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# Estimating Disturbances and Model Uncertainty in Model Validation for Robust Control

Tom Oomen and Okko Bosgra

**Abstract**—Deterministic approaches to model validation for robust control are investigated. In common deterministic model validation approaches, a trade-off between disturbances and model uncertainty is present, resulting in an ill-posed problem. In this paper, an approach to model validation is presented that attempts to remedy the ill-posedness. By employing accurate, non-parametric, deterministic disturbance models in conjunction with enforcing averaging properties of deterministic disturbances, a novel framework enabling model validation for robust control is obtained. The approach results in a realistically estimated model uncertainty and a disturbance model, and is illustrated in a simulation example.

## I. INTRODUCTION

Model validation is a crucial step in any modeling procedure, since a model is useless if it has not been confronted with measurement data from the true system. In classical approaches to model validation, where the goal of the model is prediction, simulation, *etc.*, the model residual is entirely attributed to additive disturbances. Such disturbances are often represented in a stochastic framework, since this provides an accurate description for many realistic disturbances [1].

If the goal of the model is subsequent control design, then low complexity models are often desirable. In particular, the complexity of the resulting controller commonly depends on the complexity of the model. The model should thus only represent relevant phenomena. Indeed, a robust feedback control design can cope with large systematic modeling errors in certain frequency ranges [2], *e.g.*, caused by under-modeling. Deterministic perturbation models, *e.g.*,  $\mathcal{H}_\infty$ -norm bounded perturbations, are the most natural representation for systematic modeling errors, since the nominal model residual, *e.g.*, due to undermodeling, depends deterministically on the input signal.

The development of robust control design methodologies [3], [4], [2] has led to deterministic model validation approaches, both in the time domain [5] and in the frequency domain [6], [7], [8], [9] where besides deterministic perturbation models, deterministic disturbance models are employed. Such disturbance models typically allow unknown but bounded disturbances, as opposed to stochastic disturbance descriptions in the classical approaches. In the deterministic model validation problem, the uncertain model is invalidated if there does not exist a perturbation model and disturbance signal in a certain bounded set such that the uncertain model can reproduce the measurement data.

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Otherwise, the uncertain model is not invalidated by the measurement data.

Although the deterministic model validation problem delivers a perturbation model that is compatible with robust control design methodologies, a straightforward application of the deterministic model validation problem leads to debatable results. In particular, the nominal model residual can be attributed either to model uncertainty or to disturbances, hence a trade-off is present, see [10], [11] for the time domain case and [12] for the frequency domain case.

Presence of a trade-off in the model validation problem implies that the problem is ill-posed. Conceptually, disturbances are signals that are independent of the system input. Otherwise, these signals are the result of input-output behavior and should be considered as model uncertainty. In a deterministic model validation framework, deterministic disturbances are allowed to reproduce the model residual even if the residual depends on the input, resulting in overly optimistic results. Pursuing a worst-case, *i.e.*, pessimistic, approach to model validation [13], which is closely related to identification in  $\mathcal{H}_\infty$  [14], does not resolve the issue, since in this case the disturbance will work perfectly against the input and hence is still dependent on the input. Set-based deterministic white noise descriptions [15] provide a means to properly define disturbances. However, the approach is computationally intensive and hence may be infeasible for large data sets, as is typically encountered in model validation. Employing deterministic perturbation models in conjunction with stochastic disturbance models provides an opportunity to enforce independence between inputs and disturbances in a straightforward manner, yet a mixed deterministic/probabilistic approach is computationally hard, *e.g.*, [16]. The present paper investigates the deterministic model validation problem in further detail. For related model validation and model error modeling approaches in a stochastic framework, see [17] and references therein.

The main contribution of the present paper is a model validation approach that attempts to resolve the ill-posedness of deterministic model validation approaches. The presented approach addresses the trade-off that is present in common deterministic model validation tools by appropriately defining the notion of a disturbance. Specifically, a nonparametric, deterministic disturbance modeling approach in the frequency domain is proposed in conjunction with averaging properties of the disturbances with respect to the input signal. The key idea is to consider an appropriate input design, *i.e.*, periodic signals are employed. The advantages of periodic input signals are well established in a stochastic framework [18], [1], yet not in a deterministic model validation

framework. The approach is applicable to MIMO open-loop and closed-loop systems by employing a coprime factor-based approach. Coprime factor-based approaches have been presented for SISO and SIMO systems in [8], [19], the present approach extends the results to the MIMO case.

