod_{l=P+1}^{N} a(\lambda_k,z_{i_l}) \det \mathcal{B}_P(\{\lambda_1,\ldots,\lambda_P\},\{z_{i_1},\ldots,z_{i_P}\}) \\ &\quad \times \det \mathcal{B}_{N-P}(\{\lambda_{P+1},\ldots,\lambda_N\},\{z_{i_{P+1}},\ldots,z_{i_N}\}),\end{aligned} \tag{3.11}$$

where $\{i_1,\ldots,i_P\}\cap\{i_{P+1},\ldots,i_n\}=\oslash$ and $\mathcal{B}_M(\{\underline{\lambda}\},\{\underline{z}\})$ is an $M\times M$ matrix with elements

$$(\mathcal{B}_M(\{\underline{\lambda}\},\{\underline{z}\}))_{\alpha\beta} = b(\lambda_\alpha,z_\beta)\prod_{\gamma=1}^{\alpha-1} a(\lambda_\gamma,z_\beta). \tag{3.12}$$

In Eq. (3.11), the function $G(\{\underline{z}\})$ is defined by

$$G(\{\underset{\sim}{z}\}) \equiv \prod_{l>k=1}^{P} (2a(z_{i_k},z_{i_l})) \prod_{l>k=P+1}^{N} (2a(z_{i_k},z_{i_l}))\langle 1|_{(N)} \cdots \langle 1|_{(P+1)}\langle 2|_{(P)} \cdots \langle 2|_{(1)} \times F_{1\cdots N}^{-1}E_{(i_1)}^{23} \cdots E_{i_P}^{23}E_{(i_{P+1})}^{13} \cdots E_{(i_N)}^{13}|0\rangle_{(1)} \cdots |0\rangle_{(N)}. \tag{3.13}$$

Using the matrix $F_{1\cdots N}^{-1}$ defined by Eqs. (2.15)–(2.18), one may simplify $G(\{\underset{\sim}{z}\})$ as follows:

$$\begin{aligned} G(\{\underset{\sim}{z}\}) = &\prod_{l>k=1}^{P} (2a(z_{i_k},z_{i_l})) \prod_{l>k=P+1}^{N} (2a(z_{i_k},z_{i_l})) \\ &\times \langle 0|_{(N)} \cdots \langle 0|_{(1)} E_{(N)}^{31} \cdots E_{(P+1)}^{31} E_{(P)}^{32} \cdots E_{(1)}^{32} \sum_{\sigma \in \mathcal{S}_2} S(c,\sigma,\alpha_\sigma) \\ &\times R_{\sigma(1\cdots N)}^{\sigma^{-1}} P_{(\sigma(1))}^{\alpha_{\sigma(1)}} \cdots P_{(\sigma(N))}^{\alpha_{\sigma(N)}} \prod_{i<j} \Delta_{ij}^{-1} E_{(i_1)}^{23} \ldots E_{i_P}^{23} E_{(i_{P+1})}^{13} \cdots E_{(i_N)}^{13} |0\rangle_{(1)} \cdots |0\rangle_{(N)} \\ = &\prod_{k=1}^{P} \prod_{l=P+1}^{N} a^{-1}(z_{i_k},z_{i_l}) \prod_{k>l=1}^{P} (2a(z_{i_k},z_{i_l}))^{-1} \prod_{k>l=P+1}^{N} (2a(z_{i_k},z_{i_l}))^{-1} \\ &\times \langle 0|_{(N)} \cdots \langle 0|_{(1)} E_{(N)}^{31} \cdots E_{(P+1)}^{31} E_{(P)}^{32} \cdots E_{(1)}^{32} \sum_{\sigma \in \mathcal{S}_N} S(c,\sigma,\alpha_\sigma) \\ &\times R_{\sigma(1\cdots N)}^{\sigma^{-1}} E_{(i_1)}^{23} \cdots E_{i_P}^{23} E_{(i_{P+1})}^{13} \cdots E_{(i_N)}^{13} |0\rangle_{(1)} \cdots |0\rangle_{(N)} \\ = &\prod_{k=1}^{P} \prod_{l=P+1}^{N} a^{-1}(z_{i_k},z_{i_l}) \prod_{k>l=1}^{P} (2a(z_{i_k},z_{i_l}))^{-1} \prod_{k>l=P+1}^{N} (2a(z_{i_k},z_{i_l}))^{-1} \\ &\times \sum_{\sigma \in S_N} (-1)^{\mathrm{sign}(\sigma'(i_1,\ldots,i_N)=(1,\ldots,N))} S(c,\sigma,\alpha_\sigma) (R_{\sigma(1\cdots N)}^{\sigma^{-1}})_{2\cdots 21\cdots 1}^{\alpha_1\cdots\alpha_P\alpha_{P+1}\cdots\alpha_N}, \end{aligned} \tag{3.14}$$

