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Revealed Likelihood and Knightian Uncertainty

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Abstract

Nonadditive expected utility models were developed for explaining preferences in settings where probabilities cannot be assigned to events. In the absence of probabilities, difficulties arise in the interpretation of likelihoods of events. In this paper we introduce a notion of revealed likelihood that is defined entirely in terms of preferences and that does not require the existence of (subjective) probabilities. Our proposal is that decision weights rather than capacities are more suitable measures of revealed likelihood in rank-dependent expected utility models and prospect theory. Applications of our proposal to the updating of beliefs and to the description of attitudes towards ambiguity are presented.

Key words: Ellsberg paradox, nonadditive probability, Choquet-expected utility, rank-dependent utility, prospect theory, updating of beliefs

JEL Classification: D81

It has long been recognized that there is a distinction between risk, where probabilities are known, and uncertainty, where probabilities are unknown (Keynes, 1921; Knight, 1921). In a seminal work, Savage (1954) argued that for a rational agent such a distinction is not relevant. In his framework, probabilities measure the likelihood of events. A key idea in Savage's theory is that probabilities are revealed from preferences rather than from introspection or verbal reports.

There is, however, a large body of empirical evidence that contradicts Savage's subjective expected utility model (Camerer and Weber, 1992). In particular, Ellsberg (1961) showed that Savage's method for revealing probability leads to inconsistencies, i.e., probabilities cannot always be assigned to events in that manner. Ellsberg's concerns about the inadequacy of probability have been satisfactorily addressed by the nonadditive probability model developed by Schmeidler (1989) and Gilboa (1987). This model is called Choquet-expected utility (CEU) hereafter, and is assumed throughout our analysis. Our analysis can also be applied to cumulative prospect theory developed by Starmer and Sugden (1989), Luce and Fishburn (1991), and Tversky and Kahneman (1992). Cumulative prospect theory generalizes CEU by permitting decision weights for gains to be different than decision weights for losses and has a number of empirical advantages. Our analysis can be applied to gains and losses separately.

We propose a notion of revealed likelihood that corresponds with decision weights, rather than capacities, of events. As will be demonstrated, the revealed likelihood of an event thus depends on the "dominating" event, i.e., the event that yields superior consequences. Our proposal introduces some, be it minimal, dependency of beliefs on tastes, which may be the price to pay for giving up expected utility. Revealed likelihood sheds new light on a number of issues in rank-dependent theories, such as a duality paradox, the various definitions of null events and updating, and the interpretation of capacities as belief. Let us emphasize that revealed likelihood is entirely based on preferences, i.e., refers to subjective perceptions in a Savagean sense.

In Section 1, we review the notion of likelihood in subjective expected utility theory. Section 2 discusses the discrepancy between likelihood revealed from bets on and bets against events that is commonly found in the Ellsberg examples. In Section 3, we argue that in CEU, one needs to distinguish between revealed likelihoods derived from bets on events and revealed likelihoods derived from bets against events. This distinction is already a first step towards the dependency of revealed likelihood on dominating events, proposed later in this paper. Section 4 shows that in the derivation of CEU one may use preference conditions in a consistent way so long as one employs the appropriate notion of revealed likelihood. This solves a duality paradox noted in the literature. In Section 5, we generalize revealed likelihood to the multiple consequences case. We argue that, if revealed likelihood should "tell you where to put your money," then decision weights are the proper measure of revealed likelihood under CEU. Section 5 also sheds new light on axiom P2 of Gilboa (1987). It shows how that axiom can be used to empirically elicit orderings of decision weights, thus revealed likelihood as proposed in this paper.

An attractive property of expected utility is independence of beliefs from tastes. In Section 6, we argue that to some degree independence of revealed likelihood from consequences can be maintained in CEU so long as one specifies a dominating event. In Section 7, we argue that decision weights have some distinct advantages over capacities in measuring revealed likelihood. Section 8 illustrates an application of our measure of revealed likelihood in defining null events which is an important issue for updating and for the definition of Nash equilibrium in game theory. Several other properties of decision weights as measure of revealed likelihood are discussed. For example, a new interpretation is provided for the case of probabilistic sophistication (Machina and Schmeidler, 1992; Epstein and Le Breton, 1993). Section 9 discusses updating if new information is gathered. Several proposals for updating in the literature can be explained as different choices of the dominating events introduced in Section 6. In Section 10, we discuss the interpretation of revealed likelihood as a measure of belief. Revealed likelihood may depend both on beliefs and on decision attitudes. Finally, Section 11 presents conclusions. Proofs are presented in the appendix.

1. Subjective expected utility

In subjective expected utility (SEU), the likelihood of an event is measured by its subjective probability. Thus,

Event A is more likely than event B
if and only if
the probability of A is greater than the probability of B.

In the above statement the likelihood judgments are quantified by a probability measure. Thus, we write

$$A > B \text{ if only if } P(A) > P(B). \tag{1}$$

Subjective probabilities are often interpreted as a measure of degree of belief, reflecting the state of information of the decision maker. For the two consequence case, the likelihood relation can be operationalized in either of the following two equivalent ways:

A is more likely than B if one prefers a bet on A to a bet on B (Figure 1a).

A is more likely than B if one prefers a bet against A less than a bet against B (Figure 1b).

Thus, "more likely than" judgments are revealed through preference comparisons between various "win-lose" bets. Implicit in the betting method for revealing likelihood is the assumption that the likelihood comparison of events A and B is independent of the pairs of consequences used. This independence is ensured through Savage's axiom P4.

It is easy to verify that for an SEU maximizer either preferences in Figure 1a or Figure 1b would reveal the same likelihood relation. The preferences in Figure 1a reveal $P(A)>P(B)$ and those in Figure 1b reveal $P(A^c)=1-P(A)<P(B^c)=1-P(B)$, each leading to the conclusion that A is revealed to be more likely than B.

The desirability of eliciting likelihoods using bets on events or bets against events depends on the decision context. In theoretical analyses, likelihoods have mostly been inferred using bets on events, as in Figure 1a (Sarin and Wakker, 1992; Epstein and Le Breton, 1993). They can, however, just as well be elicited using bets against events as in Figure 1b. There is no prior reason to prefer one method over the other, though in practical applications one of the two methods may be more convenient. A large part of our risky decisions concerns avoidance of unfavorable events, in which case it is natural to think in terms of bets against events. Examples are health care, safety measures, and insurance. In the next section the choice of method will be more than a matter of practical convenience and will lead to conceptual differences.

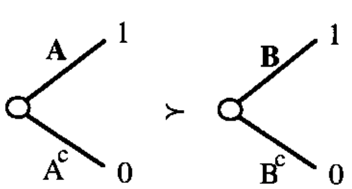

Figure 1a.