The paper is organized as follows. In Section II the model validation problem is stated. Section III presents an approach for nonparametric deterministic disturbance modeling. Mild stochastic assumptions are imposed that are appropriate for many realistic systems. Section IV establishes the independence properties between input signals and disturbances by suitable input design. Section V briefly describes the validation test. Section VI contains a simulation example, illustrating the results in the paper. Finally, conclusions are drawn in Section VII.

*Notation.*  $\mathcal{H}_\infty$  denotes the Hardy space of  $\mathcal{L}_\infty(\delta\mathbb{D})$  functions analytic in  $\mathbb{D}_E$ ,  $\mathcal{L}_\infty(\delta D)$  denotes the space of bounded functions on  $\delta\mathbb{D}$ , where  $\|F\|_\infty := \text{ess sup}_{\omega \in \mathbb{R}} \bar{\sigma}(F(e^{j\omega}))$ ,  $\mathbb{D} := \{z|z < 1\}$ ,  $\delta\mathbb{D} := \{z|z = 1\}$ , and  $\mathbb{D}_E := \{z|z > 1\}$ . In addition, the prefix  $\mathcal{R}$  denotes real-rational and  $\mathcal{B}$  denotes the open unit ball in a normed space. The transfer function of a system  $F$  is denoted by its  $Z$ -transform  $F(z) = \sum_{k=-\infty}^{\infty} f(k)z^{-k}$ . The discrete Fourier transform of a signal  $x$  is given by  $X_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^N x(t)e^{j\omega t}$ , with DFT grid  $\Omega = \{\frac{2\pi p}{N}, p = 0, 1, \dots, N-1\}$ . The argument  $\omega$  is often omitted to facilitate the notation. The upper linear fractional transformation (LFT) is given by  $\mathcal{F}_u(M, \Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}$ . All signals and systems evolve in discrete time with a normalized sampling frequency  $h = 1$ , generalization to continuous time systems is straightforward by imposing suitable assumptions.

## II. MODEL VALIDATION PROBLEM

The model validation setup considered in this paper is depicted in Figure 1.  $M_o$  denotes the true system with manipulated input  $w \in \mathbb{R}^{n_w}$  and measured output  $z_m \in \mathbb{R}^{n_z}$ , where  $z_m$  is affected by disturbances, including unmeasured inputs and measurement noise. It is assumed that the true plant can be represented by a Linear Time Invariant (LTI) nominal model and an LTI perturbation model. The uncertain model is represented by

$$z = \mathcal{F}_u(\hat{M}, \Delta_u)w + v, \quad (1)$$

where  $\hat{M}$  contains the nominal plant model  $\hat{P}$ , model uncertainty structure, and weighting filters [4]. In addition,  $\Delta_u$  is an  $\mathcal{H}_\infty$ -norm bounded perturbation block representing model uncertainty, *i.e.*,  $\|\Delta_u\|_\infty < \gamma$ , either structured or unstructured, *i.e.*,  $\Delta_u \in \mathbf{\Delta}_u$ , where  $\mathbf{\Delta}_u$  is defined as in [3]. The disturbance model is represented by  $v \in \mathbb{R}^{n_z}$ .  $\mathcal{L}_\infty$ -norm bounded deterministic disturbances are considered, which are elaborated on in more detail in Section III.

The following Model Validation Optimization Problem (MVOP) is the main problem that is addressed in this paper.

**Problem 1 (MVOP)** *Given the uncertain model (1), a norm-bounded  $v$ , and measurements  $w, z_m$ , determine the minimum value of  $\gamma$  such that the uncertain model is consistent with the data.*

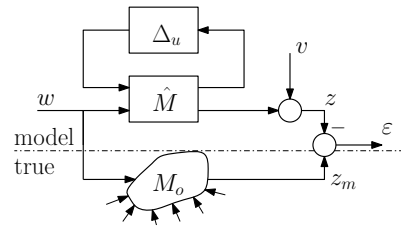


Fig. 1. Model validation setup.

In the deterministic model validation problem, the uncertain model (1) is consistent with the data if there exists a  $\Delta_u$  and  $v$  in a bounded set such that the residual  $\varepsilon$  equals zero.

