Safety Neighbourhoods for the Kronecker Canonical Form*

Juan-Miguel Gracia† and Inmaculada de Hoyos‡
Departamento de Matemática Aplicada y Estadística e I.O.
Universidad del País Vasco, Facultad de Farmacia
Apartado 450
E-01080 Vitoria-Gasteiz, Spain

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Abstract

We give safety neighbourhoods for the necessary conditions in the change of the Kronecker canonical form of a matrix pencil under small perturbations.

1 Introduction

Let $\lambda F - G$ be a pencil of rectangular complex matrices. It is known that there exists a neighbourhood $\mathcal{V}$ of $\lambda F - G$ in the space of matrix pencils such that for all $\lambda F' - G' \in \mathcal{V}$, the Kronecker canonical form of $\lambda F' - G'$ is *very related* with that of $\lambda F - G$. This result has been obtained independently by several authors, being the paper of Pokrzywa (1986) [13] the first appeared. Some of the other authors are de Hoyos [8], Marques de Sá [12], Hinrichsen and O'Halloran [7], Boley [1].

We define a distance between the pencils $\lambda F' - G'$ and $\lambda F - G$ by

$$\mathrm{d}(\lambda F' - G', \lambda F - G) := \|F' - F\| + \|G' - G\|,$$

where $\|\cdot\|$ is the *spectral matrix norm*. In this paper we give a theoretical method for finding a positive real number $\varepsilon(\lambda F - G)$ such that

$$\mathrm{d}(\lambda F' - G', \lambda F - G) < \varepsilon(\lambda F - G)$$

implies that the pencil $\lambda F' - G'$ has a Kronecker canonical form *very related* with that of $\lambda F - G$.

For special pencils $\lambda F - G$, it is sufficient to use the singular value decomposition theorem to obtain $\varepsilon(\lambda F - G)$. In the general case, it is also necessary to solve a constrained optimization problem to achieve $\varepsilon(\lambda F - G)$.

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†mepgrmej@vc.ehu.es
‡mepdeizi@vc.ehu.es

The results of this paper belong to the *quantitative theory* of perturbation of the Kronecker canonical form of pencils. We are interested in obtaining metrical information about pencils of matrices. The *qualitative theory* deals with the characterization of all possible changes that happen in the Kronecker form of the matrix pencils that belong to a sufficiently small neighbourhood $\mathcal{V}$ of the given pencil $\lambda F - G$. It also deals with the proof that every possible change is attained by matrix pencils $\lambda F' - G'$ as close as one wants to $\lambda F - G$. In both cases, one insists on existence theorems: of $\mathcal{V}$, or of $\lambda F' - G'$; the obtained information has a topological character. An article on quantitative theory is [4, Elmroth,Kågström].

Our methodology consists in formulating in the simplest possible form the integral invariants of the strict equivalence of matrix pencils by means of ranks of matrices; also we use the duality of partitions of integers. Thus, we utilize the Weyr characteristic instead of the Segre characteristic with respect to the eigenvalues of the matrix pencil. In the same way, we employ "conjugate" partitions of the column and row Kronecker minimal indices of the matrix pencil. We think that this formulation simplifies very much the exposition and permits to understand the core of the problems.

The organization of this paper is as follows. The first section is the present introduction. Section 2 is about definitions and notations; here in Subsection 2.1 we recall the definition of partition of an integer, its conjugate (or dual) partition, the order relations $\prec$ (majorization) and $\prec\!\!\prec$ (weak majorization) between two partitions, that the map which takes a partition into its conjugate partition is non-increasing monotone with respect to $\prec$, and that the conjugate of the union of partitions is equal to the sum of the conjugate partitions; these last two properties permit us to pass from Segre characteristic to Weyr characteristic and reciprocally.

In Subsection 2.2 we give definitions of rectangular matrix pencils, strict equivalence of pencils, normal rank of a pencil, regular and right (or left) regular pencils. We describe a complete system of invariants for the strict equivalence relation: Kronecker minimal indices and finite and infinite elementary divisors. We also give the "conjugate" partitions of the Kronecker minimal indices, and the exponents of the elementary divisors associated with each eigenvalue. We recall the definitions of Segre and Weyr characteristic.

In Section 3 we give a characterization of the integral invariants, mentioned before, by ranks of some like block Toeplitz matrices, which dates back to Gantmacher [5], but developed by Karcanias and Kalogeropoulos [9, 10] and Pokrzywa [13]. We also present some lemmas about decompositions and nullities of analogous block Toeplitz matrices due to de Hoyos [8].

In Section 4 we reformulate the necessary conditions satisfied by the Kronecker canonical form of every pencil $\mathcal{H}' = \lambda F' - G'$ sufficiently close to a fixed pencil $\mathcal{H} = \lambda F - G$. This is posed in Theorem 4.2. At the end of this section, we set the main objective of this paper: *a quantitative version* of Theorem 4.2.

In Section 5 we reformulate the underlying inverse problem in the perturbations of the Kronecker canonical form of a matrix pencil.

Finally, Section 6 is dedicated to obtaining safety radius of balls centered at pencil $\mathcal{H}$ that assure the validity of necessary conditions in Theorem 4.2: In Theorem 6.3 we give a radius so that the normal rank does not decay; in Theorem 6.4 we give a smaller radius that guarantees the "lower semicontinuity" with respect to $\prec\!\!\prec$ of the "conjugate" partitions of the column and row minimal indices; in Theorem 6.6 we give a smaller radius that warrants the "upper semicontinuity" with respect to $\prec\!\!\prec$ of the Weyr partitions corresponding to the perturbed eigenvalues *close* to the eigenvalues of $\mathcal{H}$; and in Theorem 6.9, when ∞ is an eigenvalue of $\mathcal{H}$, we give —with the help of a minimization problem— a smaller radius to bound upperly the Weyr partitions of perturbed eigenvalues that are distant from those of $\mathcal{H}$; and this last question remains unanswered when ∞ is not an eigenvalue of $\mathcal{H}$.

2 Definitions and Notations

2.1 Partitions of Integers

A *partition* is a finite or infinite sequence of nonnegative integers

$$a = (a_1, a_2, \ldots)$$

ordered in a nonincreasing order and such that there is only a finite number of them different from zero,

$$a_1 \geq a_2 \geq \cdots \geq a_{l(a)} > 0 = a_{l(a)+1} = \cdots .$$

We call *length* of a, $l(a)$, the number of terms of a different from zero.

If a is a given partition, we define the *conjugate partition*, $\bar{a}$, as the partition whose ith component is

$$\bar{a}_i = \mathrm{Card}\{j : a_j \geq i\}, \quad i = 1, 2, \ldots .$$

If a and b are partitions and $m := \max\{l(a), l(b)\}$ we say that a is *majorized* by b and we denote it by $a \prec b$ if

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i, \quad k = 1, 2, \ldots, m-1,$$

and

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{k} b_i;$$

we say that a is *weakly majorized* by b and we denote it by $a \prec\!\!\prec b$ if

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i, \quad k = 1, 2, \ldots .$$

It is well known that

$$a \prec b \Leftrightarrow \bar{b} \prec \bar{a}.$$

The *sum* of a and b is denoted by $a + b$ and it is the partition whose ith component is $a_i + b_i$. The *union* of a and b is denoted by $a \cup b$ and it is the partition obtained by reordering all the components of a and b in nonincreasing order. It is also known that

$$\overline{a \cup b} = \bar{a} + \bar{b}.$$

2.2 Matrix Pencils

A matrix pencil is a matrix polynomial of degree less than or equal to one; we denote it by $\lambda F - G$ or briefly by $\mathcal{H}$; the matrices F and G belong to $\mathbb{C}^{m\times n}$ and, so, $\lambda F - G \in \mathbb{C}[\lambda]^{m\times n}$.

The subset of $\mathbb{C}[\lambda]^{m\times n}$ formed by all the matrix pencils of size $m \times n$, will be denoted by $\mathcal{P}_{m\times n}$.

We say that the matrix pencils $\lambda F_1 - G_1, \lambda F_2 - G_2 \in \mathcal{P}_{m\times n}$ are *strictly equivalent* if there exist invertible matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ such that $P(\lambda F_1 - G_1)Q = \lambda F_2 - G_2$.

The *normal rank* of the pencil $\mathcal{H} = \lambda F - G \in \mathcal{P}_{m\times n}$ is the order of its greatest minor different from the polynomial zero. We denote it by $\mathrm{nrk}(\lambda F - G)$ or $\mathrm{nrk}(\mathcal{H})$.

The normal rank, so defined, coincides with the ordinary rank of $\lambda F - G$ as matrix whose entries belong to $\mathbb{C}(\lambda)$, the quotient field of $\mathbb{C}[\lambda]$.

We say that a pencil $\mathcal{H}$ is *regular* if $\mathrm{nrk}(\mathcal{H}) = n = m$. We say that $\mathcal{H}$ is *right regular* if $\mathrm{nrk}(\mathcal{H}) = n \leq m$. We say that $\mathcal{H}$ is *left regular* if $\mathrm{nrk}(\mathcal{H}) = m \leq n$. Consequently, a pencil is regular if and only if it is right and left regular. In the three cases, $\mathrm{nrk}(\mathcal{H})$ is full (i.e. equal to $\min\{m, n\}$) and, conversely, if $\mathrm{nrk}(\mathcal{H})$ is full the pencil $\mathcal{H}$ must be of some (or several) of the indicate kinds.

The following theorem is well known [5, Gantmacher, Section XII.5].

Theorem 2.1 *Two pencils are strictly equivalent if and only if they have the same (column and row) minimal indices and the same (finite and infinite) elementary divisors.*

Therefore, a complete system of invariants for the relation of strict equivalence of pencils is formed by the following types of invariants, associated with each pencil $\mathcal{H}$:

(1) *Column minimal indices* denoted by

$$\varepsilon_1 \geq \cdots \geq \varepsilon_{r_1} > \varepsilon_{r_1+1} = \cdots = \varepsilon_{r_0} = 0.$$

We define for $i = 0, 1, 2, \ldots$

$$r_i := \mathrm{Card}\{j : \varepsilon_j \geq i\}.$$

The numbers $r_0, r_1, r_2, \ldots$ will be called the r-*numbers* of the pencil $\mathcal{H}$. From the definition of the r_i's we deduce that the partitions $(\varepsilon_1, \ldots, \varepsilon_{r_1}, 0, \ldots)$ and $\mathrm{r}(\mathcal{H}) := (r_1, \ldots, r_{\varepsilon_1}, 0, \ldots)$ are conjugate. If $r_1 = 0, r_2 = 0, \ldots$, we put $\mathrm{r}(\mathcal{H}) := 0$, where $0 := (0, 0, \ldots)$ is the null partition; in this case r_0 may be zero or not. From the concept of normal rank we have that $r_0 = n - \mathrm{nrk}(\mathcal{H})$. We will denote by $\mathrm{ci}(\mathcal{H})$ the number, r_0, of column minimal indices of $\mathcal{H}$.