where $\alpha = 1$ or 2, the subscribes of α are indices of space, $\mathrm{sign}(\sigma) = 1$ if σ is odd, and $\mathrm{sign}(\sigma) = 0$ if σ is even.

The procedure of computing the homogeneous limit is similar to that for the $gl(1|1)$ vertex model. Here we only give the results. In the homogeneous limit, i.e., when $\lambda_1 = \cdots = \lambda_N \equiv \lambda$ and $z_1 = \cdots = z_N \equiv z$, the partition function (3.11) becomes

$$Z_N = (a(\lambda,z))^{P(N-P)}(b(\lambda,z))^{P^2}(b(\lambda,z))^{(N-P)^2} \lim_{z_1,\ldots,z_N \to z} \sum_{i_1<\cdots<i_P} \sum_{i_{P+1}<\cdots<i_N} G(\{\underset{\sim}{z}\}). \tag{3.15}$$

By using the decomposition law (2.13), the R matrix R^σ in Eq. (3.14) can be decomposed to elementary R matrices. In the homogeneous limit, the elements of the elementary R matrix can be tended to

$$R(z,z)_{21}^{12} = R(z,z)_{12}^{21} = -1, \quad R(z,z)_{11}^{11} = R(z,z)_{22}^{22} = -1, \quad R(z,z)_{12}^{12} = R(z,z)_{21}^{21} = 0. \tag{3.16}$$

Therefore, for the last factor in Eq. (3.15) involving $G(\{\underset{\sim}{z}\})$, one may easily obtain

$$\lim_{z_1,\ldots,z_N \to z} \sum_{i_1<\cdots<i_P} \sum_{i_{P+1}<\cdots<i_N} G(\{\underset{\sim}{z}\}) = \lim_{z_1,\ldots,z_N \to z} \sum_{i_1<\cdots<i_P} \sum_{i_{P+1}<\cdots<i_N} \prod_{k=1}^{P} \prod_{l=P+1}^{N} a^{-1}(z_{i_k},z_{i_l}) = C_N^P, \tag{3.17}$$

where C_N^P is the combinatorial number.

In summary, in the homogeneous limit the DW partition function Z_N is

$$Z_N = C_N^P (a(\lambda,z))^{P(N-P)} (b(\lambda,z))^{P^2+(N-P)^2}. \tag{3.18}$$

It is easily seen that when $P=0$ or $P=N$, the DW partition function of the $gl(2|1)$ vertex model reduces to that of the $gl(1|1)$ vertex model, confirming a statement made in Ref. 24 on DW partition functions of vertex models based on A_n-type algebras. However, when $P\neq 0$, N, one obtains new partition functions of the true $gl(2|1)$ supersymmetric vertex model with DW boundary conditions.

IV. CONCLUSION AND DISCUSSION

In this paper, we have proposed the $gl(1|1)$ and $gl(2|1)$ supersymmetric vertex models with the so-called DW boundary conditions. The DW partition functions of the models have been computed by means of the approach of the Drinfeld twists. We have found that in the homogeneous limit, the partition functions degenerate to simple functions. For the 2D square $gl(2|1)$ lattice model, we note here that the definition of the DW boundary condition is not unique. For the other cases, by using the same procedure, one may find that their partition functions are similar with those in this paper.

We have demonstrated, by working out the DW $gl(1|1)$ and $gl(2|1)$ supersymmetric vertex models as examples, that by means of the Drinfeld twist method, one can actually derive directly, rather than conjecture a formula and then verify it as usually done, the determinant representations of DW partition functions.

It is widely known that determinant representations of partition functions are closely related to some pure mathematical problems, such as algebraic combinations and tilings of the Aztec diamond.[5,22] In our further work, we will study the mathematical problems arising from the present models. The results in this paper will also be useful for simplifying the correlation functions of the supersymmetric *t*-*J* model obtained in Ref. 23 and for further studying physical properties of the model.

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