The motivation to solve the MVOP in the frequency domain is threefold: 1) a frequency response-based approach using a discrete frequency grid provides a necessary and sufficient test for validation using  $\mathcal{H}_\infty$ -norm bounded perturbations, see Section V-A, 2) a frequency response-based approach allows usage of accurate nonparametric disturbance models, which do not require a cumbersome parametrization step, and 3) frequency response-based problems result in constant matrix problems, whereas the MVOP involves operators that are more difficult to handle computationally. In the frequency response-based approach, an uncertain model is invalidated if it is inconsistent with the data for at least one  $\omega_i \in \Omega$ , otherwise it is not invalidated. The Frequency Domain Model Validation Optimization Problem (FDMVOP) amounts to solving the MVOP at each frequency and is defined as follows.

**Problem 2 (FDMVOP)** *Given the uncertain model (1), an  $\mathcal{L}_\infty$  norm-bounded disturbance model  $V(\omega_i)$ , and measurements  $W(\omega_i), Z_m(\omega_i)$ , determine the minimum value of  $\gamma(\omega_i)$ ,  $\gamma(\omega_i) = \bar{\sigma}(\Delta_u(\omega_i))$  such that the uncertain model is consistent with the data.*

The FDMVOP is solved by performing a bisection over  $\gamma(\omega_i)$  and solving a series of Frequency Domain Model Validation Decision Problems (FDMVDPs).

**Problem 3 (FDMVDP)** *Given the uncertain model (1), an  $\mathcal{L}_\infty$  norm-bounded disturbance model  $V(\omega_i)$ , measurements  $W(\omega_i), Z_m(\omega_i)$ , and  $\gamma(\omega_i)$ ,  $\gamma(\omega_i) = \bar{\sigma}(\Delta_u(\omega_i))$ , is the uncertain model consistent with the data at frequency  $\omega_i$ ?*

A solution to Problem 3 is presented in Section V. First, several assumptions are imposed to ensure that the model validation problem is well-defined.

Well-posedness of the model validation problem requires well-posedness of the LFT  $\mathcal{F}_u(\hat{M}, \Delta_u)$ , which is assumed throughout and formalized in the following assumption.

**Assumption 4**  $\det(I - \hat{M}_{11}\Delta_u) \neq 0 \forall \Delta_u \in \mathbf{\Delta}_u, \|\Delta_u\|_\infty < \gamma$ .

In addition, the following assumption ensures that the model uncertainty affects the relevant model outputs.

**Assumption 5**

$Z_m - \hat{M}_{22}W \in \text{Im} \left( \hat{M}_{21}\Delta_u(I - \hat{M}_{11}\Delta_u)^{-1}\hat{M}_{12} \right)$  for a certain  $\Delta_u \in \mathbf{\Delta}_u$ .

Both Assumptions 4 and 5 are ensured if an appropriate perturbation model structure is selected, *e.g.*, additive structures for open-loop systems [4] and dual-Youla structures for closed-loop systems [20]. The following assumption ensures that the model validation problem is not trivially solved.

**Assumption 6**  $z_m - \hat{M}_{22}w \neq 0$ .

### III. DISTURBANCE MODEL

In this section, the disturbance model, *i.e.*, the term  $v$  in (1) is discussed in more detail. The motivation for considering nonparametric disturbance models is twofold: 1) the model validation problem is performed at a discrete frequency grid  $\omega_i \in \Omega^{\text{val}}$ , hence a nonparametric model suffices, and 2) nonparametric models can be estimated straightforwardly and accurately from data, since a parametrization, model order selection, and numerical optimization step are not required, *e.g.*, compared to [1].

A deterministic model validation procedure requires a deterministic disturbance model. Estimating a deterministic disturbance model from data requires deterministic prior assumptions regarding the data. Selection of accurate deterministic assumptions regarding signals is difficult in many realistic situations, *e.g.*, outliers are not allowed in such cases. Indeed, in many realistic situations, stochastic disturbance models are appropriate descriptions for disturbances, see also [1], [18], especially if the measurement time is large.

The nonparametric disturbance model is estimated based on mild stochastic assumptions regarding the time domain signals. These assumptions lead to a frequency domain disturbance model with certain favorable properties. In particular, the stochastic disturbance model can be converted nonconservatively to a deterministic disturbance model by selecting a suitable confidence interval.

The theoretical stochastic disturbance model is described in Section III-A, the conversion to a deterministic disturbance model is the topic of Section III-B. Estimating disturbance models from finite time data is discussed in Section III-C.

#### A. Theoretical stochastic disturbance model

Consider the stochastic disturbance model

$$v_s = H_o e, \quad (2)$$

where  $e \in \mathbb{R}^{n_z}$  is a sequence of independent, identically distributed random vectors with zero mean, unit covariance, and bounded moments of all orders,  $H_o \in \mathcal{RH}_{\infty}^{n_z \times n_z}$ , and  $v_s \in \mathbb{R}^{n_z}$  represents a stochastic disturbance assumed additive to the true plant output. Under this assumption, the following theorem applies, where  $V_{s,N}(e^{j\omega_i})$  denotes the DFT of  $v_s$ , see Section I.