(2) *Row minimal indices* denoted by

$$\eta_1 \geq \cdots \geq \eta_{s_1} > \eta_{s_1+1} = \cdots = \eta_{s_0} = 0.$$

We define for $i = 0, 1, 2, \ldots$

$$s_i := \mathrm{Card}\{j : \eta_j \geq i\}.$$

The numbers $s_0, s_1, s_2, \ldots$ will be called the s-*numbers* of the pencil $\mathcal{H}$. From the definition of the s_i's we deduce that the partitions $(\eta_1, \ldots, \eta_{s_1}, 0, \ldots)$ and

$\mathrm{s}(\mathcal{H}) := (s_1, \dots, s_{\eta_1}, 0, \dots)$ are conjugate. If $s_1 = 0, s_2 = 0, \dots$, we put $\mathrm{s}(\mathcal{H}) := 0$, where $0 := (0, 0, \dots)$ is the null partition; in this case s_0 may be zero or not. From the concept of normal rank we have that $s_0 = m - \mathrm{nrk}(\mathcal{H})$. We will denote by $\mathrm{ri}(\mathcal{H})$ the number, s_0, of row minimal indices of $\mathcal{H}$.

(3) *Infinite elementary divisors* of the form

$$\mu^{n_{\infty 1}}, \dots, \mu^{n_{\infty \nu_\infty}}, \quad \text{with} \quad n_{\infty 1} \geq \dots \geq n_{\infty \nu_\infty} \geq 1.$$

We will say that

$$\mathrm{Segre}(\infty, \mathcal{H}) := (n_{\infty 1}, \dots, n_{\infty \nu_\infty}, 0, \dots)$$

is the partition of the Segre characteristic of the pencil $\mathcal{H}$ for the infinite eigenvalue and its conjugate partition

$$\mathrm{Weyr}(\infty, \mathcal{H}) := \overline{\mathrm{Segre}(\infty, \mathcal{H})} := (m_{\infty 1}, m_{\infty 2}, \dots)$$

is the partition of the Weyr characteristic of the pencil $\mathcal{H}$ for the infinite eigenvalue.

Therefore $m_{\infty 1} = \nu_\infty$. If ∞ is not an eigenvalue of $\mathcal{H}$, we write $\mathrm{Segre}(\infty, \mathcal{H}) := 0$ and $\mathrm{Weyr}(\infty, \mathcal{H}) := 0$.

(4) *Finite elementary divisors* of the form

$$(\lambda - \lambda_1)^{n_{\lambda_1 1}}, \dots, (\lambda - \lambda_1)^{n_{\lambda_1 \nu_1}}, \dots, (\lambda - \lambda_u)^{n_{\lambda_u 1}}, \dots, (\lambda - \lambda_u)^{n_{\lambda_u \nu_u}}$$

with $n_{\lambda_i 1} \geq \dots \geq n_{\lambda_i \nu_i} \geq 1 \quad (i = 1, \dots, u)$.

We will say that

$$\mathrm{Segre}(\lambda_i, \mathcal{H}) := (n_{\lambda_i 1}, \dots, n_{\lambda_i \nu_i}, 0, \dots)$$

is the partition of the Segre characteristic of the pencil $\mathcal{H}$ corresponding to the eigenvalue λ_i $(i = 1, \dots, u)$. Its conjugate partition

$$\mathrm{Weyr}(\lambda_i, \mathcal{H}) := \overline{\mathrm{Segre}(\lambda_i, \mathcal{H})} := (m_{\lambda_i 1}, m_{\lambda_i 2}, \dots)$$

will be called the partition of the Weyr characteristic of the pencil $\mathcal{H}$ corresponding to the eigenvalue λ_i $(i = 1, \dots, u)$. Consequently, $m_{\lambda_i 1} = \nu_{\lambda_i}$ $(i = 1, \dots, u)$.

Let $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. The subset of $\overline{\mathbb{C}}$ formed by all the eigenvalues of $\mathcal{H}$ will be called the *spectrum* of the pencil $\mathcal{H}$, and will be denoted by $\sigma(\mathcal{H})$.

The *Segre characteristic* of $\mathcal{H}$ is the system of partitions

$$\mathrm{Segre}(\mathcal{H}) := (\mathrm{Segre}(\alpha, \mathcal{H}))_{\alpha \in \sigma(\mathcal{H})}.$$

The *Weyr characteristic* of $\mathcal{H}$ is the system of partitions

$$\mathrm{Weyr}(\mathcal{H}) := (\mathrm{Weyr}(\alpha, \mathcal{H}))_{\alpha \in \sigma(\mathcal{H})}.$$

We generalize the notations of (3) and (4): If $z \in \overline{\mathbb{C}}$ we define

$$\mathrm{Segre}(z, \mathcal{H}) := \begin{cases} \mathrm{Segre}(z, \mathcal{H}) & \text{if} \quad z \in \sigma(\mathcal{H}) \\ 0 \text{ (null partition)} & \text{if} \quad z \notin \sigma(\mathcal{H}) \end{cases}$$

and analogously

$$\mathrm{Weyr}(z, \mathcal{H}) := \begin{cases} \mathrm{Weyr}(z, \mathcal{H}) & \text{if} \quad z \in \sigma(\mathcal{H}) \\ 0 \text{ (null partition)} & \text{if} \quad z \notin \sigma(\mathcal{H}) \end{cases}.$$

Remark 2.2 A right regular pencil $\mathcal{H}$ has not column minimal indices, since $r_0 = n - \text{nrk}(\mathcal{H})$ and $n = \text{nrk}(\mathcal{H})$. Analogously, a left regular pencil has not row minimal indices. Thus a regular pencil has as unique invariants the elementary divisors (finite and infinite).

Given a pencil $\mathcal{H}$ with the invariants described above, we can associate with it a pencil in *Kronecker canonical form* that we are going to define [5, Gantmacher]:

(1) If ε_j is a column minimal index > 0, we put

$$R_{\varepsilon_j} := \begin{bmatrix} \lambda & -1 & & \\ & \ddots & \ddots & \\ & & \lambda & -1 \end{bmatrix} \in \mathcal{P}_{\varepsilon_j \times (\varepsilon_j+1)}.$$

(2) If η_j is a row minimal index > 0, we put

$$L_{\eta_j} := R_{\eta_j}^T \in \mathcal{P}_{(\eta_j+1)\times\eta_j},$$

where ^{T} denotes transpose.

(3) If $\mu^{n_{\infty j}}$ is an infinite elementary divisor,

$$J_{n_{\infty j}}(\infty) := \begin{bmatrix} -1 & \lambda & & \\ & \ddots & \ddots & \\ & & -1 & \lambda \\ & & & -1 \end{bmatrix} \in \mathcal{P}_{n_{\infty j} \times n_{\infty j}};$$

that is to say,

$$J_{n_{\infty j}}(\infty) = \lambda J_{n_{\infty j}}(0) - I_{n_{\infty j}},$$

$J_{n_{\infty j}}(0) \in \mathbb{C}^{n_{\infty j} \times n_{\infty j}}$ being a Jordan block associated with the eigenvalue 0.

(4) If $(\lambda - \alpha)^{n_{\alpha j}}$ is a finite elementary divisor,

$$JF_{n_{\alpha j}}(\alpha) := \begin{bmatrix} \lambda-\alpha & -1 & & \\ & \ddots & \ddots & \\ & & \lambda-\alpha & -1 \\ & & & \lambda-\alpha \end{bmatrix} \in \mathcal{P}_{n_{\alpha j} \times n_{\alpha j}};$$

that is to say,

$$JF_{n_{\alpha j}}(\alpha) = \lambda I_{n_{\alpha j}} - J_{n_{\alpha j}}(\alpha)$$

$J_{n_{\alpha j}}(\alpha) \in \mathbb{C}^{n_{\alpha j} \times n_{\alpha j}}$ being a Jordan block associated with the eigenvalue α.

The *Kronecker canonical form* of the pencil $\mathcal{H}$ is the pencil

$$C_K(\mathcal{H}) := \text{diag}(R, L, J(\infty), JF),$$

where

$$R := \left[\text{diag}(R_{\varepsilon_1}, \ldots, R_{\varepsilon_{r_1}}), 0_{(\varepsilon_1+\cdots+\varepsilon_{r_1})\times(r_0-r_1)}\right],$$

$$L := \begin{bmatrix} \text{diag}(L_{\eta_1}, \ldots, L_{\eta_{s_1}}) \\ 0_{(s_0-s_1)\times(\eta_1+\cdots+\eta_{s_1})} \end{bmatrix},$$

$$J(\infty) := \text{diag}\left(J_{n_{\infty 1}}(\infty), \ldots, J_{n_{\infty \nu_\infty}}(\infty)\right)$$

and

$$JF := \operatorname{diag}\left(JF_{n_{\lambda_1 1}}(\lambda_1), \ldots, JF_{n_{\lambda_1 \nu_1}}(\lambda_1), \ldots, JF_{n_{\lambda_u 1}}(\lambda_u), \ldots, JF_{n_{\lambda_u \nu_u}}(\lambda_u)\right).$$

By $0_{p\times q}$ we denote the $p \times q$ zero matrix.