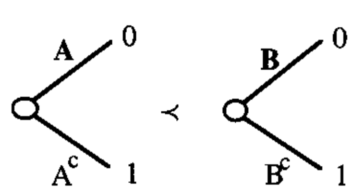

Figure 1b.

2. Revealed qualitative likelihood

Ellsberg (1961) showed that empirically the two ways of operationalizing likelihood as in Figure 1 do not lead to the same result for some events. To illustrate this violation of SEU, consider two urns, one containing 50 white and 50 black balls, and the other containing a total of 100 white and black balls in unknown proportion (Figure 2). From each of the two urns a ball is randomly drawn. People often prefer a bet on event K (white from known urn) to a bet on event U (white from unknown urn), while preferring a bet on K^c (black from known urn) to a bet on U^c (black from unknown urn).

The pattern of preferences in Figures 2a and 2b implies that event K is *revealed more likely* than event U when one derives the likelihood relation from bets on events (Figure 2a), and event K is *revealed less likely* than event U when one derives it from bets against events (Figure 2b). In Figure 2b, one loses (fails to win) on events K and U and the inference that K is less likely than U is guided by the intuition that one should prefer to lose on the less likely event.

The example demonstrates that a revealed likelihood relation derived from bets on events may differ from that derived from bets against events. To distinguish these two notions of revealed likelihood we introduce the following notation. We write $\succcurlyeq^{\uparrow}$ for revealed likelihood derived from bets on events. That is, $A \succcurlyeq^{\uparrow} B$ if there exist consequences $x \succ y$ such that

$$(A,x;\, A^c,y) \succcurlyeq (B,x;\, B^c,y) \tag{2}$$

where $\succcurlyeq$ denotes weak preference and $(A,x;A^c,y)$ denotes the act yieding x if A occurs and y otherwise. (Conditions will later be studied that guarantee that (2) holds for all $x \succ y$ as soon as it holds for some $x \succ y$.) We write $\succ^{\uparrow}$ instead of $\succcurlyeq^{\uparrow}$ if the preference in (2) is strict.

Similarly, we write $\succcurlyeq^{\downarrow}$ for revealed likelihood derived from bets *against* events. That is, $A \succcurlyeq^{\downarrow} B$ if there exist consequences $x \succ y$ such that

$$(A^c,x;\, A,y) \preccurlyeq (B^c,x;\, B,y)$$

where $\preccurlyeq$ denotes reversed preference ($f \preccurlyeq g$ meaning $g \succcurlyeq f$). Again, $\succ^{\downarrow}$ denotes strict preference. Under SEU, $\succcurlyeq^{\uparrow}$ and $\succcurlyeq^{\downarrow}$ coincide. In the elicitation of $\succcurlyeq^{\uparrow}$ a superior consequence is associated with events A and B, that is, A and B play the role of *good-news events*. They describe the receipt of a consequence or anything better than that conse-

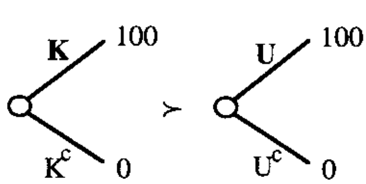

Figure 2a.

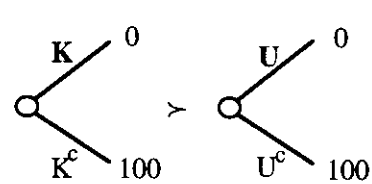

Figure 2b.

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quence. In contrast, in the elicitation of $\succcurlyeq^{\downarrow}$ an inferior consequence is associated with events A and B, hence these events play the role of *bad-news events.* They describe the receipt of a consequence or anything worse than that consequence. The preference pattern observed in the Ellsberg paradox implies $K \succ^{\uparrow} U$ but $U \succ^{\downarrow} K$ and thus constitutes a violation of SEU. The pattern can be explained by pessimism with respect to unknown probabilities (ambiguity aversion), where the likelihood of winning with unknown probability is downplayed and the likelihood of losing with unknown probability is exaggerated.

It is useful to note the following duality between $\succcurlyeq^{\uparrow}$ and $\succcurlyeq^{\downarrow}$:

$$A \succcurlyeq^{\uparrow} B \Leftrightarrow B^c \succcurlyeq^{\downarrow} A^c. \tag{3}$$

The left-hand side says that a bet on A is preferred to a bet on B. As a bet on A is a bet against A^c and a bet on B is a bet against B^c, this means that a bet against B^c is preferred less than a bet against A^c, which is the right-hand side. In other words, both the left-hand side and the right-hand side describe the preference in Figure 1a. The preferences are dual ways for describing the same empirical phenomenon.

3. Choquet-expected utility

We assume that consequences are amounts of money and that preferences satisfy monotonicity, i.e., higher amounts are preferred to lower amounts. S denotes the *state space*; S may be finite or infinite. *Events* A, B, etc. are subsets of S. We restrict attention to simple acts (i.e., acts that take only finitely many different consequences) throughout the paper. In Choquet-expected utility, a "capacity" ν is used instead of the additive probability measure P of SEU. It is assumed that ν assigns value 0 to the impossible event and value 1 to the universal event S, and that $A \supset B$ implies $\nu(A) \geq \nu(B)$. The CEU value of an act $(A_1,x_1;\cdots;A_n,x_n)$, with $x_1 \geq \cdots \geq x_n$, is given by

$$\sum_{i=1}^{n} \pi_i U(x_i) \tag{4}$$

where U is the utility function as in SEU and the π_is denote *decision weights*, defined by

$$\pi_i = \nu(A_1 \cup \cdots \cup A_i) - \nu(A_1 \cup \cdots \cup A_{i-1}). \tag{5}$$

Note that the decision weight of A_i depends on the rank-ordering of consequences or, at least, on the "dominating" event $A_1 \cup \cdots \cup A_{i-1}$. That dependence is not expressed in notation but should be kept in mind. Similar formulas are used for cumulative prospect theory, except that the capacity for gains can be different than the capacity for losses. CEU permits the preference patterns observed in the Ellsberg paradox by setting $\nu(K) > \nu(U)$ and $\nu(K^c) > \nu(U^c)$. Under CEU the following results hold:

(i) $A \succcurlyeq^{\uparrow} B$ if and only if $\nu(A) \geq \nu(B)$.
(ii) $A \succcurlyeq^{\downarrow} B$ if and only if $1\text{-}\nu(A^c) \geq 1\text{-}\nu(B^c)$.