**Theorem 7** Consider  $v_s$  given by (2). Then, for  $N \rightarrow \infty$ ,  $V_{s,N}(e^{j\omega_i})$  converges in distribution to

$$\begin{cases} \mathcal{N}_c(0, C_{v_s}(\omega_i)) & \omega_i \neq k\pi, k \in \mathbb{Z} \\ \mathcal{N}(0, 2C_{v_s}(\omega_i)) & \omega_i = k\pi, k \in \mathbb{Z}, \end{cases} \quad (3)$$

in addition,  $V_{s,N}(e^{j\omega_i})$  and  $V_{s,N}(e^{j\omega_j})$  are asymptotically independent for  $i \neq j$ ,  $\omega_i, \omega_j \in [0, \pi]$ .

*Proof:* The proof is based on the central limit theorem, see [21, Theorem 4.4.1] and [18, Theorem 14.25]. ■

Throughout, only the case  $\omega_i \neq k\pi, k \in \mathbb{Z}$  is considered for notational convenience. In (3),  $\mathcal{N}_c(0, C_v(\omega_i))$  denotes the circular complex normal distribution, which is defined next.

#### Definition 8 (Circular complex normal distribution)

[22] For a complex random vector  $z$ , the circular complex normal distribution  $\mathcal{N}_c(m_z, C_z)$  is defined by its mean  $m_z = \mathbb{E}\{z\}$ , covariance matrix  $C_z = \mathbb{E}\{(z - m_z)(z - m_z)^*\}$ ,  $C_z = C_z^* \geq 0$ , and the property

$$\mathbb{E}\{(z - m_z)(z - m_z)^T\} = 0. \quad (4)$$

Throughout, the nondegenerate case is considered, *i.e.*,  $C_z > 0$ . To interpret the circularity property (4), write  $z = z_r + jz_i$  and assume  $m_z = 0$ . Then, (4) becomes

$$\mathbb{E}\{z_r z_r^T - z_i z_i^T + j(z_r z_i^T + (z_r z_i^T)^T)\} = 0, \quad (5)$$

implying that the autocovariance of the real and imaginary part of  $z$  are equal and the cross-covariance between the real and imaginary part of  $z$  is skew-symmetric. Skew-symmetry implies the real and imaginary parts of each element  $z$  are uncorrelated, however, correlation between the real and imaginary part of distinct elements of  $z$  may be present. Due to the circularity property (4), correlation between the elements of  $z$  may be removed by introducing the coordinate transformation

$$\tilde{z} = T_z^* z. \quad (6)$$

In (6),  $T_z$  follows from the eigenvalue decomposition  $C_z = T_z \Sigma_z T_z^*$ , where  $T_z T_z^* = T_z^* T_z = I$  and  $\Sigma_z$  is a diagonal matrix. As a result,

$$\mathbb{E}\{\tilde{z}\} = T_z^* m_z, \quad \mathbb{E}\{(\tilde{z} - m_{\tilde{z}})(\tilde{z} - m_{\tilde{z}})^*\} = \Sigma_z, \quad (7)$$

hence the elements of  $\tilde{z}$  are independent, circularly complex normally distributed random variables. The following lemma regarding circularly complex normally distributed random variables is useful for converting the stochastic disturbance model to a deterministic disturbance model in Section III-B.

**Lemma 9** Let  $z$  be a circular complex random variable, *i.e.*,  $z \in \mathcal{N}_c(m_z, C_z)$ ,  $m_z \in \mathbb{C}$ ,  $C_z \in \mathbb{R}$ , and  $\alpha \in [0, 1)$ . Then,

$$\mathcal{P}(|z - m_z| < \sqrt{\frac{1}{2} C_z \chi_2^2(\alpha)}) = \alpha, \quad (8)$$

*Proof:* Suppose  $m_z = 0$  without loss of generality, then the result follows by using the fact  $\mathbb{E}\{z_r z_r^T\} = \mathbb{E}\{z_i z_i^T\} = \frac{1}{2} \mathbb{E}\{z z^*\}$  and properties of real-valued bivariate normal and  $\chi^2$  distributions. ■