Remark 2.3 The relation between the number of columns (respectively, the number of rows) of an $m \times n$ pencil $\mathcal{H}$ and its integral invariants is the following:

$$\sum_{\alpha\in\sigma(\mathcal{H})} \sum_{k=1}^{l(\mathrm{Weyr}(\alpha,\mathcal{H}))} m_{\alpha k} + \sum_{i=1}^{l(\mathrm{r}(\mathcal{H}))} r_i + \sum_{i=1}^{l(\mathrm{s}(\mathcal{H}))} s_i + \mathrm{ci}(\mathcal{H}) = n,$$

respectively,

$$\sum_{\alpha\in\sigma(\mathcal{H})} \sum_{k=1}^{l(\mathrm{Weyr}(\alpha,\mathcal{H}))} m_{\alpha k} + \sum_{i=1}^{l(\mathrm{r}(\mathcal{H}))} r_i + \sum_{i=1}^{l(\mathrm{s}(\mathcal{H}))} s_i + \mathrm{ri}(\mathcal{H}) = m.$$

3 Characterization of the Integer Invariants by Ranks

A characterization of $\mathrm{Weyr}(\alpha, \mathcal{H})$ for $\alpha \in \overline{\mathbb{C}}$ and $\mathcal{H} = \lambda F - G$ a regular pencil (or right or left regular pencil) with F and G real matrices, is given in [9, Karcanias, Kalogeropoulos]. These results can be generalized to any pencil of complex matrices [13, Pokrzywa].

Let $\lambda F - G \in \mathcal{P}_{m\times n}$ and α be any complex number. We define for $k = 1, 2, \ldots$

$$P_\alpha^k(F,G) := \begin{bmatrix} \alpha F - G & 0 & \ldots & \ldots & 0 \\ F & \alpha F - G & \ddots & & \vdots \\ 0 & F & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha F - G & 0 \\ 0 & \ldots & 0 & F & \alpha F - G \end{bmatrix} \in \mathbb{C}^{km\times kn}.$$

Observe that

$$P_\alpha^k(F,G) = \begin{pmatrix} 0 & 0 \\ I_{k-1} & 0 \end{pmatrix} \otimes F + I_k \otimes (\alpha F - G),$$

where $\otimes$ is the Kronecker product.

If we consider ∞ instead of α we define for $k = 1, 2, \ldots$

$$P_\infty^k(F,G) := \begin{bmatrix} F & 0 & \ldots & \ldots & 0 \\ G & F & \ddots & & \vdots \\ 0 & G & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & F & 0 \\ 0 & \ldots & 0 & G & F \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ I_{k-1} & 0 \end{pmatrix} \otimes G + I_k \otimes F \in \mathbb{C}^{km\times kn}.$$

For any $\mu \in \overline{\mathbb{C}}$, we also write $P_\mu^k(\mathcal{H}) := P_\mu^k(F,G)$.

Theorem 3.1 *Let $\lambda F - G \in \mathcal{P}_{m\times n}$. Then, for all $\alpha \in \overline{\mathbb{C}}$ and for all $k = 1, 2, \ldots$ it follows that*

$$\nu(P_\alpha^k(F,G)) = \sum_{i=1}^{k} m_{\alpha i} + k r_0,$$

where $(m_{\alpha 1}, m_{\alpha 2}, \ldots) := \mathrm{Weyr}(\alpha, \lambda F - G)$, $r_0 := \mathrm{ci}(\lambda F - G)$ and $\nu(M) := \dim \mathrm{Ker}\, M$ denotes the nullity of any complex matrix M.

We also have characterizations of the r-numbers and s-numbers of a pencil from sequences of nullities, as it is proved in [10, Karcanias, Kalogeropoulos] for the real case and in [13, Pokrzywa] for the complex case.

Let $\lambda F - G \in \mathcal{P}_{m\times n}$. We define for $k = 1, 2, \ldots$

$$T_k(F,G) := \begin{bmatrix} F & 0 & \ldots & \ldots & 0 \\ G & F & \ddots & & \vdots \\ 0 & G & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & F & 0 \\ \vdots & & \ddots & G & F \\ 0 & \ldots & \ldots & 0 & G \end{bmatrix} = \begin{pmatrix} 0 \\ I_k \end{pmatrix} \otimes G + \begin{pmatrix} I_k \\ 0 \end{pmatrix} \otimes F \in \mathbb{C}^{(k+1)m\times kn}.$$

Theorem 3.2 *Let $\lambda F - G \in \mathcal{P}_{m\times n}$. Then, for all $k = 1, 2, \ldots$ one has*

$$\nu(T_k(F,G)) = k r_0 - \sum_{i=1}^{k} r_i,$$

where $r_0 := \mathrm{ci}(\lambda F - G)$ and $(r_1, r_2, \ldots) := \mathrm{r}(\lambda F - G)$.

Theorem 3.3 *Let $\lambda F - G \in \mathcal{P}_{m\times n}$. Then, for all $k = 1, 2, \ldots$ one has*

$$\nu(T_k(F^T, G^T)) = k s_0 - \sum_{i=1}^{k} s_i,$$

where $s_0 := \mathrm{ri}(\lambda F - G)$ and $(s_1, s_2, \ldots) := \mathrm{s}(\lambda F - G)$.

The next results are necessary in Section 6. Firstly, we define some matrices like $P_\alpha^k(F,G)$ and that contain these as certain submatrices. Secondly, we will give the results, which throw some light on Theorem 3.1.

Let $\lambda F - G \in \mathcal{P}_{m\times n}$ and $\alpha_1, \ldots, \alpha_q$ be arbitrary complex numbers, with $\alpha_i \neq \alpha_j$ for $i \neq j,\ i, j \in \{1, \ldots, q\}$. We define for $k_1, \ldots, k_q \in \{1, 2, \ldots\}$

$$P_{\alpha_1,\ldots,\alpha_q}^{k_1,\ldots,k_q}(F,G) := \begin{bmatrix} P_{\alpha_1}^{k_1}(F,G) & 0 & \ldots & \ldots & 0 \\ Q_{k_2,k_1}(F) & P_{\alpha_2}^{k_2}(F,G) & \ddots & & \vdots \\ 0 & Q_{k_3,k_2}(F) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & P_{\alpha_{q-1}}^{k_{q-1}}(F,G) & 0 \\ 0 & \ldots & 0 & Q_{k_q,k_{q-1}}(F) & P_{\alpha_q}^{k_q}(F,G) \end{bmatrix}$$

where

$$Q_{k_{j+1},k_j}(F) := \begin{bmatrix} 0 & \dots & 0 & F \\ 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix} \in \mathbb{C}^{(k_{j+1}m)\times(k_j n)}$$

for $j = 1, \dots, q-1$, and

$$P^{k_1,\dots,k_q}_{\alpha_1,\dots,\alpha_q}(\mathcal{H}) := P^{k_1,\dots,k_q}_{\alpha_1,\dots,\alpha_q}(F,G) \in \mathbb{C}^{[(k_1+\dots+k_q)m]\times[(k_1+\dots+k_q)n]},$$

if $\mathcal{H} := \lambda F - G$.

Lemma 3.4 *Let $\lambda F - G \in \mathcal{P}_{m\times n}$. Then, for any distinct $\alpha_1, \dots, \alpha_q \in \mathbb{C}$ and for any $k_1, \dots, k_q \in \{1, 2, \dots\}$ it follows that*

$$\nu(P^{k_1,\dots,k_q}_{\alpha_1,\dots,\alpha_q}(F,G)) = \nu(\mathrm{diag}(P^{k_1}_{\alpha_1}(F,G), \dots, P^{k_q}_{\alpha_q}(F,G))).$$

This lemma was proved by transforming the matrix $P^{k_1,\dots,k_q}_{\alpha_1,\dots,\alpha_q}(F,G)$ into the block diagonal matrix $\mathrm{diag}(P^{k_1}_{\alpha_1}(F,G), \dots, P^{k_q}_{\alpha_q}(F,G))$ by means of block elementary operations [8, I. de Hoyos].

Lemma 3.5 [8, I. de Hoyos]. *Let $\lambda F - G \in \mathcal{P}_{m\times n}$. Let any distinct $\alpha_1, \dots, \alpha_q \in \mathbb{C}$ and $k_1, \dots, k_q \in \{1, 2, \dots\}$. Let $k := k_1 + \dots + k_q$.*

If $\alpha_i \notin \sigma(\lambda F - G)$ for every $i = 1, \dots, q$, then

$$\mathrm{rank}(P^{k_1,\dots,k_q}_{\alpha_1,\dots,\alpha_q}(F,G)) = k \cdot \mathrm{nrk}(\lambda F - G).$$

When we put any complex numbers $\beta_1, \dots, \beta_k$ (not necessarily distinct) instead of $\alpha_1, \dots, \alpha_q$, we define

$$P_{\beta_1,\dots,\beta_k}(F,G) := \begin{bmatrix} \beta_1 F - G & 0 & \dots & \dots & 0 \\ F & \beta_2 F - G & \ddots & & \vdots \\ 0 & F & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \beta_{k-1}F - G & 0 \\ 0 & \dots & 0 & F & \beta_k F - G \end{bmatrix}$$

and we have analogous results to Lemmas 3.4 and 3.5.

Lemma 3.6 [8]. *Let $\lambda F - G \in \mathcal{P}_{m\times n}$. Let $\beta_1, \dots, \beta_k \in \mathbb{C}$ (not necessarily distinct) such that k_i of them are equal to α_i, $(i = 1, \dots, q)$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. Then*

(i) $\nu(P_{\beta_1,\dots,\beta_k}(F,G)) = \nu(\mathrm{diag}(P^{k_1}_{\alpha_1}(F,G), \dots, P^{k_q}_{\alpha_q}(F,G)))$,

(ii) *if $\beta_j \notin \sigma(\lambda F - G))$ for all $j = 1, \dots, k$, then*

$$\mathrm{rank}(P_{\beta_1,\dots,\beta_k}(F,G)) = k \cdot \mathrm{nrk}(\lambda F - G).$$

4 Necessary Conditions for Perturbations of Matrix Pencils

As it is known we can obtain a compactification of $\mathbb{C}$ adding to it a point at ∞: $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. If $\alpha \in \mathbb{C}$, the open ball $B(\alpha, \rho) := \{z \in \mathbb{C} : |z - \alpha| < \rho\}$ where $\rho > 0$. And, the open ball of center ∞ and radius ρ is

$$B(\infty, \rho) := \{z \in \mathbb{C} : |z| > \rho^{-1}\} \cup \{\infty\}.$$

This basis of neighbourhoods makes of $\overline{\mathbb{C}}$ a compact topological space.