Thus, $\nu(A)$ represents the $\succcurlyeq^{\uparrow}$ ordering, derived from bets on events, and its dual $1\text{-}\nu(A^c)$ represents the $\succcurlyeq^{\downarrow}$ ordering, derived from bets against events. For this reason, we write $\nu^{\uparrow}(A)$ for $\nu(A)$ and $\nu^{\downarrow}(A)$ for $1\text{-}\nu(A^c)$. $\nu^{\uparrow}$ is the capacity for events in the role of good-news events (i.e., an event describing the receipt of a consequence or anything better) and $\nu^{\downarrow}$ is the capacity for events in the role of bad-news events. In SEU, $\nu^{\uparrow} = \nu^{\downarrow} = P$. In CEU, however, $\nu^{\uparrow}$ and $\nu^{\downarrow}$ need not be identical.

The previous discussion is based on a duality between good- and bad-news events. As there has been confusion about this duality in the literature, and it is central for our measure of likelihood, we discuss it in some detail. The duality has also been discussed for Choquet integration. In the literature, an alternative way for defining Choquet integrals that is dual to Formula (5) has been used. This dual Choquet integral is obtained by defining

$$\pi_i = \nu(A_i \cup \cdots \cup A_n) - \nu(A_{i+1} \cup \cdots \cup A_n) \tag{6}$$

instead of (5), and using the π_is of (6) in Formula (4). The decision weight π_1 in (6) now is equal to $1 - \nu(A_2 \cup \cdots \cup A_n)$ instead of $\nu(A_1)$ as it was in (5). Equivalently, one can order consequences alternatively by $x_1 \leq \cdots \leq x_n$ and then use Formula (5). Reversing the rank-ordering of consequences and using Formula (5) gives the same result as keeping the rank-ordering of this paper and using Formula (6).

The method of integration through (6) is called the *lower Choquet integral*, and similarly the method of integration through (5) is called the *upper Choquet integral*. Clearly, these may yield different orderings of acts. Thus the question arises which formula for computing CEU is the "right" one and how the seeming inconsistency between (5) and (6) can be resolved. There is no inconsistency, however, between (5) and (6) if the relevance of the role of events is recognized. That is, (5) entails good-news events $A_1 \cup \cdots \cup A_i$ (receive x_i or more) and therefore $\nu^{\uparrow}$ should be used there. Formula (6) entails bad-news events $A_i \cup \cdots \cup A_n$ (receive x_i or less) and therefore $\nu^{\downarrow}$ should be used. In this manner, the two methods for computing CEU yield identical results. Note that this consistency is obtained in general and it does not impose restrictions on capacities such as "symmetry."

Imagine now that a person uses the capacity $\nu^{\uparrow}$, elicited from bets on events, but uses Formula (6) to calculate CEU. In this case the capacity $\nu^{\uparrow}$ for good-news events is applied to bad-news events in (6). For *symmetric* capacities ($\nu(A) = 1 - \nu(A^c)$, i.e., $\nu^{\uparrow} = \nu^{\downarrow}$), the scheme results in the correct CEU values after all. For non-symmetric capacities, this mis-matching of capacity and integration will produce wrong results (Gilboa, 1989a). The question of which capacity to use, $\nu^{\uparrow}$ or $\nu^{\downarrow}$, and the question of which method of integration to use, (5) or (6), are not meaningful in isolation. They must be considered jointly and applied consistently.

The following linguistic example may illustrate the idea of mis-matching. It is now well-accepted that an author may use male-specific pronouns (he/his/him) or female-specific pronouns (she/her) to designate an abstract person (decision maker, agent, defendant). There is no reason to prefer a choice of "he" to a choice of "she," and there is no reason to prefer a choice of "him" to a choice of "her." These two choices, however, are intertwined and cannot be made independently. An argument to the effect that "he" could be replaced by "she" without recognizing the interdependence of the he/she choice with the his/her choice would lead to anomalies such as "he maximizes her utility." Clearly a mis-match of the pronouns along the way yields an unintended implication of altruism. The sentences "he maximizes his utility" and "she maximizes her utility" are truly dual to each other and either one is acceptable.

The main point in the above discussion has been that the revealed likelihood ordering ($\succcurlyeq^{\uparrow}$ or $\succcurlyeq^{\downarrow}$), the capacity ($\nu^{\uparrow}$ or $\nu^{\downarrow}$), and the manner of integration (upper or lower) should be consistent with the role of events. For the good-news events $\succcurlyeq^{\uparrow}$, $\nu^{\uparrow}$, and upper integration should be used. For the bad-news events $\succcurlyeq^{\downarrow}$, $\nu^{\downarrow}$, and lower integration should be used. Good-news or bad-news events are dual in the same way as the male or female gender are in the linguistic example. There is a complete freedom to choose the role of events in CEU and the gender in the linguistic example, as long as consistency is maintained throughout.

4. Cumulative dominance

In Sarin and Wakker (1992), CEU is characterized by using a cumulative dominance condition. *Cumulative dominance* states that act f is weakly preferred to act g whenever, for all consequences x, the good-news event of receiving x or more under f is revealed at least as likely as the good-news event of receiving x or more under g. As this formulation employs good-news events, the revealed likelihood for good-news events ($\succcurlyeq^{\uparrow}$) should be adopted. We display the condition:

$$f \succcurlyeq g \text{ whenever, for all consequences x, } [f \geq x] \succcurlyeq^{\uparrow} [g \geq x]. \tag{7}$$

Next an equivalent dual formulation is given, in terms of bad-news events. *Dual cumulative dominance* hold if

$$f \succcurlyeq g \text{ whenever, for all consequences x, } [g \leq x] \succcurlyeq^{\downarrow} [f \leq x]. \tag{8}$$

The condition states that act f is weakly preferred to act g whenever, for all consequences x, the bad-news event of receiving x or less under g is revealed at least as likely as the bad-news event of receiving x or less under f. As this formulation employs bad-news events, the revealed likelihood relation for bad-news events ($\succcurlyeq^{\downarrow}$) is adopted. The proof of the following observation illustrates the duality, hence is presented in the main text. It shows that the two dominance conditions are not just logically equivalent in the presence of other axioms but, moreover, the two conditions provide dual ways for describing exactly the same empirical phenomenon.

Observation 4.1 Cumulative dominance holds if and only if dual cumulative dominance holds.