#### B. Deterministic disturbance model

Circularity of the theoretical stochastic disturbance model in Theorem 7 in conjunction with Lemma 9 provides an opportunity to accurately convert the stochastic model into a deterministic one. For each  $\omega_i \in \Omega$ , introduce the coordinate transformation

$$\tilde{V}_{s,N}(e^{j\omega_i}) = T_V^*(\omega_i) V_{s,N}(e^{j\omega_i}), \quad (9)$$

where  $T^*(\omega_i)$  is obtained from an eigenvalue decomposition of  $C_{v_s}(\omega_i)$  for each  $\omega_i$ . As a result of the transformation,

$$\tilde{V}_{s,N}(e^{j\omega_i}) \in \mathcal{N}_c(0, \Sigma_{s,N}(\omega_i)), \quad (10)$$

where  $\Sigma_{s,N} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{n_v})$ . Given a probability level  $\alpha \in [0, 1)$ , let

$$\tilde{V}_q(\omega_i) := \sqrt{\frac{1}{2}\lambda_q(\omega_i)\chi_2^2(\alpha)} \quad (11)$$

and let  $\tilde{V}(\omega_i)$  be a matrix with the  $q^{\text{th}}$  diagonal element equal to  $\tilde{V}_q(\omega_i)$ ,  $q = 1, 2, \dots, n_v$ . In addition, define the block structure

$$\Delta_v = \left\{ \text{diag}(\delta_{v,1}, \delta_{v,2}, \dots, \delta_{v,n_z}) \mid \delta_{v,q} \in \mathbb{C}, q = 1, \dots, n_z \right\}. \quad (12)$$

Then,  $\tilde{V}(\omega_i)\mathbf{1}$  constitutes a deterministic  $\mathcal{L}_\infty$  norm-bounded disturbance model in the transformed coordinates, where

$$\tilde{\mathbf{V}}(\omega_i) = \left\{ \Delta_v \tilde{V}(\omega_i) \mid \Delta_v \in \mathcal{B}\Delta_v \right\}, \quad (13)$$

where  $\mathbf{1}$  denotes a vector with all elements equal to one. In the original coordinates,  $\mathbf{V}(\omega_i)\mathbf{1}$  constitutes a deterministic  $\mathcal{L}_\infty$  norm-bounded disturbance model, where

$$\mathbf{V}(\omega_i) = \left\{ T_V(\omega_i)\Delta_v \tilde{V}(\omega_i) \mid \Delta_v \in \mathcal{B}\Delta_v \right\}. \quad (14)$$

This disturbance model (14) is computed for each frequency  $\omega_i \in \Omega^{\text{val}}$  and is employed in the model validation procedure as discussed in the subsequent sections. Note that due to the conversion from stochastic to deterministic bounds,  $V(\omega_i) \in \mathbf{V}(\omega_i)\mathbf{1}$  with probability  $\alpha$ . In addition, note that correlation between different elements and different frequencies of  $V(\omega_i)$  is allowed. However, in Section IV, it is shown that the disturbance averages out, hence the optimism decreases with increasing measurement time.

### C. Estimating nonparametric disturbance models

The development of the disturbance model in the preceding sections requires knowledge of  $C_{v_s}(\omega_i)$ ,  $\omega_i \in \Omega^{\text{val}}$ . Assuming  $v_s$  is associated with the output,  $z_m$  is given by

$$z_m = M_o w + v_s. \quad (15)$$

The approach to estimate the statistical properties of  $v_s$  is to keep  $w$  constant during repeated experiments. Then, the deterministic contribution  $M_o w$  in (15) is constant, whereas  $v_s$  will vary. Specifically, let  $n_{\text{exp}}$  measurements be performed, each with measurement time  $N$ . Applying the DFT results in

$$W_r(\omega_i) = W(\omega_i), \quad Z_{m,r}(\omega_i), \quad (16)$$

where  $r = 1, \dots, n_{\text{exp}}$  and  $\omega_i \in \Omega$ . In virtue of Theorem 7,  $\mathbb{E}\{V_s(\omega_i)\} = 0$ , leading to the estimator

$$\hat{C}_{v_s}(\omega_i) = \frac{1}{n_{\text{exp}} - 1} \sum_{r=1}^{n_{\text{exp}}} \left( (Z_{m,r}(\omega_i) - m_{Z_{m,r}}(\omega_i)) \cdot (Z_{m,r}(\omega_i) - m_{Z_{m,r}}(\omega_i))^* \right) \quad (17)$$

$$m_{Z_{m,r}}(\omega_i) = \frac{1}{n_{\text{exp}}} \sum_{r=1}^{n_{\text{exp}}} Z_{m,r}(\omega_i). \quad (18)$$