We will use the *spectral matrix norm* (or the norm associated to the vector Euclidean norm $\|\cdot\|_2$ seeing the matrix as a linear operator):

$$\|M\| := \max_{\|x\|_2=1} \|Mx\|_2$$

where $M \in \mathbb{C}^{m\times n}$.

It is well-known that $\|M\| = \sigma_1(M)$, the greatest singular value of M.

The vector space $\mathcal{P}_{m\times n}$ formed by the $m \times n$ complex matrix pencils has the structure of normed space if we consider the following norm on it: for every $\lambda F - G \in \mathcal{P}_{m\times n}$

$$\|\lambda F - G\| := \|F\| + \|G\|.$$

Let $\mathcal{H} := \lambda F - G \in \mathcal{P}_{m\times n}$. Given a real number $\delta > 0$, we define the *δ-neighbourhood of the spectrum of* $\mathcal{H}$ as

$$\mathcal{V}_\delta(\mathcal{H}) := \bigcup_{\alpha\in\sigma(\mathcal{H})} B(\alpha, \delta)$$

whenever the balls $B(\alpha, \delta)$ are pairwise disjoint. Thus we will always suppose that $\delta > 0$ is sufficiently small to hold the meaning of the above definition.

If the matrix A is square, the eigenvalues of A are "continuous"functions of A as A varies. This property fails for the eigenvalues of a pencil. Let us take the following example from [14, Van Dooren (1979)] to illustrate this.

Let us consider the pencil

$$\begin{bmatrix} \lambda & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

whose invariants are $\varepsilon_1 = 1$ and $\eta_1 = 2$.

A possible perturbation of this pencil is -for example-

$$\begin{bmatrix} \lambda & -1 & 0 & 0 \\ 0 & k_4\lambda + k_3 & -1 & 0 \\ 0 & k_2 & \lambda & -1 \\ k_0 & k_1 & 0 & \lambda \end{bmatrix}$$

(taking k_0, k_1, k_2, k_3 and k_4 complex numbers sufficiently small in absolute value). The determinant of the perturbed pencil is

$$k_4\lambda^4 + k_3\lambda^3 + k_2\lambda^2 + k_1\lambda + k_0.$$

Thus, unless a constant factor, we can obtain any polynomial of degree less or equal to four, and therefore, we can make the perturbed pencil have arbitrary eigenvalues; the infinite eigenvalue included, which will appear if we take $k_4 = 0$.

We may observe that the normal rank of the pencil has changed when we have perturbed it. But, if we restrict our attention to perturbed pencils that keep constant the normal rank, then we have continuity for the eigenvalues:

Lemma 4.1 *(Continuity of the eigenvalues of a matrix pencil.) Let $\mathcal{H} := \lambda F - G \in \mathcal{P}_{m\times n}$ and $\delta > 0$. There exists a neighbourhood $\mathcal{V}$ of $\mathcal{H}$ in $\mathcal{P}_{m\times n}$ such that if $\mathcal{H}' = \lambda F' - G' \in \mathcal{V}$ and $\mathrm{nrk}(\mathcal{H}') = \mathrm{nrk}(\mathcal{H})$, then*

$$\sigma(\mathcal{H}') \subset \mathcal{V}_\delta(\mathcal{H}).$$

In the next theorem we collect all the *necessary conditions* of perturbation of the Kronecker canonical form for matrix pencils, with the above notations.

Theorem 4.2 *Let $\mathcal{H} \in \mathcal{P}_{m\times n}$ and $\delta > 0$. Then there exists a neighbourhood $\mathcal{V}$ of $\mathcal{H}$ in $\mathcal{P}_{m\times n}$ such that if $\mathcal{H}' \in \mathcal{V}$ it follows that $\mathrm{nrk}(\mathcal{H}') \geq \mathrm{nrk}(\mathcal{H})$ and moreover:*

(i)

$$\mathrm{r}(\mathcal{H}) \prec\!\!\prec \mathrm{r}(\mathcal{H}') + (h, \overset{(l)}{\dots}, h),$$

(ii)

$$\mathrm{s}(\mathcal{H}) \prec\!\!\prec \mathrm{s}(\mathcal{H}') + (h, \overset{(l)}{\dots}, h),$$

(iii) *for each $\alpha \in \sigma(\mathcal{H})$ there exists an open neighbourhood Λ_α of α, contained in $B(\alpha, \delta)$, such that*

$$\bigcup_{\beta\in\Lambda_\alpha} \mathrm{Weyr}(\beta, \mathcal{H}') \prec\!\!\prec \mathrm{Weyr}(\alpha, \mathcal{H}) + (h, \overset{(l)}{\dots}, h),$$

(iv)

$$\bigcup_{\beta\in K_0} \mathrm{Weyr}(\beta, \mathcal{H}') \prec\!\!\prec (h, \overset{(l)}{\dots}, h),$$

where K_0 can be taken as $\overline{\mathbb{C}} - \mathcal{V}_\delta(\mathcal{H})$ or as $\overline{\mathbb{C}} - \bigcup_{\alpha\in\sigma(\mathcal{H})} \Lambda_\alpha$, Λ_α being the open neighbourhood of α that appears in (iii). *If $\sigma(\mathcal{H}) = \emptyset$, then $K_0 := \overline{\mathbb{C}}$. The number h is $\mathrm{nrk}(\mathcal{H}') - \mathrm{nrk}(\mathcal{H}) = \mathrm{ci}(\mathcal{H}) - \mathrm{ci}(\mathcal{H}')$ and $l := \min\{m, n\}$.*

Remark 4.3 If $\mathcal{H} := \lambda 0 - 0$ (the zero pencil)$\in \mathcal{P}_{m\times n}$, then the conditions (i) to (iv) are satisfied for *all* pencil $\mathcal{H}' \in \mathcal{P}_{m\times n}$, close to $\mathcal{H}$ or not.

Objective: We intend to find a quantitative version of this theorem in the following sense. We want to find explicitly a positive number $\varepsilon(\mathcal{H}, \delta)$, which depends on $\mathcal{H}$ and δ, such that if

$$\|\mathcal{H}' - \mathcal{H}\| < \varepsilon(\mathcal{H}, \delta)$$

then $\mathrm{nrk}(\mathcal{H}') \geq \mathrm{nrk}(\mathcal{H})$ and the consequences (i) to (iv) of Theorem 4.2 are true.

But, before that let us give our version of the converse of Theorem 4.2 in the following section.

5 Underlying Inverse Problem in the Perturbations of Matrix Pencils

The necessary conditions of perturbation of Kronecker canonical form are also sufficient (in a suitable sense) to find, as close as we want of a given pencil, a pencil with some invariants fixed beforehand.

Theorem 5.1 *Let $\mathcal{H} \in \mathcal{P}_{m\times n}$ and $\delta > 0$. For each $\alpha \in \sigma(\mathcal{H})$ let t_α be a given integer ≥ 0 and let $m'(\alpha,1),\dots,m'(\alpha,t_\alpha)$ be given partitions. For each $\beta \in \overline{\mathbb{C}} - \mathcal{V}_\delta(\mathcal{H}) =: K_0$ let $m'(\beta)$ be a given partition, where only a finite number of $m'(\beta)$ are different from the null partition. Let r' and s' be given partitions. Let r'_0 be a nonnegative integer less than or equal to $\mathrm{ci}(\mathcal{H})$. Let $s'_0 := r'_0 + m - n$.*

There exists in every neighbourhood of $\mathcal{H}$ a pencil $\mathcal{H}' \in \mathcal{P}_{m\times n}$ such that

(a) *$r' = \mathrm{r}(\mathcal{H}')$ and $r'_0 = \mathrm{ci}(\mathcal{H}')$,*

(b) *$s' = \mathrm{s}(\mathcal{H}')$ and $s'_0 = \mathrm{ri}(\mathcal{H}')$,*

(c) *there exists an open neighbourhood Λ_α of α, contained in $B(\alpha,\delta)$, such that $\mathcal{H}'$ has just t_α eigenvalues $\mu_{\alpha 1},\dots,\mu_{\alpha t_\alpha}$ in Λ_α, and*

$$m'(\alpha,j) = \mathrm{Weyr}(\mu_{\alpha j},\mathcal{H}') \qquad (j=1,\dots,t_\alpha);$$

for each $\alpha \in \sigma(\mathcal{H})$,

(d)

$$m'(\beta) = \mathrm{Weyr}(\beta,\mathcal{H}')$$

for each $\beta \in K_0$

if and only if the following conditions are satisfied

(i)

$$\mathrm{r}(\mathcal{H}) \prec\!\!\prec r' + (h,\overset{(l)}{\dots},h),$$

(ii)

$$\mathrm{s}(\mathcal{H}) \prec\!\!\prec s' + (h,\overset{(l)}{\dots},h),$$

(iii)

$$\bigcup_{j=1}^{t_\alpha} m'(\alpha,j) \prec\!\!\prec \mathrm{Weyr}(\alpha,\mathcal{H}) + (h,\overset{(l)}{\dots},h)$$

for each $\alpha \in \sigma(\mathcal{H})$,

(iv)

$$\bigcup_{\beta\in K_0} m'(\beta) \prec\!\!\prec (h,\overset{(l)}{\dots},h),$$

(v)

$$\sum_{\alpha\in\sigma(\mathcal{H})}\sum_{j=1}^{t_\alpha}\sum_{k=1}^{l(m'(\alpha,j))} m'(\alpha,j)_k + \sum_{\beta\in K_0}\sum_{k=1}^{l(m'(\beta))} m'(\beta)_k + \sum_{i=1}^{l(r')} r'_i + \sum_{i=1}^{l(s')} s'_i + r'_0 = n,$$

where $h := \mathrm{ci}(\mathcal{H}) - r'_0$ and $l := \min\{m,n\}$.

The problem about the existence of the pencil $\mathcal{H}'$ is called here the underlying inverse problem; inverse, because we prescribe beforehand some restrictions on $\mathcal{H}'$; underlying, since it is something hidden behind this approach. The demonstrations of the existence of $\mathcal{H}'$ are "constructive", taking that we exactly know the Kronecker canonical form of $\mathcal{H}$ for granted.

6 Safety Neighbourhoods for the perturbation of the Kronecker Canonical Form

Now, we are going to formulate our answer to the objective written at the end of Section 4. Before that, we give two lemmas about the matrices $T_k(F, G)$ in Section 3 and other matrices related with them.