Proof In the dual formulation, the premise requires for every consequence x that $[g \leq x] \succcurlyeq^{\downarrow} [f \leq x]$. By (3), that is equivalent to $[f \leq x]^c \succcurlyeq^{\uparrow} [g \leq x]^c$, i.e., $[f > x] \succcurlyeq^{\uparrow} [g > x]$. In view of the finite ranges of f and g, it is readily seen that the requirement is equivalent to the requirement that for every consequence y, $[f \geq y] \succcurlyeq^{\uparrow} [g \geq y]$. (This equivalence holds for general consequence sets, not just for the reals.) In other words, the premises of the two dominance conditions describe the same empirical phenomenon. □

The important point to note in the preceding analysis is that the revealed likelihood relation should be consistent with the role of the events. To further illustrate that point, we discuss a variation of cumulative dominance in which the preference condition involves bad-news events but the revealed likelihood-relation adopted is the one for good-news events. In other words:

$$f \succcurlyeq g \text{ whenever, for all consequences x, } [g \leq x] \succcurlyeq^{\uparrow} [f \leq x]. \tag{9}$$

Consider two-consequence acts $f=(A,x; A^c,y)$ and $g=(B,x; B^c,y)$, $x > y$. Clearly, $f \succcurlyeq g$ if and only if $A \succcurlyeq^{\uparrow} B$. Condition (9), however, would require that $f \succcurlyeq g$ if $B^c \succcurlyeq^{\uparrow} A^c$, i.e., (by Formula 3) if $A \succcurlyeq^{\downarrow} B$. Thus, $A \succcurlyeq^{\downarrow} B$ would imply $A \succcurlyeq^{\uparrow} B$ which was precisely the restriction we wished to relax to accommodate the Ellsberg paradox. In other words, the mismatch of (bad-news) events and the (good-news) likelihood relation in (9) leads to unwarranted implications (Nehring, 1994). This consitutes the same mis-matching as described at the end of Section 3 and illustrated there by the linguistic example.

Next we demonstrate that cumulative dominance and dual cumulative dominance are necessary conditions for CEU. The elementary proof (given in the appendix) further clarifies the duality between the two dominance conditions and shows that this duality is a qualitative analog of the duality between upper and lower Choquet integration.

Observation 4.2 Cumulative dominance and dual cumulative dominance are necessary conditions for CEU. □

Cumulative dominance has a resemblance to stochastic dominance when probabilities are given. Although this resemblance makes the condition transparent, it should be understood that cumulative dominance does not have the normative appeal of stochastic dominance. This is because, unlike stochastic dominance, cumulative dominance cannot be derived from a statewise monotonicity condition.

Let us summarize the discussion in Sections 2, 3, and 4. Section 2 discusses the duality between "good-news" and "bad-news" events in CEU. In a quantitative setting, this duality was discussed by Gilboa (1989a) and in a qualitative setting it was discussed by Nehring (1994). Our discussion starts in the qualitative context of revealed likelihood orderings and addresses the issue of whether these orderings should be inferred from bets on or bets against events. In Section 3, the same issue is discussed in its quantitative version, i.e., whether a capacity or its dual should be used to measure revealed likelihood. The same duality also underlies the discussion whether one should do Choquet integration in the "upper" version or in the dual, "lower," version. In Section 4, we present a preference condition, cumulative dominance, that was used to characterize CEU by Sarin and Wakker (1992). The distinctin between cumulative dominance and its dual is analogous to the distinction between upper and lower integration. Again, the good-news likelihood ordering should be used for cumulative dominance and the bad-news likelihood ordering should be used for dual cumulative dominance. Let us finally mention that the analysis of Sections 2, 3, and 4 holds for general consequence sets and need not be restricted to monetary consequences.

The approach developed so far boils down to a simple prescription: When defining revealed likelihood and capacities, and applying these to preference conditions and Choquet integration, one should be consistent regarding the role of events. This prescription will be elaborated in the next part of the paper.

5. Events with intermediate consequences

So far we have discussed revealed likelihood of events when they are associated with best or worst consequences. In the more general multiple-consequence case, some events have intermediate consequences. We examine revealed likelihoods of such events.

REMARK. From now on, in the rest of the paper, we assume CEU. □

We will mainly consider "connected" events; these will turn out to be especially suited for our interpretations of revealed likelihood. An event is *connected* if each state outside the event either is lower in rank-ordering than all states of the event, or is higher in rank-ordering than all states of the event, but never is in between the states of the event. For example, for a given act f the event $\{s \in S: x \leq f(s) \leq y]$ is connected. Every event that has a constant consequence is connected. Throughout this paper, many notions depend on the considered act or, more precisely, on the rank-ordering of states that is presupposed. That also holds for the definition of connected events.

To illustrate the general idea of revealing likelihood for intermediate events, assume an indifference

$$(A_1,10; \mathbf{A_2,2;}\ A_3,1) \sim (B_1,12; \mathbf{B_2,2;}\ B_3,0).$$

In this case, events A_2 and B_2 are associated with an intermediate consequence and our interest is in comparing the revealed likelihoods of A_2 and B_2. Suppose we ask the question what is preferred, receiving an additional dollar under A_2 or under B_2. That is, what is the preference between

$(A_1,10;\ \mathbf{A_2,3;}\ A_3,1)$ and $(B_1,12;\ \mathbf{B_2,3;}\ B_3,0)$?

An intuitive reply may be that the additional dollar is preferred for the "more likely" event. Thus, if the left act is preferred then A_2 is "more likely" than B_2. In this context, A_2 and B_2 are neither good-news events nor bad-news events as they are associated with intermediate consequences. An SEU maximizer will prefer the left act if and only if $P(A_2) > P(B_2)$. The initial indifference and the preference for the left act together imply that the SEU increment for the left act, $P(A_2)(U(3)-U(2))$, is higher than $P(B_2)(U(3)-U(2))$, the SEU increment for the right act. A CEU maximizer will prefer the left act if and only if $\pi(A_2)>\pi(B_2)$, where $\pi(A_2)$ denotes the decision weight of A_2 and $\pi(B_2)$ the decision weight of B_2. This is because the CEU increment for the left act, $\pi(A_2)(U(3)-U(2))$ is higher than $\pi(B_2)(U(3)-U(2))$, the CEU increment for the right act. Since decision weights reflect where one would stake the bet, they can be a plausible measure of revealed likelihood.

We further discuss the issue of interpretation after stating a preference condition for comparing revealed likelihoods through decision weights. The condition is based on Gilboa's (1987) condition P2* (see also Gilboa, 1989a)[1] that contains an intuitive and empirically valuable idea for CEU: It shows a way for comparing decision weights.

Suppose that $\beta > \alpha$ and

$$(A_1,x_1;\ \cdots;\ A_{i-1},x_{i-1};\ \mathbf{A_i},\alpha;\ A_{i+1},\ x_{i+1};\ \cdots;\ A_n,x_n)$$

$$\sim$$

$$(B_1,y_1;\ \cdots;\ B_{j-1},y_{j-1};\ \mathbf{B_j},\alpha;\ B_{j+1},y_{j+1};\ \cdots;\ B_m,y_m)$$

where $x_1 \geq \cdots \geq x_{i-1} \geq \beta > \alpha \geq x_{i+1} \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_{j-1} \geq \beta > \alpha \geq y_{j+1} \geq \cdots \geq y_m$.