The resulting estimate  $\hat{C}_v(\omega_i)$ ,  $\omega_i \in \Omega$  can be used directly in the deterministic disturbance model as discussed in Section III-B. To obtain an efficient procedure for model validation, a periodic input signal  $w$  is applied with period  $N$ . Then, the estimator of (17) can be applied, where  $n_{\text{exp}} := n_{\text{per}}$ ,  $n_{\text{per}}$  denoting the number of periods that have been measured. The same data set can be used for nonparametric disturbance modeling and model validation. Transient effects are neglected, which is formalized in the following assumption.

**Assumption 10** *If a periodic input  $w$  is applied with period  $N$ , then it is assumed  $M_o w$  is periodic with period  $N$ .*

## IV. AVERAGING IN A DETERMINISTIC FRAMEWORK

Optimal solutions to deterministic problems typically correspond to situations where disturbances perfectly depend on the input, *e.g.*, [23]. However, the notion of disturbance prohibits this dependence. In this section, an approach is presented that ensures that the notion of disturbance is well-defined in a deterministic framework.

Consider the uncertain model set of Figure 1. In the frequency response-based model validation problem, the model error  $\varepsilon$  is given by

$$\varepsilon = Z_m - (\mathcal{F}_u(\hat{M}, \Delta_u)W + \mathbf{V}\mathbf{1}). \quad (19)$$

In the frequency response-based model validation problem, the goal is to determine the minimum  $\gamma$  such that

$$\varepsilon = 0 \quad \forall \omega_i \in \Omega^{\text{val}}. \quad (20)$$

In (19), the nominal model error  $\varepsilon^{\text{nom}} := Z_m - \hat{M}_{22}W$  has to be explained by the disturbance term ( $V$ ) and the model uncertainty term ( $\Delta$ ). The following proposition is the main result of this section.

**Proposition 11** *Let  $W$  increase with a factor  $\alpha$ . Then, the part of the nominal modeling error  $\varepsilon^{\text{nom}}$  that can be attributed to disturbances decreases by a factor  $\alpha$ .*

*Proof:* Omitted due to space limitations. ■

The key idea in a frequency response-based approach is that besides increasing the amplitude in the time domain, the size of  $W$  can be increased by applying a periodic input signal. Specifically, suppose that  $w(t)$  is periodic with period  $N$  and suppose that  $n_{\text{per}}$  periods are applied. Then for  $\omega_i \in \left\{ \frac{2\pi p}{N}, p = 0, 1, \dots, N-1 \right\}$ ,

$$W_{n_{\text{per}}N}(\omega_i) = \frac{1}{\sqrt{n_{\text{per}}N}} \sum_{t=1}^{n_{\text{per}}N} w(t)e^{i\omega_i t} = \sqrt{n_{\text{per}}}W_N(\omega_i), \quad (21)$$

where  $W_N(\omega_i)$  is the discrete Fourier transform of  $w(t)$  over one period. Hence, in virtue of Proposition 11, the model uncertainty averages out with a factor  $n_{\text{per}}$  if the measurement time is increased with a factor  $n_{\text{per}}$ .

## V. VALIDATION TEST

## A. Motivation for a frequency response based approach

In section II, inconsistency of the FDMVDP for at least one frequency  $\omega_i \in \Omega$  is a sufficient condition for model invalidation. The following proposition reveals it is also necessary, providing a strong motivation for a frequency response based approach.

**Proposition 12** *A model is not invalidated, i.e., a  $\Delta_u \in \mathcal{RH}_\infty, \|\Delta_u\|_\infty < \gamma$  exists, if and only if the FDMVDP has a positive answer  $\forall \omega_i \in \Omega^{\text{val}}$  and  $\gamma$ .*

*Proof:* Only a sketch of the proof is given. ( $\Rightarrow$ ) Follows trivially from the definition of the  $\mathcal{H}_\infty$  norm. ( $\Leftarrow$ ) Follows by considering the bilinear transformation  $z = \frac{1}{wr}$  and the right tangential Nevanlinna-Pick interpolation problem [24, Chapter 18]. ■