For each $k = 1, 2, \ldots$, the matrix

$$T_k(F,G) = \begin{bmatrix} F & 0 & \ldots & \ldots & 0 \\ G & F & \ddots & & \vdots \\ 0 & G & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & F & 0 \\ \vdots & & \ddots & G & F \\ 0 & \ldots & \ldots & 0 & G \end{bmatrix}$$

is equal to

$$\begin{bmatrix} I_k \\ 0_{1\times k} \end{bmatrix} \otimes F + \begin{bmatrix} 0_{1\times k} \\ I_k \end{bmatrix} \otimes G.$$

Since

$$\|A \otimes B\| = \|A\|\|B\|$$

for any complex matrices A and B and we consider the spectral norm $\|\cdot\|$, we have the following lemma.

Lemma 6.1 *Let $M, F, G \in \mathbb{C}^{m\times n}$. Then*

$$\|I_k \otimes M\| = \|M\|$$

$$\|T_k(F,G)\| \leq \|F\| + \|G\|.$$

Lemma 6.2 *Let $M_1, M_2, \ldots, M_q, N \in \mathbb{C}^{m\times n}$ and let $k_1, k_2, \ldots, k_q$ positive integers such that $k_1 + k_2 + \cdots + k_q = k$. Define the matrix*

$$\mathbb{M}^{k_1,\ldots,k_q}_{(M_1,\ldots,M_q,N)} \in \mathbb{C}^{km\times kn}$$

as follows

$$
\left[\begin{array}{cccccccccccccccccc}
M_1 & 0 & \dots & \dots & 0 & & & & & & & & & & & & & \\
N & M_1 & \ddots & & \vdots & & & & & & & & & & & & & \\
0 & N & \ddots & \ddots & \vdots & & & & & & & & & & & & & \\
\vdots & \ddots & \ddots & M_1 & 0 & & & & & & & & & & & & & \\
0 & \dots & 0 & N & M_1 & & & & & & & & & & & & & \\
 & & & & N & M_2 & 0 & \dots & \dots & 0 & & & & & & & & \\
 & & & & & N & M_2 & \ddots & & \vdots & & & & & & & & \\
 & & & & & 0 & N & \ddots & \ddots & \vdots & & & & & & & & \\
 & & & & & \vdots & \ddots & \ddots & M_2 & 0 & & & & & & & & \\
 & & & & & 0 & \dots & 0 & N & M_2 & & & & & & & & \\
 & & & & & & & & & N & M_3 & \dots & & & & & & \\
 & & & & & & & & & & N & \dots & & & & & & \\
 & & & & & & & & & & \vdots & & & & & & & \\
 & & & & & & & & & & & \ddots & & & & & & \\
 & & & & & & & & & & & & \vdots & & & & & \\
 & & & & & & & & & & & \dots & 0 & & & & & \\
 & & & & & & & & & & & \dots & M_{q-1} & & & & & \\
 & & & & & & & & & & & & N & M_q & 0 & \dots & \dots & 0 \\
 & & & & & & & & & & & & & N & M_q & \ddots & & \vdots \\
 & & & & & & & & & & & & & 0 & N & \ddots & \ddots & \vdots \\
 & & & & & & & & & & & & & \vdots & \ddots & \ddots & M_q & 0 \\
 & & & & & & & & & & & & & 0 & \dots & 0 & N & M_q
\end{array}\right]
$$

where M_1 appears in k_1 block columns and k_1 block rows,..., M_q appears in k_q block columns and k_q block rows.

Then

$$\|\mathbb{M}^{k_1,\dots,k_q}_{(M_1,\dots,M_q,N)}\| \leq \sum_{i=1}^{q} \|M_i\| + q\|N\|.$$

Proof:

$$\|\mathbb{M}^{k_1,\dots,k_q}_{(M_1,\dots,M_q,N)}\| \leq$$

$$
\| \begin{bmatrix}
M_1 & 0 & \dots & \dots & 0 \\
N & M_1 & \ddots & & \vdots \\
0 & N & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & M_1 & 0 \\
\vdots & & \ddots & N & M_1 \\
0 & \dots & \dots & 0 & N
\end{bmatrix} \| + \| \begin{bmatrix}
M_2 & 0 & \dots & \dots & 0 \\
N & M_2 & \ddots & & \vdots \\
0 & N & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & M_2 & 0 \\
\vdots & & \ddots & N & M_2 \\
0 & \dots & \dots & 0 & N
\end{bmatrix} \| + \cdots
$$

$$+\left\| \begin{bmatrix} M_{q-1} & 0 & \dots & \dots & 0 \\ N & M_{q-1} & \ddots & & \vdots \\ 0 & N & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & M_{q-1} & 0 \\ \vdots & & \ddots & N & M_{q-1} \\ 0 & \dots & \dots & 0 & N \end{bmatrix} \right\| + \left\| \begin{bmatrix} M_q & 0 & \dots & \dots & 0 \\ N & M_q & \ddots & & \vdots \\ 0 & N & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & M_q & 0 \\ 0 & \dots & 0 & N & M_q \end{bmatrix} \right\| =$$

$$\|T_{k_1}(M_1, N)\| + \|T_{k_2}(M_2, N)\| + \cdots + \|T_{k_{q-1}}(M_{q-1}, N)\| +$$

$$\left\| I_{k_q} \otimes M_q + \begin{bmatrix} 0_{1\times(k_q-1)} & 0 \\ I_{k_q-1} & 0_{(k_q-1)\times 1} \end{bmatrix} \otimes N \right\| \leq$$

$$(\|M_1\| + \|N\|) + (\|M_2\| + \|N\|) + \cdots + (\|M_{q-1}\| + \|N\|) + (\|M_q\| + \|N\|) =$$

$$\sum_{i=1}^{q} \|M_i\| + q\|N\|.$$

□

I. Now, let $\mathcal{H} := \lambda F - G \in \mathcal{P}_{m\times n}$ with $\mathcal{H} \neq 0$ (by Remark 4.3, the case $\mathcal{H} = 0$ is trivial) and suppose that $\rho := \mathrm{nrk}(\lambda F - G)$. We want to find a real number $\varepsilon_\rho > 0$ such that if $\mathcal{H}' = \lambda F' - G' \in \mathcal{P}_{m\times n}$ satisfies

$$\|F' - F\| + \|G' - G\| < \varepsilon_\rho$$

then

$$\rho \leq \mathrm{nrk}(\lambda F' - G').$$

Given that ρ is the normal rank of $\lambda F - G$, there exists a minor of order ρ of $\mathcal{H}$ that is not equal to the zero polynomial. That is to say, there exists integers $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_\rho \leq m$, $1 \leq \beta_1 < \beta_2 < \cdots < \beta_\rho \leq n$ such that

$$\det\mathcal{H}[\alpha|\beta] := \det(\lambda F[\alpha|\beta] - G[\alpha|\beta]) \not\equiv 0$$

where $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_\rho)$ and $\beta := (\beta_1, \beta_2, \dots, \beta_\rho)$. [Here, we are using the standard notation: the sequence α indicates the rows and the sequence β indicates the columns of the $\rho \times \rho$ submatrix $F[\alpha|\beta]$ of F; the analogous for $G[\alpha|\beta]$ with regard to G and $\mathcal{H}[\alpha|\beta]$ with regard to $\mathcal{H}$.]

Thus, the pencil $\lambda F[\alpha|\beta] - G[\alpha|\beta] \in \mathcal{P}_{\rho\times\rho}$ is regular. Therefore, due to Theorem 3.2 the rank of the matrix

$$T_\rho(F[\alpha|\beta], G[\alpha|\beta]) := \begin{bmatrix} F[\alpha|\beta] & 0 & \dots & \dots & 0 \\ G[\alpha|\beta] & F[\alpha|\beta] & \ddots & & \vdots \\ 0 & G[\alpha|\beta] & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & F[\alpha|\beta] & 0 \\ \vdots & & \ddots & G[\alpha|\beta] & F[\alpha|\beta] \\ 0 & \dots & \dots & 0 & G[\alpha|\beta] \end{bmatrix} \in \mathbb{C}^{[(\rho+1)\rho]\times\rho^2}$$

is equal to ρ^2. Let $\sigma_{\rho^2}(T_\rho(F[\alpha|\beta], G[\alpha|\beta]))$ be the minimum positive singular value of the matrix $T_\rho(F[\alpha|\beta], G[\alpha|\beta])$.

For example: If $\rho = 3$, the matrix

$$T_3(F[\alpha|\beta], G[\alpha|\beta]) = \begin{bmatrix} F[\alpha|\beta] & 0 & 0 \\ G[\alpha|\beta] & F[\alpha|\beta] & 0 \\ 0 & G[\alpha|\beta] & F[\alpha|\beta] \\ 0 & 0 & G[\alpha|\beta] \end{bmatrix} \tag{6.1}$$

is a submatrix of the matrix

$$T_3(F, G) = \begin{bmatrix} F & 0 & 0 \\ G & F & 0 \\ 0 & G & F \\ 0 & 0 & G \end{bmatrix};$$

the matrix (6.1) is just the submatrix corresponding to the rows $\alpha_1,\ \alpha_2,\ \dots,\ \alpha_\rho$, $m+\alpha_1,\ m+\alpha_2,\ \dots,\ m+\alpha_\rho,\ 2m+\alpha_1,\ 2m+\alpha_2,\ \dots,\ 2m+\alpha_\rho,\ 3m+\alpha_1$, $3m+\alpha_2,\ \dots,\ 3m+\alpha_\rho$ and to the columns $\beta_1,\ \dots,\ \beta_\rho,\ n+\beta_1,\ \dots,\ n+\beta_\rho$, $2n+\beta_1,\ \dots,\ 2n+\beta_\rho$ of $T_3(F, G)$.