Under CEU,

$$(A_1,x_1;\ \cdots;\ A_{i-1},x_{i-1};\ \mathbf{A_i},\beta;\ A_{i+1},\ x_{i+1};\ \cdots;\ A_n,x_n)$$

$$\succcurlyeq$$

$$(B_1,y_1;\ \cdots;\ B_{j-1},y_{j-1};\ \mathbf{B_j},\beta;\ B_{j+1},y_{j+1};\ \cdots;\ B_m,y_m)$$

if and only if the decision weights satisfy $\pi(A_i)\geq\pi(B_j)$.

In the condition, the incremental impact of A_i is equal to $\pi(A_i)(U(\beta)-U(\alpha))$ whereas the incremental impact of B_j is $\pi(B_j)(U(\beta)-U(\alpha))$. One therefore prefers to stake an additional amount of money on the event with the higher decision weight. It is in this sense that one could interpret that the revealed likelihood of A_i is higher than that of B_j

in this decision context. In the next two sections we elaborate on measuring revealed likelihood through decision weights.

We end this section by discussing empirical evidence regarding the likelihood of intermediate events. It has been observed that intermediate events have less impact than extreme events. In other words, the revealed likelihood of an event is lower when it is associated with intermediate consequences than when it is associated with extreme (best or worst) consequences. This phenomenon has not yet been the subject of many theoretical investigations, but there is ample empirical evidence. The phenomenon is described by "bounded subadditivity" for the uncertainty case (Curley and Yates, 1989; Tversky and Fox, 1995; Tversky and Wakker, 1995; Fox, Rogers, and Tversky, 1996; Wu and Gonzalez, 1997) and by inverse S-shaped probability transformation for the risk case (Preston and Baratta, 1948; Yaari, 1965; Cohen and Jaffray, 1988; Viscusi, 1989; Karni and Safra, 1990; Birnbaum, Coffey, Mellers, and Weiss, 1992; Lattimore, Baker, and Witte, 1992; Kachelmeier and Shehata, 1992; Tversky and Kahneman, 1992; Bernasconi, 1994; Camerer and Ho, 1994; Wu, 1994; Tversky and Fox, 1995; Prelec, 1995; Wu and Gonzalez, 1996; Abdellaoui, 1998). Bounded subadditivity underlies the coexistence of insurance and gambling. Counter-evidence has been provided by Birnbaum and McIntosh (1996) and Birnbaum and Chavez (1997).

6. Independence of beliefs from tastes

A well-known property of SEU is the "independence of beliefs from tastes." It means that the likelihood of an event, i.e., its probability, is independent of the consequences that are associated with the event, thus is independent of the particular acts. If one defines revealed likelihood of an event through its decision weight, as we propose, then the revealed likelihood of the event depends on the acts. More precisely, a revealed likelihood is relevant in the evaluation of acts that generate, through their consequences, a given rank-ordering over the state space. Such a subset of acts that generate the same rank-ordering of states is called *comonotonic*. Some degree of independence from tastes is achieved here because the decision weights do not depend on the exact magnitudes of consequences so long as the rank-ordering of the consequences remains constant. In particular, we note that CEU satisfies Savage's P4. That is, if $A \succcurlyeq^{\uparrow} B$ is revealed through $(x,A; y,A^c) \succcurlyeq (x,B; y,B^c)$ for some $x > y$, then for all $x' > y'$, $(x',A; y',A^c) \succcurlyeq (x',B; y',B^c)$, confirming $A \succcurlyeq^{\uparrow} B$. Nevertheless, dependence of decision weights on the rank-ordering of consequences may be considered excessively flexible. It entails a considerable dependence of beliefs on tastes and makes it hard to think of revealed likelihood as a property of events.

We now illustrate how the dependence of the revealed likelihood of events on the rank-order of the consequences can be reduced to such a degree that it becomes better possible to consider revealed likelihood as a property of events. To do so, we introduce the following definition. Event D is a *dominating event* for event A if its states are rank-ordered higher than those of A, and the remaining states are rank-ordered lower than A.

Thus, given an act f, event D is dominating for event A if $A \cap D = \emptyset$ and $f(t) \geq f(s) \geq f(t')$ for all $t \in D$, $s \in A$, and $t' \in (A \cup D)^c$. Because the rank-ordering of states with equivalent consequences can be chosen arbitrarily, we can choose a rank-ordering of states that is compatible with f and is such that the states in D are ranked higher than the states in A, and the latter are ranked higher than those in $(A \cup D)^c$. As we shall see, for a large class of events, the revealed likelihood of A only depends on what the dominating event D of A is. That is, the revealed likelihood is relevant for the subset of all acts for which D is a dominating event for A. Of course, one could equivalently express dependence of likelihood on the dominated, "inferior," event I, i.e. the event yielding inferior consequences, by substituting $D = (A \cup I)^c$.

For simplicity, first think of the case in which A describes the receipt of a single consequence. Then the decision weight of A is given by

$$\nu(A \cup D) - \nu(D), \tag{10}$$

where D denotes the dominating event. The dependence of the decision weight of an event A on the dominating event D can be expressed in notation by writing $\pi(A,D)$. When no confusion can arise, the event D is sometimes suppressed. Implicit in this notation is that A and D are disjoint. The decision weight of an event A can vary depending on whether the dominating event D is $\emptyset$, A^c, or some other event. Thus decision weight, as a measure of revealed likelihood, is a two-argument-function, depending on two events — the event itself and the dominating event. Interpreted thus, decision weights are to a high degree independent of consequences and they depend only on events. We think that a desirable feature of rank-dependent theories is that they obtain this high degree of independence of revealed likelihood from consequences.

The observations just made also hold for connected events that yield more than one consequence. For more general, nonconnected events, decision weights can still be used as an index of revealed likelihood but their dependency on the rank-ordering of the other events is more complex and cannot be described merely by one dominating event. Of course, the decision weight of a nonconnected event can be derived from the decision weights of the separate connected components through summation. We restrict most of the discussion of revealed likelihood in this paper to the class of connected events. The class is rich enough to cover the majority of cases in which likelihood is relevant.

In the Ellsberg example presented in Section 2, $\pi(K,D) > \pi(U,D)$ when $D = \emptyset$ (Figure 2a) and $\pi(K,D') < \pi(U,D'')$ when D' and D'' represent complementary events K^c and U^c respectively (Figure 2b). The following example considers a variation of the three-color Ellsberg paradox.