## B. Solution to the FDMVDP

In this section, a solution to the FDMVDP is provided. The FDMVDP amounts to verifying consistency of the model with the data at each frequency  $\omega_i \in \Omega^{\text{val}}$ . The consistency equation, see (19), becomes

$$0 = Z_m - \mathcal{F}_u(M, \Delta)W - T_v \Delta_v \bar{V} \mathbf{1}, \quad (22)$$

where  $\Delta_v \in \mathcal{B} \Delta_v, \Delta_u \in \Delta_u, \bar{\sigma}(\Delta_u) < \gamma$ . Straightforward matrix manipulations reveal that (22) is equivalent to

$$\mathcal{F}_u(\bar{M}, \bar{\Delta})\alpha = 0, \quad (23)$$

where

$$\bar{M} = \left[ \begin{array}{cc|c} 0 & 0 & \bar{V} \mathbf{1} \\ 0 & \gamma M_{11} & -M_{12} W \\ \hline -T_V & \gamma M_{21} & Z_m - M_{22} W \end{array} \right] \quad (24)$$

$$\bar{\Delta} \in \mathcal{B} \bar{\Delta} \quad (25)$$

$$\bar{\Delta} = \left\{ \left[ \begin{array}{cc} \Delta_v & 0 \\ 0 & \Delta_u \end{array} \right] \mid \Delta_v \in \Delta_v, \Delta_u \in \Delta_u \right\}, \quad (26)$$

and  $\alpha \in \mathbb{C} \setminus \{0\}$ . The FDMVDP amounts to verifying whether there exist nonzero signals that satisfy the LFT (23). This resembles a Structured Singular Value (SSV) test [25], however, instead of considering an autonomous feedback interconnection, the LFT in (23) is implicit, i.e., it has an output equal to zero and an input. This motivates the following generalization of the SSV.

**Definition 13 (Generalization of the SSV)** [26] *Given complex matrices  $M, N$  of appropriate sizes,  $\bar{\mu}_\Delta(M, N)$  is defined as*

$$\bar{\mu}_\Delta(M, N) := \left( \min_{\Delta} \{ \bar{\sigma}(\Delta) \mid \Delta \in \Delta, \text{Ker} \begin{pmatrix} I - \Delta M \\ N \end{pmatrix} \neq \{0\} \} \right)^{-1} \quad (27)$$

*unless  $\text{Ker} \begin{pmatrix} I - \Delta M \\ N \end{pmatrix} \neq \{0\}, \forall \Delta \in \Delta$ , in which case  $\bar{\mu}_\Delta(M, N) := 0$ .*

This leads to the following necessary and sufficient test for the FDMVDP, where  $\bar{X}_{22}$  exists under Assumption 6.

**Proposition 14** *In the FDMVDP, the model is not invalidated if and only if  $\bar{\mu}_\Delta(\bar{M}_{11} - \bar{M}_{12} \bar{X}_{22} \bar{M}_{21}, \bar{M}_{21} - \bar{X}_{22} \bar{M}_{22} \bar{M}_{21}) > 1$ , where  $\bar{X}_{22}$  is a matrix that satisfies  $\bar{X}_{22} \bar{M}_{22} = I$ .*

*Proof:* Proceeds along the same lines as in [26]. ■

## C. Algorithm

The computational complexity of  $\bar{\mu}_\Delta(M, N)$  is comparable to  $\mu$ , which is proven to be NP-hard, see [27]. Upper and lower bounds for  $\bar{\mu}_\Delta(M, N)$  have been suggested in [26] and are implemented in [28]. Testing whether the upper bound  $\bar{\mu}_\Delta^u(N, M) \leq 1$  is a sufficient but not necessary condition for model invalidation, hence testing whether  $\bar{\mu}_\Delta^u(N, M) > 1$  is an optimistic approach to model validation.

## VI. EXAMPLE

In this section, an example is provided to illustrate 1) the estimation of nonparametric disturbance models, 2) the FDMVDP, and 3) the averaging properties in an optimistic, deterministic model validation.

The considered true system  $M_o$  and model  $\hat{M}$ , see (15) and (1) are given by

$$M_o = \begin{bmatrix} 1 & 0.1 \\ -0.1 & 1 \end{bmatrix}, \quad \hat{M} = \left[ \begin{array}{c|c} 0_{2 \times 2} & I_2 \\ \hline I_2 & I_2 \end{array} \right], \quad (28)$$

respectively, i.e., both are static MIMO systems and  $\hat{M}$  is equipped with an unweighted, unstructured additive perturbation model [4]. The true disturbance system  $H_o$ , see (2), is dynamic and is defined by the state-space realization

$$H_o = \left[ \begin{array}{c|cc} 0.1 & 1 & 1 \\ \hline 0.45 & 0.5 & 0.5 \\ 0.225 & 0.25 & 1.25 \end{array} \right]. \quad (29)$$

Problem 2 is considered for one frequency, i.e.,  $\omega_i = 0.4\pi$ , and one input direction, hence the input  $w$  is defined as

$$w(t) = 0.4 \begin{bmatrix} 1 & 1 \end{bmatrix}^T \sin(\omega_i t). \quad (30)$$

To illustrate accuracy of the estimator (17), validity of the circularity condition (4), and averaging properties of the deterministic disturbance model, see Section IV, the model validation problem is considered for  $n_{\text{per}} = \{10, 100, 1000, 10000\}$ .