We take

$$\varepsilon_\rho := \sigma_{\rho^2}(T_\rho(F[\alpha|\beta], G[\alpha|\beta]);$$

then if $\lambda F' - G' \in \mathcal{P}_{m\times n}$ is a pencil such that

$$\|F' - F\| + \|G' - G\| < \varepsilon_\rho$$

it follows that

$$\|F'[\alpha|\beta] - F[\alpha|\beta]\| + \|G'[\alpha|\beta] - G[\alpha|\beta]\| \leq \|F' - F\| + \|G' - G\| < \varepsilon_\rho$$

and, therefore,

$$\|T_\rho(F'[\alpha|\beta], G'[\alpha, \beta]) - T_\rho(F[\alpha|\beta], G[\alpha|\beta])\| = \text{(in the example)}$$

$$\| \begin{bmatrix} F'[\alpha|\beta] & 0 & 0 \\ G'[\alpha|\beta] & F'[\alpha|\beta] & 0 \\ 0 & G'[\alpha|\beta] & F'[\alpha|\beta] \\ 0 & 0 & G'[\alpha|\beta] \end{bmatrix} - \begin{bmatrix} F[\alpha|\beta] & 0 & 0 \\ G[\alpha|\beta] & F[\alpha|\beta] & 0 \\ 0 & G[\alpha|\beta] & F[\alpha|\beta] \\ 0 & 0 & G[\alpha|\beta] \end{bmatrix} \| \leq$$

$$\|F'[\alpha|\beta] - F[\alpha|\beta]\| + \|G'[\alpha|\beta] - G[\alpha|\beta]\| < \varepsilon_\rho.$$

Consequently,

$$\operatorname{rank} T_\rho(F'[\alpha|\beta], G'[\alpha|\beta]) = \rho^2$$

and, accordingly, the pencil

$$\lambda F'[\alpha|\beta] - G'[\alpha|\beta] \in \mathcal{P}_{\rho\times\rho}$$

is regular; hence

$$\det(\lambda F'[\alpha|\beta] - G'[\alpha|\beta]) \not\equiv 0$$

and

$$\operatorname{nrk}(\lambda F' - G') \geq \rho.$$

We have so proved the following theorem.

Theorem 6.3 *With the above notations, if*

$$\|F' - F\| + \|G' - G\| < \varepsilon_\rho,$$

then

$$\operatorname{nrk}(\lambda F - G) \leq \operatorname{nrk}(\lambda F' - G').$$

II. Now we search a real number $\varepsilon_1 > 0$ such that

$$\|F' - F\| + \|G' - G\| < \varepsilon_1$$

implies

$$\operatorname{nrk}(\lambda F - G) \leq \operatorname{nrk}(\lambda F' - G'),$$

$$\operatorname{r}(\lambda F - G) \prec\!\!\prec \operatorname{r}(\lambda F' - G') + (h, \overset{(l)}{\ldots}, h),$$

$$\operatorname{s}(\lambda F - G) \prec\!\!\prec \operatorname{s}(\lambda F' - G') + (h, \overset{(l)}{\ldots}, h)$$

where

$$h := \operatorname{nrk}(\lambda F' - G') - \operatorname{nrk}(\lambda F - G) \text{ and } l := \min\{m, n\}.$$

For each $k = 1, \ldots, l$, let

$$\sigma_{\min}(T_k(F, G))$$

be the minimum positive singular value of the matrix $T_k(F, G)$. Let $r_0 := ci(\mathcal{H}), r_1, r_2, \ldots$ and $r'_0 := ci(\mathcal{H}'), r'_1, r'_2, \ldots$ be the r-numbers of the pencils $\lambda F - G$ and $\lambda F' - G'$, respectively. By Theorem 3.2, we have

$$\nu(T_k(F,G)) = kr_0 - \sum_{i=1}^{k} r_i$$

$$\nu(T_k(F',G')) = kr'_0 - \sum_{i=1}^{k} r'_i.$$

Hence, if $\|F'-F\| + \|G'-G\| < \sigma_{\min}(T_k(F,G))$ then

$$\|T_k(F',G') - T_k(F,G)\| \leq \|F'-F\| + \|G'-G\| < \sigma_{\min}(T_k(F,G)),$$

which implies that

$$\operatorname{rank} T_k(F',G') \geq \operatorname{rank} T_k(F,G);$$

consequently

$$\nu(T_k(F',G')) \leq \nu(T_k(F,G)).$$

It follows that

$$kr'_0 - \sum_{i=1}^{k} r'_i \leq kr_0 - \sum_{i=1}^{k} r_i,$$

which is equivalent to

$$\sum_{i=1}^{k} r_i \leq \sum_{i=1}^{k} r'_i + k(r_0 - r'_0).$$

As this is true for $k = 1, 2, \ldots, l$ and $h = r_0 - r'_0 = ci(\mathcal{H}) - ci(\mathcal{H}')$, we deduce

$$\mathrm{r}(\lambda F - G) \lll \mathrm{r}(\lambda F' - G') + (h, \overset{(l)}{\ldots}, h).$$

For each $k = 1, \ldots, l$ let

$$\sigma_{\min}(T_k(F^T, G^T))$$

be the minimum positive singular value of the matrix $T_k(F^T, G^T)$. Then, by Theorems 3.2 and 3.3, we have the following theorem.

Theorem 6.4 *With the current notations, let*

$$\varepsilon_1 := \min\{\varepsilon_\rho, \min_{1\leq k\leq l} \sigma_{\min}(T_k(F,G)), \min_{1\leq k\leq l} \sigma_{\min}(T_k(F^T,G^T))\}.$$

Then, if $\lambda F' - G' \in \mathcal{P}_{m\times n}$ satisfies the inequality

$$\|F'-F\| + \|G'-G\| < \varepsilon_1,$$

it follows that

$$\operatorname{nrk}(\lambda F - G) \leq \operatorname{nrk}(\lambda F' - G'),$$

(i)

$$\mathrm{r}(\lambda F - G) \prec\!\!\prec \mathrm{r}(\lambda F' - G') + (h, \overset{(l)}{\ldots}, h),$$

(ii)

$$\mathrm{s}(\lambda F - G) \prec\!\!\prec \mathrm{s}(\lambda F' - G') + (h, \overset{(l)}{\ldots}, h).$$

III. We continue the study of the problem of finding a safety neighbourhood of the pencil $\mathcal{H} = \lambda F - G \in \mathcal{P}_{m\times n}$ such that if $\mathcal{H}'$ belongs to that neighbourhood, then the following inequalities are satisfied

$$\mathrm{nrk}(\mathcal{H}) \leq \mathrm{nrk}(\mathcal{H}'),$$

(i)

$$\mathrm{r}(\mathcal{H}) \prec\!\!\prec \mathrm{r}(\mathcal{H}') + (h, \overset{(l)}{\ldots}, h),$$

(ii)

$$\mathrm{s}(\mathcal{H}) \prec\!\!\prec \mathrm{s}(\mathcal{H}') + (h, \overset{(l)}{\ldots}, h),$$

and

(iii) for each $\alpha \in \sigma(\mathcal{H})$ there exists an open neighbourhood $\Lambda_\alpha := B(\alpha, \delta_\alpha) \subset B(\alpha, \delta)$ of α such that

$$\bigcup_{\beta \in \Lambda_\alpha} \mathrm{Weyr}(\beta, \mathcal{H}') \prec\!\!\prec \mathrm{Weyr}(\alpha, \mathcal{H}) + (h, \overset{(h)}{\ldots}, h)$$

This requires some precision. Let us better set the problem for the condition (iii). *Let us suppose that* $\sigma(\mathcal{H}) \neq \emptyset$; in other case the condition (iii) always holds trivially.

We want to choose $\delta > 0$ such that the open balls $B(\alpha, \delta)$, $\alpha \in \sigma(\mathcal{H})$ are pairwise disjoint and δ is the best possible (i.e. the greatest possible). We previously analyze the possibilities about $\sigma(\mathcal{H})$ which can happen:

Case 1: $\infty \in \sigma(\mathcal{H})$:

Subcase 1.1: $\sigma(\mathcal{H}) \neq \{\infty\}$, i.e. $\sigma(\mathcal{H}) = \{\lambda_1, \ldots, \lambda_u, \infty\}$, $\lambda_i \in \mathbb{C}$ for $i = 1, \ldots, u$.

Subcase 1.2: $\sigma(\mathcal{H}) = \{\infty\}$.

Case 2: $\infty \notin \sigma(\mathcal{H})$, i.e. $\sigma(\mathcal{H}) = \{\lambda_1, \ldots, \lambda_u\}$ (only finite eigenvalues!):

Subcase 2.1: $u > 1$, i.e. the pencil $\mathcal{H}$ has two or more eigenvalues.

Subcase 2.2: $u = 1$, i.e. the pencil $\mathcal{H}$ has an only eigenvalue.

Case 1: $\infty \in \sigma(\mathcal{H})$.

Subcase 1.1: $\sigma(\mathcal{H}) = \{\lambda_1, \ldots, \lambda_u, \infty\}$ with $\lambda_1, \ldots, \lambda_u \in \mathbb{C}$ $(u \geq 1)$.

Subcase 1.1.1: $0 \notin \sigma(\mathcal{H})$.

Let $|\lambda_k| := \max_{1\leq i\leq u} |\lambda_i|$, with $k \in \{1, \ldots, u\}$.

Let us choose δ to be equal to

$$\delta_{111} := \min\left\{ \frac{\sqrt{|\lambda_k|^2 + 4} - |\lambda_k|}{2}, \min_{\substack{i \neq j \\ i,j \in \{1,\ldots,u\}}} \frac{1}{2}|\lambda_i - \lambda_j|, \min_{\substack{i \neq j \\ i,j \in \{1,\ldots,u\}}} \frac{1}{2}\left|\frac{1}{\lambda_i} - \frac{1}{\lambda_j}\right|, \min_{1\leq i\leq u} \frac{1}{2}\left|\frac{1}{\lambda_i}\right| \right\}.$$

Subcase 1.1.2: $0 \in \sigma(\mathcal{H})$.

Let us choose

$$\delta := \delta_{112} := \min \left\{ \delta_{111}, \frac{\sqrt{\left|\frac{1}{\lambda_m}\right|^2 + 4} - \left|\frac{1}{\lambda_m}\right|}{2} \right\},$$

where

$$|\lambda_m| := \min_{\substack{1 \leq i \leq u \\ \lambda_i \neq 0}} |\lambda_i|.$$

Subcase 1.2: $\sigma(\mathcal{H}) = \{\infty\}$. Then the pencil $F - \mu G \in \mathbb{C}^{m \times n}$ has as only eigenvalue the zero: $\sigma(F - \mu G) = \{0\}$. We interchange the roles of F and G and we proceed as in Subcase 2.2 later (i.e. we choose δ being any positive real number).