Example 6.1 Consider an urn containing 30 red (R) balls and 60 yellow (Y) and white (W) balls in unknown proportion. Four bets are illustrated in Table 1.

One may prefer bet 2 over bet 1 and bet 3 over bet 4. The first preference shows that Y is revealed more likely than R when the dominating event is W; i.e., $\pi(Y,W) > \pi(R,W)$. The second preference reveals the reverse ordering; i.e., R is revealed more likely than Y when the dominating event is null ($\pi(R,\emptyset) > \pi(Y,\emptyset)$). □

Table 1.

	R	Y	W
bet 1	**90**	0	100
bet 2	0	**90**	100
bet 3	**90**	0	0
bet 4	0	**90**	0

Figure 3 depicts the decision weight of an event E as a function of the dominating event, for the case of bounded subadditivity. For the purpose of illustration, the dominating events are depicted as if they lie on one line. The decision weight of an event E is largest when the dominating event is maximal (E^c), i.e., all other events are dominating. Then E yields the minimal consequences and its role as compared to the other events is salient. Similarly, the decision weight of E is also large when the dominating event is minimal ($\emptyset$), i.e., no other events are dominating and E yields the best consequences. Then again E's role is salient. The decision weight of E is smaller when the dominating event is neither maximal nor minimal, i.e., when E is associated with intermediate consequences. In this case the role of E in comparison to the other events is less salient.

We concede that decision weights as measures of revealed likelihood in the CEU model are not as elegant as probabilities in the SEU model. For a comonotonic class (fixed rank-ordering), however, decision weights share some common features with probabilities. For example, decision weights sum to one. As a result, if in an n-fold partition of the universal event all decision weights are the same then one immediately concludes that

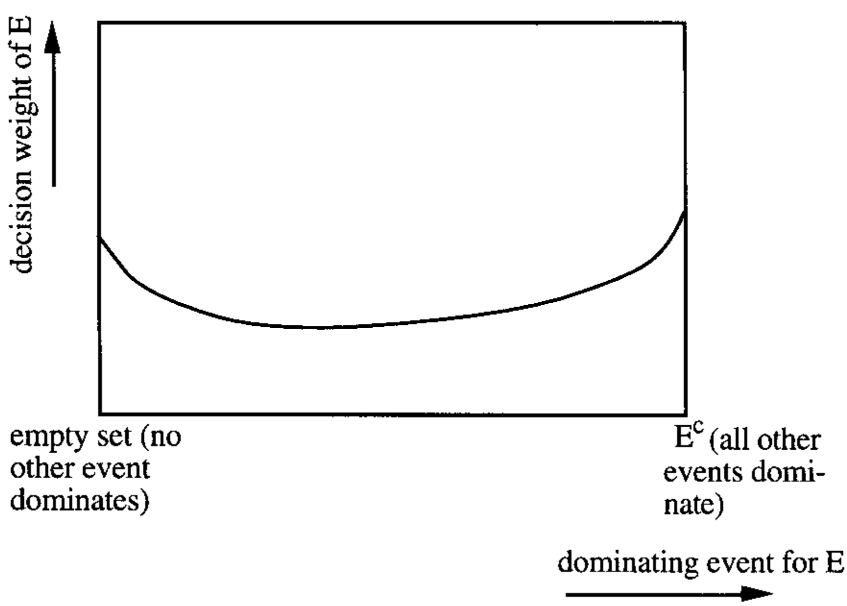

Figure 3. Decision weight of an event E as a function of the dominating event.

they are all 1/n. For a subset of acts for which the dominating event D associated with an event A remains the same, the decision weight of A does not change. Clearly, in comparison to SEU, where the probability of A is independent of what goes on outside of A, revealed likelihood in CEU is more complicated. In CEU, willingness to bet on an event A depends on the dominating event. Since revealed likelihood is elicited from preferences, there seems to be no escape from dependence on dominating events.

To further underscore the analogy between decision weights and probabilities, consider an act $(A_1,x_1;\cdots;A_n,x_n)$ with $x_1 > \cdots > x_n$ and define $U(x_i) = u_i$. The act can be represented as $(A_1,u_1;\cdots;A_n,u_n)$, $u_1 > \cdots > u_n$. In SEU, with U the SEU value of an act, $\delta U/\delta u_i = p_i = P(A_i)$. In a similar manner, $\delta U/\delta u_i = \pi_i = \pi(A_i)$ in CEU, where U represents the CEU value of an act. This observation illustrates once more that decision weights in CEU are the analogs of probabilities in many respects.

7. Capacities versus decision weights

In this section we argue that decision weights have some distinct advantages over capacities in measuring revealed likelihood. We begin with the simple Ellsberg example given in Section 2 to illustrate our viewpoint.

Example 7.1 The capacity-interpretation and the decision-weight interpretation agree on the preference in Figure 2a suggesting a higher revealed likelihood for K than for U. However, the conclusion that K be revealed more likely than U cannot be made in general and is not appropriate in Figure 2b. The preference in Figure 2b illustrates that one prefers to lose on event K rather than on event U. Therefore, event K is revealed less likely than event U. The decision weight of K in Figure 2b is indeed smaller than the decision weight of U. As pointed out at the beginning of this example, the capacity of K is larger than the capacity of U. Therefore the decision weight seems a better measure of revealed likelihood than the capacity. From our perspective, capacities measure revealed likelihood only for events in the role of good-news events and not otherwise. □

In Figure 2b one could use the dual capacity to compare the revealed likelihood of K and U because it so turns out that the dual capacity is indeed the decision weight. In the multiple-consequence case, however, neither the capacity nor its dual will suffice as an index of revealed likelihood. This is illustrated by the following example.

Example 7.2 Consider the preferences in Table 2. Suppose that in the first indifference situation the person is asked if he prefers to receive an additional dollar on event A_2 or on event B_2. Suppose that the person prefers the extra dollar on B_2, as shown in preference 2. Such a preference reveals that the person considers B_2 to be more likely than A_2.

Indeed, the decision weight of B_2 is higher than that of A_2. The capacity, however, produces the reverse ordering of revealed likelihood as shown in preference 3. If the decision situations in preferences 1 and 2 are relevant to us, where A_2 and B_2 play the role of intermediate event, then we think that B_2 is revealed more likely than A_2. For such

Table 2.

	A_1	$\mathbf{A_2}$	A_3		B_1	$\mathbf{B_2}$	B_3
pref. 1	10	**2**	1	~	12	**2**	0
pref. 2	10	**3**	1	$\prec$	12	**3**	0
pref. 3	0	**1**	0	$\succ$	0	**1**	0
pref. 4	10	**1**	10	$\prec$	10	**1**	10

multiple-consequence cases the capacity is not an appropriate index of revealed likelihood. It may be noted that in this example the dual capacity would not be an appropriate index of revealed likelihood either, because of preference 4. Losing on A_2 is less preferred than losing on B_2 which implies that A_2 is revealed more likely than B_2 when they are both bad-news events.