The first step in the model validation problem is the estimation of the nonparametric disturbance model for frequency  $\omega_i$ . The 2-norm of the difference between the estimator (17) and the theoretical limit for  $N \rightarrow \infty$  of  $C_{v_s}(\omega_i)$ , which is given by  $C_{v_s}(\omega_i) = H_o(\omega_i)H_o^*(\omega_i)$ , is depicted in Figure 2. It is concluded that the error of the estimator converges to zero for increasing  $n_{\text{per}}$ . Additionally, a similar estimator as (17) is used to verify the circularity condition (4). The 2-norm of the estimate is depicted in Figure 2. Clearly, the circularity condition is satisfied for sufficiently large  $n_{\text{per}}$ .

Next, the estimate  $\hat{C}_{v_s}(\omega_i)$  is used to construct a deterministic disturbance model as described in Section III-B. Then, the minimum value of  $\gamma$  is determined such that the uncertain model is not invalidated by the data using the approach in Section V. The results for varying  $n_{\text{per}}$  are depicted in

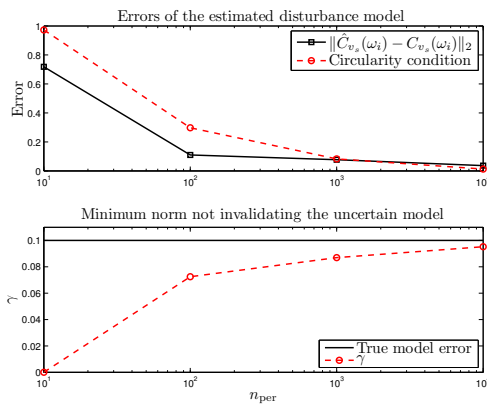


Fig. 2. Simulation results.

Figure 2. For  $n_{\text{per}} = 10$ ,  $\gamma = 0$ , implying that there are no indications in the data set that the nominal model is incorrect, *i.e.*, the nominal model error can be fully attributed to disturbances. By increasing  $n_{\text{per}}$ ,  $\gamma$  converges to the value  $\bar{\sigma}(M_o - \hat{M}_{22}) = 0.1$  for this input direction. Hence, by an appropriate input design in conjunction with a nonparametric, deterministic disturbance model, the disturbances average out in an optimistic model validation setting.

## VII. CONCLUSIONS

In this paper, model validation for robust control has been investigated. Typical deterministic model validation approaches are ill-posed, since a trade-off between disturbances and model uncertainty exists. The fundamental reason for this trade-off is that the notion of a disturbance is poorly defined in a deterministic framework. Instead, the main contribution of this paper is a well-defined framework for deterministic model validation by employing accurate, nonparametric, deterministic disturbance models and enforcing averaging properties of deterministic disturbances.

In the disturbance modeling, there is freedom with respect to the confidence interval parameterized by  $\alpha$ . A large value of  $\alpha$  results in more optimistic results. In realistic situations, it is expected that the value of  $\alpha$  is of little significance and should be chosen close to 1. In particular,  $\alpha$  close to 1 ensures that a poor data set does not result in an overly large systematic modeling error. A high quality data set in turn typically corresponds to a large time interval, allowing the disturbance to average out, resulting in a realistically estimated systematic model uncertainty.

As discussed in the example, model validation for multivariable systems results in a validated model for a particular input direction. To validate the entire model, multiple validation experiments have to be performed such that the entire input space is spanned [18]. In addition, the structure of  $\Delta$  in the multivariable situation involves further research, since in this case the uncertainty can be attributed to different blocks of the perturbation model, introducing a trade-off.

The resulting frequency dependent value of the FDMVOP can be used for uncertainty modeling by overbounding it for each block of  $\Delta_u$  by a bistable transfer function. Concluding, the procedure results in an accurate uncertainty and disturbance model, suitable for subsequent control design.

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