Case 2: $\infty \notin \sigma(\mathcal{H})$; that is to say $\sigma(\mathcal{H}) = \{\lambda_1, \ldots, \lambda_u\}$ with $\lambda_1, \ldots, \lambda_u \in \mathbb{C}$.

Subcase 2.1: $u > 1$; i.e. the pencil $\mathcal{H}$ has two or more finite eigenvalues. Then we choose

$$\delta := \delta_{21} := \min_{\substack{i \neq j \\ i,j \in \{1,\ldots,u\}}} \frac{1}{2}|\lambda_i - \lambda_j|.$$

Subcase 2.2: $u = 1$; i.e. the pencil $\mathcal{H}$ has an only finite eigenvalue: $\sigma(\mathcal{H}) = \{\lambda_1\}$ with $\lambda_1 \in \mathbb{C}$. Then we choose

δ :=an arbitrary positive real number.

Now, we leave out this classification and consider that the complex number α is an eigenvalue of the pencil $\mathcal{H} = \lambda F - G$. *We try to determine two real numbers* $\varepsilon(\mathcal{H}, \alpha, \delta) > 0$ *and* $\Delta(\mathcal{H}, \alpha, \delta) > 0$, *with* $\Delta(\mathcal{H}, \alpha, \delta) \leq \delta$ *such that if a pencil* $\mathcal{H}' = \lambda F' - G' \in \mathcal{P}_{m \times n}$ *satisfies*

$$\|F' - F\| + \|G' - G\| < \varepsilon(\mathcal{H}, \alpha, \delta),$$

then we have:

$$\mathrm{nrk}(\mathcal{H}) \leq \mathrm{nrk}(\mathcal{H}')$$

and

$$\bigcup_{\beta \in B(\alpha, \Delta(\mathcal{H}, \alpha, \delta))} \mathrm{Weyr}(\beta, \mathcal{H}') \prec\!\!\prec \mathrm{Weyr}(\alpha, \mathcal{H}) + (h, \overset{(l)}{\ldots}, h) \qquad \text{(iii)}$$

where $h := \mathrm{nrk}(\mathcal{H}') - \mathrm{nrk}(\mathcal{H})$ and $l := \min\{m, n\}$.

Remark 6.5 It can happen that $\mathcal{H}'$ has no eigenvalue in $B(\alpha, \Delta(\mathcal{H}, \alpha, \delta))$; in which case $\mathrm{Weyr}(\beta, \mathcal{H}') = 0$ for all $\beta \in B(\alpha, \Delta(\mathcal{H}, \alpha, \delta))$ and (iii) is trivially satisfied.

The following theorem shows us a way to find $\varepsilon(\mathcal{H}, \alpha, \delta)$ and $\Delta(\mathcal{H}, \alpha, \delta)$.

Theorem 6.6 *Let $\mathcal{H} = \lambda F - G \in \mathcal{P}_{m\times n}$ be a pencil which has finite eigenvalues. Let $\rho := \operatorname{nrk}(\mathcal{H})$ and choose $\delta > 0$ as we say in the above classification (Cases 1 and 2, subcases 1.1, 1.2, etc.). Let $\alpha \in \sigma(\mathcal{H}) \cap \mathbb{C}$. Let $l := \min\{m, n\}$ and for each $k = 1, \cdots, l$, let*

$$\sigma_{\min}^{k,\alpha} := \text{ minimum positive singular value of the matrix } P_\alpha^k(\mathcal{H});$$

and we take

$$\sigma_{\min}^{\alpha} := \min_{1\le k\le l} \sigma_{\min}^{k,\alpha}, \quad \text{if the pencil } \mathcal{H} \text{ has not the shape } \begin{bmatrix} \lambda I - \alpha I & 0 \\ 0 & 0 \end{bmatrix},$$

$$\sigma_{\min}^{\alpha} := \min_{2\le k\le l} \sigma_{\min}^{k,\alpha}, \quad \text{if the pencil } \mathcal{H} \text{ has the shape } \begin{bmatrix} \lambda I - \alpha I & 0 \\ 0 & 0 \end{bmatrix}.$$

Let η be a number such that $0 < \eta < 1$, and let

$$\Delta(\mathcal{H}, \alpha, \delta) := \min\{\frac{\sigma_{\min}^{\alpha}/2l}{|\alpha| + \|F\| + 1}, \delta, \eta\}$$

$$\varepsilon(\mathcal{H}, \alpha, \delta) := \min\{\varepsilon_1, \Delta(\mathcal{H}, \alpha, \delta)\}$$

where ε_1 is the number defined in the statement of Theorem 6.4.

Then, for all pencil $\mathcal{H}' = \lambda F' - G' \in \mathcal{P}_{m\times n}$ such that

$$\|F' - F\| + \|G' - G\| < \varepsilon(\mathcal{H}, \alpha, \delta)$$

it follows that

$$\operatorname{nrk}(\mathcal{H}) \le \operatorname{nrk}(\mathcal{H}')$$

(i)

$$\operatorname{r}(\mathcal{H}) \prec\!\!\prec \operatorname{r}(\mathcal{H}') + (h, \overset{(l)}{\ldots}, h)$$

(ii)

$$\operatorname{s}(\mathcal{H}) \prec\!\!\prec \operatorname{s}(\mathcal{H}') + (h, \overset{(l)}{\ldots}, h)$$

and

(iii)

$$\bigcup_{\beta \in B(\alpha, \Delta(\mathcal{H},\alpha,\delta))} \operatorname{Weyr}(\beta, \mathcal{H}') \prec\!\!\prec \operatorname{Weyr}(\alpha, \mathcal{H}) + (h, \overset{(l)}{\ldots}, h),$$

where $h := \operatorname{nrk}(\mathcal{H}') - \operatorname{nrk}(\mathcal{H})$.

Proof: Let $\mathcal{H}' = \lambda F' - G' \in \mathcal{P}_{m\times n}$ be a pencil such that $\|F' - F\| + \|G' - G\| < \varepsilon(\mathcal{H}, \alpha, \delta)$.

Fix a $k \in \{1, \ldots, l\}$ and let q be an integer, $q \le k$ and let $k_1, \ldots, k_q$ be positive integers such that $k_1 + \ldots + k_q = k$.

Let $z_1, \ldots, z_q \in \mathbb{C}$ (distinct). Then, by Lemma 6.2,

$$\|P^{k_1,\ldots,k_q}_{z_1,\ldots,z_q}(\mathcal{H}') - P^k_\alpha(\mathcal{H})\| \le q\|F' - F\| + \sum_{i=1}^{q} \|(z_i F' - \alpha F) - (G' - G)\|$$

$$\le l(\|F' - F\| + \|G' - G\|) + \sum_{i=1}^{q} \|z_i F' - \alpha F\|. \tag{6.2}$$

For each $i = 1, \ldots, q$ we have the identity $z_i F' - \alpha F = \alpha(F' - F) + (z_i - \alpha)F + (z_i - \alpha)F + (z_i - \alpha)(F' - F)$; hence,

$$\|z_i F' - \alpha F\| \le |\alpha|\|F' - F\| + |z_i - \alpha|\|F\| + |z_i - \alpha|\|F' - F\|. \tag{6.3}$$

If $|z_i - \alpha| < \Delta(\mathcal{H}, \alpha, \delta)$, as $\|F' - F\| < \Delta(\mathcal{H}, \alpha, \delta)$, we have

$$\|z_i F' - \alpha F\| \le |\alpha|\Delta(\mathcal{H}, \alpha, \delta) + \Delta(\mathcal{H}, \alpha, \delta)\|F\| + \Delta(\mathcal{H}, \alpha, \delta)^2$$

$$= \Delta(\mathcal{H}, \alpha, \delta)[|\alpha| + \|F\| + \Delta(\mathcal{H}, \alpha, \delta)] < \Delta(\mathcal{H}, \alpha, \delta)[|\alpha| + \|F\| + 1] \tag{6.4}$$

because $\Delta(\mathcal{H}, \alpha, \delta) < 1$. By (6.4) and the definition of $\Delta(\mathcal{H}, \alpha, \delta)$ it follows that

$$\|z_i F' - \alpha F\| < \frac{\sigma^\alpha_{\min}}{2l}. \tag{6.5}$$

Now, we define Δ_α as the open ball of center α and radius $\Delta(\mathcal{H}, \alpha, \delta)$:

$$\Lambda_\alpha := B(\alpha, \Delta(\mathcal{H}, \alpha, \delta)).$$

Let $\beta_1, \ldots, \beta_t$ be the eigenvalues of $\mathcal{H}'$ that are in the ball Λ_α. Let the partition

$$d := \bigcup_{\beta \in \Lambda_\alpha} \mathrm{Weyr}(\beta, \mathcal{H}') = \bigcup_{i=1}^{t} \mathrm{Weyr}(\beta_i, \mathcal{H}') = (d_1, d_2, \ldots, d_k, \ldots).$$

Denote for $\beta \in \Lambda_\alpha$, $\mathrm{Weyr}(\beta, \mathcal{H}') =: (m'_{\beta 1}, m'_{\beta 2}, \ldots)$.

By the definition of union of partitions for each $k \in \{1, \ldots, l\}$ there exist $i_1, \ldots, i_q \in \{1, \ldots, t\}$ such that

$$\sum_{i=1}^{k} d_i = \sum_{j=1}^{k_1} m'_{\beta_{i_1} j} + \ldots + \sum_{j=1}^{k_q} m'_{\beta_{i_q} j} \tag{6.6}$$

where $k_1 + \ldots + k_q = k$.