Preferences as described occur when the overweighting of low likelihoods and the underweighting of high likelihoods is more pronounced for the A-events than for the B-events. Then the A-events receive relatively more decision weights than the B-events when they are associated with extreme consequences, and they receive relatively less decision weights when they are associated with intermediate consequences. This phenomenon was characterized by Tversky and Wakker (1995), and is commonly found if the A events are ambiguous and the B events are unambiguous (see Tversky and Wakker, 1998, and the references therein). □

The next example considers null events.

Example 7.3 See Table 3. Here the capacity-interpretation of revealed likelihood suggests, according to the first indifference, that Y is null, which agrees with the decision-weight interpretation for good-news events. We think, however, that the claim that Y be null cannot be accepted in the second preference, where the person strictly prefers receiving an additional dollar on Y if it is the worst event. The decision weight of Y is indeed positive in this case. The preferences in the table result for an extremely ambiguity averse person, where the proportion of R is 1/3 and the proportions of Y and W are unknown (Gilboa and Schmeidler, 1993, example in the introduction). □

Capacities resemble probabilities because they preserve independence of beliefs from tastes. In CEU, however, using capacities as a measure of likelihood introduces arbitrariness. From our perspective, it means that events are implicitly assumed to be good-news events. The capacity has a seductive appeal as a measure of likelihood since it does not depend on the rank-ordering of consequences. In CEU, insisting on a measure of revealed likelihood that is entirely independent of the rank-ordering of consequences (or a domi-

Table 3.

	R	**Y**	W		R	**Y**	W
pref. 1	0	**1**	0	~	0	**0**	0
pref. 2	9	**1**	9	$\succ$	9	**0**	9

nating event) is akin to throwing out the baby with the bath water. This is because, in CEU, preferences depend on the rank-ordering of consequences and if the revealed likelihood is derived solely through preferences, there seems to be no escape from revealed likelihood to depend on the rank-ordering as well. Our three examples have demonstrated this dependence of revealed likelihood on rank-ordering through the dominating event.

8. Restrictions on decision weights

We have observed that, under CEU, revealed likelihood of an event measured by its decision weight depends on the dominating event, whereas under SEU, the revealed likelihood of an event is entirely independent of the dominating event. This section describes a number of cases that are intermediate between CEU and SEU in restricting the dependence of revealed likelihood on the dominating event.

We first demonstrate the application of decision weights as measure of revealed likelihood in defining null events. Loosely speaking, a null event is equally likely as the impossible event. In our interpretation it means that an event is null if its decision weight is 0. Null events are important for updating (Gilboa, 1989a; Klibanoff, 1995) and for the definition of the support of a distribution, which is central in some problems in game theory (Dow and Werlang, 1994; Eichberger and Kelsey, 1994; Epstein and Wang, 1994; Lo, 1995; Groes, Jacobsen, Sloth, and Tranaes, 1998). Haller (1995) proposed three different definitions of support, depending on how null events are interpreted, and pointed out that different definitions of equilibria in games with nonadditive measures can be explained by different definitions of support.

Whether an event is null can be inferred from preferences as follows.

Suppose that $\alpha > \beta$. Then

$$\pi(A,D) = 0 \text{ if and only if } (D,\alpha;\ A,\alpha;\ I,\beta) \sim (D,\alpha;\ A,\beta;\ I,\beta), \tag{11}$$

where $I = (A \cup D)^c$. Substitution of CEU shows that the indifference in (11) holds if and only if

$$(A_1,x_1;\ \cdots;\ A_{i-1},x_{i-1};\ A_i,\alpha;\ A_{i+1},\ x_{i+1};\ \cdots;\ A_n,x_n) \sim$$

$$(A_1,x_1;\ \cdots;\ A_{i-1},x_{i-1};\ A_i,\beta;\ A_{i+1},\ x_{i+1};\ \cdots;\ A_n,x_n)$$

for any $\alpha > \beta$, $A_i = A$, $D = A_1 \cup \cdots \cup A_{i-1}$, $I = A_{1+1} \cup \cdots \cup A_n$, and $x_1 \geq \cdots \geq x_{i-1} \geq \alpha > \beta \geq x_{i+1} \geq \cdots \geq x_n$. The simpler condition (11) is used in the following analysis.

Under general CEU, A can be null for some dominating event D but nonnull for another. As an example, for maximin behavior ($\nu(A) = 0$ whenever A is not the universal event), $\pi(A,D)$ is 1 if $D = A^c$ and A is nonempty but $\pi(A,D)$ is 0 whenever $D \neq A^c$. Therefore an event A is called *D-null* if $\pi(A,D) = 0$. We next discuss invariance of null events with respect to the dominating event D.

If it is assumed that events should be null only if they are logically impossible, then it is unsatisfactory that the logical (im)possibility of an event would depend on which other event were to yield better consequences in an act. One may want to ensure that an event A is null regardless of the dominating event. It must then be required that the decision weight $\pi(A,D)$ be zero for all dominating events D as soon as it is for one. In terms of preferences, this means that

$$(D,\alpha;\ A,\alpha;\ I,\beta) \sim (D,\alpha;\ A,\beta;\ I,\beta) \Rightarrow (D',\alpha;\ A,\alpha;\ I',\beta) \sim (D',\alpha;\ A,\beta;\ I',\beta) \qquad (12)$$

for all $\alpha > \beta$, I, D, I', D'. We call this condition *null-invariance*. It rules out phenomena such as in Example 7.3. It can be derived elementarily that the union of two null events is again null under null-invariance. Schmeidler (1989, Remark 4.3) shows that null-invariance can be characterized by a condition similar to Savage's P3.

The formulation in terms of dependence on dominating events gives clarifying alternative interpretations of several properties of capacities that have been studied in the literature. We list a number of them, leaving the proofs to the reader. In the following conditions, the terms increasing and decreasing refer to set-inclusion.

$$\nu \text{ is symmetric if and only if } \pi(A,\varnothing) = \pi(A,A^c) \text{ for all events A.} \qquad (13)$$

$$\nu \text{ is convex } (\nu(A) + \nu(B) \leq \nu(A \cup B) + \nu(A \cap B)) \text{ if and only if } \pi(A,D) \text{ is increasing in D.} \qquad (14)$$

$$\nu \text{ is concave } (\nu(A) + \nu(B) \geq \nu(A \cup B) + \nu(A \cap B)) \text{ if and only if } \pi(A,D) \text{ is decreasing in D.} \qquad (15)$$

Condition (14) illustrates pessimism, where a higher decision weight is assigned to an event as the event is lower in the rank-ordering. Similarly, condition (15) illustrates optimism. Conditions (14) and (15) are reminiscent of the characterization of convex functions through increasing derivatives and concave functions through decreasing derivatives. Note that the decision weight $\pi(A,D)$ describes the increase of ν if A is added to D.