As $|\beta_{i_1} - \alpha| < \Delta(\mathcal{H}, \alpha, \delta), \ldots, |\beta_{i_q} - \alpha| < \Delta(\mathcal{H}, \alpha, \delta)$, by (6.2) and (6.5) it follows that

$$\|P^{k_1,\ldots,k_q}_{\beta_{i_1},\ldots,\beta_{i_q}}(\mathcal{H}') - P^k_\alpha(\mathcal{H})\| < l\frac{\sigma^\alpha_{\min}}{2l} + q\frac{\sigma^\alpha_{\min}}{2l} \le \frac{\sigma^\alpha_{\min}}{2} + \frac{\sigma^\alpha_{\min}}{2} = \sigma^\alpha_{\min};$$

hence,

$$\mathrm{rank} P^{k_1,\ldots,k_q}_{\beta_{i_1},\ldots,\beta_{i_q}}(\mathcal{H}') \geq \mathrm{rank} P^k_\alpha(\mathcal{H});$$

which is equivalent to

$$\nu(P^{k_1,\ldots,k_q}_{\beta_{i_1},\ldots,\beta_{i_q}}(\mathcal{H}')) \leq \nu(P^k_\alpha(\mathcal{H})).$$

From Lemma 3.4, this is equivalent to

$$\sum_{s=1}^{q} \nu(P^{k_s}_{\beta_{i_s}}(\mathcal{H}')) \leq \nu(P^k_\alpha(\mathcal{H})).$$

This inequality, by (6.6), implies

$$\sum_{i=1}^{k} d_i + kr'_0 \leq \sum_{i=1}^{k} m_{\alpha i} + kr_0 \tag{6.7}$$

where

$$(m_{\alpha 1}, m_{\alpha 2}, \ldots) := \mathrm{Weyr}(\alpha, \mathcal{H}),$$

$$r_0 := ci(\mathcal{H}),$$

$$r'_0 := ci(\mathcal{H}') = n - \mathrm{nrk}(\mathcal{H}').$$

Recalling that $h = r_0 - r'_0$, the inequality (6.7) implies

$$\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} m_{\alpha i} + kh;$$

given that, this inequality is true for each $k = 1, \ldots, l$, we obtain

$$\bigcup_{\beta \in B(\alpha, \Delta(\mathcal{H}, \alpha, \delta))} \mathrm{Weyr}(\beta, \mathcal{H}') \lll \mathrm{Weyr}(\alpha, \mathcal{H}) + (h, \overset{(l)}{\ldots}, h).$$

□

Remark 6.7 When $\infty \in \sigma(\lambda F - G)$ is to be considered instead of $\alpha \in \mathbb{C} \cap \sigma(\lambda F - G)$, we put

$$\Delta(\lambda F - G, \infty, \delta) := \Delta(F - \mu G, 0, \delta)$$

$$\varepsilon(\lambda F - G, \infty, \delta) := \varepsilon(F - \mu G, 0, \delta)$$

where δ is adapted to the eigenvalue $\mu = 0$ of the pencil in μ, $F - \mu G$.

Remark 6.8 In any of the cases 1 and 2, taking

$$\varepsilon(\lambda F - G, \delta) := \min_{\alpha \in \sigma(\mathcal{H})} \varepsilon(\mathcal{H}, \alpha, \delta)$$

we will have that

$$\|F' - F\| + \|G' - G\| < \varepsilon(\lambda F - G, \delta)$$

implies

$$\mathrm{nrk}(\lambda F - G) \leq \mathrm{nrk}(\lambda F' - G')$$

and the inequality (iii) for all $\alpha \in \sigma(\lambda F - G)$.

But, even so we will not be sure that $\lambda F' - G'$ and $\lambda F - G$ were sufficiently close as to be able to affirm that

(iv)

$$\bigcup_{\beta \in K_0} \mathrm{Weyr}(\beta, \mathcal{H}') \prec\!\!\prec (h, \overset{(l)}{\ldots}, h)$$

where

$$K_0 := \overline{\mathbb{C}} - (\bigcup_{\alpha \in \sigma(\mathcal{H})} B(\alpha, \Delta(\mathcal{H}, \alpha, \delta))$$

is a compact set.

It is easy ti find a safety radius to assure (iv) when ∞ is an eigenvalue of the pencil $\lambda F - G$, as we are going to see next.

Theorem 6.9 *Let $\mathcal{H} = \lambda F - G \in \mathcal{P}_{m \times n}$. Let $\rho := \mathrm{nrk}(\lambda F - G)$. Suppose that $\infty \in \sigma(\mathcal{H})$. Let $l := \min\{m, n\}$ and let*

$$K_0 := \overline{\mathbb{C}} - (\bigcup_{\alpha \in \sigma(\mathcal{H})} B(\alpha, \Delta(\mathcal{H}, \alpha, \delta))).$$

Let

$$\varepsilon_2 := \min_{1 \leq k \leq l} \min_{(z_1, \ldots, z_k) \in K_0^k} \sigma_{k\rho} \begin{bmatrix} z_1 F - G & 0 & \ldots & \ldots & 0 \\ F & z_2 F - G & \ddots & & \vdots \\ 0 & F & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & z_{k-1} F - G & 0 \\ 0 & \ldots & 0 & F & z_k F - G \end{bmatrix}.$$

Let $M := \Delta(\mathcal{H}, \infty, \delta)^{-1}$.

Let $\varepsilon_3 := \min\{\varepsilon_2 / 2lM, \varepsilon(\mathcal{H}, \delta)\}$ (where $\varepsilon(\mathcal{H}, \delta)$ is given in Remark 6.8).

If $\mathcal{H}' = \lambda F' - G' \in \mathcal{P}_{m \times n}$ is such that

$$\|F' - F\| + \|G' - G\| < \varepsilon_3$$

then

$$\mathrm{nrk}(\lambda F - G) \leq (\lambda F' - G'),$$

(i)

$$\mathrm{r}(\mathcal{H}) \prec\!\!\prec \mathrm{r}(\mathcal{H}') + (h, \overset{(h)}{\ldots}, h),$$

(ii)

$$\mathrm{s}(\mathcal{H}) \prec\!\!\prec \mathrm{s}(\mathcal{H}') + (h, \overset{(l)}{\ldots}, h),$$

(iii) *for each* $\alpha \in \sigma(\mathcal{H})$

$$\bigcup_{\beta \in B(\alpha, \Delta(\mathcal{H}, \alpha, \delta))} \mathrm{Weyr}(\beta, \mathcal{H}') \prec\!\!\prec \mathrm{Weyr}(\alpha, \mathcal{H}) + (h, \overset{(l)}{\ldots}, h)$$

and

(iv)

$$\bigcup_{\beta \in K_0} \mathrm{Weyr}(\beta, \mathcal{H}') \prec\!\!\prec (h, \overset{(l)}{\ldots}, h);$$

where $h := \mathrm{nrk}(\mathcal{H}') - \mathrm{nrk}(\mathcal{H})$.

Proof: The notation $\sigma_{k\rho}(P_{z_1,\ldots,z_k}(F,G))$ denotes the $(k\rho)$th singular value of the matrix $P_{z_1,\ldots,z_k}(F,G)$ when they are ordered in nonincreasing order. If $z_1, \ldots, z_k \in K_0$, then by Lemma 3.6 (ii)

$$\mathrm{rank}(P_{z_1,\ldots,z_k}(F,G)) = k \cdot \mathrm{nrk}(\lambda F - G) = k\rho;$$

so $\sigma_{k\rho}(P_{z_1,\ldots,z_k}(F,G) > 0$. Thus, given that K_0^k is a compact set, we have that there exist the minima that define ε_2, and $\varepsilon_2 > 0$.

Let $\mathcal{H}' = \lambda F' - G' \in \mathcal{P}_{m\times n}$ be such that

$$\|F' - F\| + \|G' - G\| < \varepsilon_3.$$

Suppose that $\sigma(\mathcal{H}') \cap K_0 = \{\alpha_1, \ldots, \alpha_q\}$. Let $k_1, \ldots, k_q$ be integers > 0 such that $k_1 + \ldots + k_q =: k$. Then

$$\|P_{\alpha_1, \overset{(k_1)}{\ldots}, \alpha_1, \ldots, \alpha_q, \overset{(k_q)}{\ldots}, \alpha_q}(F', G') - P_{\alpha_1, \overset{(k_1)}{\ldots}, \alpha_1, \ldots, \alpha_q, \overset{(k_q)}{\ldots}, \alpha_q}(F, G)\|$$

$$\leq \sum_{i=1}^{q} \|\alpha_i(F' - F) - (G' - G)\| + q\|F' - F\|$$

$$\leq \|F' - F\|(\sum_{i=1}^{q} |\alpha_i|) + q(\|F' - F\| + \|G' - G\|).$$

Given that $\infty \in \sigma(\mathcal{H})$, the set $K_0 \subset \mathbb{C}$ is bounded; even more, for all $z \in K_0$,

$$|z| \leq M.$$

Therefore,

$$\sum_{i=1}^{q} |\alpha_i| \leq qM \leq lM,$$

because $q \leq l$.

So,

$$\|P_{\alpha_1, \overset{(k_1)}{\ldots}, \alpha_1, \ldots, \alpha_q, \overset{(k_q)}{\ldots}, \alpha_q}(F', G') - P_{\alpha_1, \overset{(k_1)}{\ldots}, \alpha_1, \ldots, k_q, \overset{(k_q)}{\ldots}, k_q}(F, G)\|$$

$$\leq 2lM\|F' - F\| + 2lM\|G' - G\| < 2lM\frac{\varepsilon_2}{2lM} = \varepsilon_2$$

Due to the fact that

$$\varepsilon_2 \leq \sigma_{k\rho}(P_{\alpha_1,\overset{(k_1)}{\ldots},\alpha_1,\ldots,\alpha_q,\overset{(k_q)}{\ldots},\alpha_q}(F,G)),$$

we have that

$$\nu(P_{\alpha_1,\overset{(k_1)}{\ldots},\alpha_1,\ldots,\alpha_q,\overset{(k_q)}{\ldots},\alpha_q}(F',G')) \leq \nu(P_{\alpha_1,\overset{(k_1)}{\ldots},\alpha_1,\ldots,\alpha_q,\overset{(k_q)}{\ldots},\alpha_q}(F,G)).$$

So, by Lemma 3.6 (i),

$$\cup_{\beta\in K_0}\mathrm{Weyr}(\beta,\mathcal{H}') \prec\!\!\prec (h,\overset{(h)}{\ldots},h).$$

□

Remark 6.10 When $\infty \notin \sigma(\mathcal{H})$, the corresponding set $K_0 \subset \mathbb{C}$ is unbounded (although K_0 is a compact subset of $\overline{\mathbb{C}}$), and we cannot bound the quantity

$$\sum_{i=1}^{q} |\alpha_i|.$$

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