Tversky and Wakker (1995) proposed the following conditions to reflect bounded subadditivity, stated here in a somewhat informal manner. ν satisfies *bounded subadditivity* if

(i) $\pi(A,\varnothing) \geq \pi(A,B)$ whenever $A \cup B$ is "sufficiently remote" from certainty.
(ii) $\pi(A,A^c) \geq \pi(A,B)$ whenever B is "sufficiently remote" from impossibility.

The conditions imply that decision weights with respect to intermediate dominating events are less than with respect to the extreme dominating events and have been illustrated in Figure 3.

We finally turn to the characterization of probabilistic sophistication for the context of CEU. In a general setting, probabilistic sophistication was characterization by Machina and Schmeidler (1992). They argued for a normative status of probabilistic sophistication. A generalization was provided by Epstein and Le Breton (1993). In the case of probabilistic sophistication, the ordering of revealed likelihoods of events remains invariant with the dominating event D. It turns out that, under some richness conditions, that condition is also sufficient for probabilistic sophistication under CEU. *Invariance of likelihood orderings* holds if:

$$\pi(A,D) \geq \pi(B,D) \Rightarrow \pi(A,D') \geq \pi(B,D') \tag{16}$$

for all events A, B, D, D'. In the proof of the following theorem we will show that condition (16) is equivalent to a well-known condition of qualitative probability theory, stating that the ordering between two events is not affected when the same disjoint event is added to both of them. In the next theorem, *solvability* of ν (introduced by Gilboa (1987) under the name convex-rangedness) means that for all events $A \subset C$ and $\nu(A) \leq \beta \leq \nu(C)$ there exists an event B such that $A \subset B \subset C$ and $\nu(B) = \beta$. We now state a theorem that uses condition (16) to relate capacities to probabilities in the sense that capacities are transforms of probabilities. A similar result was provided by Epstein and Le Breton (1993, Formula 2.5), under the assumption that the capacity is the minimum of a set of dominating probability measures.

Theorem 8.1 Let the collection of events be a sigma-algebra and let CEU hold. There exists a countably additive atomless probability measure P and a strictly increasing continuous transformation ϕ such that $\nu = \phi o P$ if and only if the following conditions hold:

(i) ν satisfies solvability;
(ii) (set-continuity) If $A_n \uparrow A$ (i.e., $A_{n+1} \supset A_n$ and $\cup A_n = A$) then $\lim_{j\to\infty}\nu(A_j) = \nu(A)$.
(iii) Invariance of likelihood orderings (16) holds. □

Note that invariance of likelihood orderings, characterized through probabilistic sophistication, has only obtained an independence of beliefs from tastes at an ordinal level, and only so for events with a common dominating event.

9. Updating revealed likelihood

We assume in this section that the decision maker updates preference in a dynamically consistent manner, i.e., in agreement with prior preference (Machina, 1989; Epstein and Le Breton, 1993). Then the definition of revealed conditional likelihood is straightforward if, as proposed in this paper, decision weights are taken as indices of revealed likelihood. Consider two events A and B and assume that the rank-ordering of the state space has been fixed, so that also dominating events are determined. The revealed conditional likelihood of A given B is simply defined by

$$\pi(A|B) = \frac{\pi(A \cap B)}{\pi(B)}. \tag{17}$$

The resulting number is always between 0 and 1. The definition of revealed conditional likelihood in (17) invokes the rank-ordering of states, thus is applicable within a given comonotonic domain of acts. This point is in line with the observation of Eichberger and Kelsey (1996), that with CEU preferences it is not possible to update beliefs independently of consequences. For two cases, that are sufficiently general to cover most applications, revealed conditional likelihood requires only partial information on the ranking of events. In the first case, $A \cap B$ and B are connected events; this case is discussed in most of this section. In the end we briefly discuss a second case, in which $A \cap B$ and $B \backslash A$ are connected.

Let us now consider the first case, with $A \cap B$ and B connected. In contrast to the additive probability case, we require the specification of a dominating event D for $A \cap B$ and D' for B. Thus revealed conditional likelihood is written as $\pi(A,D|B,D')$; when no confusion may arise, D and D' are suppressed. For consistency of rank-ordering, $D \supset D'$. We propose the following definition of revealed conditional likelihood.

$$\pi(A|B) = \pi(A,D|B,D') = \frac{\pi(A \cap B,D)}{\pi(B,D')} = \frac{\nu((A \cap B) \cup D) - \nu(D)}{\nu(B \cup D') - \nu(D')}. \tag{18}$$

In this definition we further assume that $\pi(B,D') \neq 0$. Consequences outside the conditioning event B are relevant in this formula because they determine what the dominating events are. This relevance of forgone consequences is the price one has to pay for giving up the separability of disjoint events that is characteristic for SEU, while still adhering to dynamic consistency (Machina, 1989).

Several definitions of revealed conditional likelihood have been proposed in the literature. Gilboa (1989a) proposed the following rule (see also Gilboa, 1989b, p.3, for non-additive measures that are not directly related to decisions)

$$\pi(A|B) = \frac{\nu(A \cap B)}{\nu(B)} = \frac{\pi(A \cap B,\varnothing)}{\pi(B,\varnothing)}. \tag{19}$$

Gilboa and Schmeidler (1993) suggested that this rule corresponds with an optimistic decision maker who assumes that the event B, of which he has been informed, corresponds with the “best of all possible worlds.” In our terminology that means that B is taken as a good-news event. In addition, given the information B, $A \cap B$ is in turn treated as a good-news event. In other words, both $D' = \varnothing$ and $D = \varnothing$, and the belonging rank-ordering of events is $A \cap B \succcurlyeq B \backslash A \succcurlyeq B^c$. In updating, the case of null-conditioning events is usually excluded. As the conditioning event is taken as a good-news event, meaning that the dominating event D' is empty, it seems appropriate that the conditioning event should not be D'-null for $D' = \varnothing$. This was indeed the definition adopted by Gilboa (1989a).

The following updating rule was proposed by Dempster (1967) and Shafer (1976) for belief functions (a special case of capacities). It was characterized and advocated by Gilboa and Schmeidler (1993) and used by Dow and Werlang (1992). This rule was also used by Lo (1995) for the multiple priors model.

$$\pi(A|B) = \frac{\nu((A \cap B) \cup B^c) - \nu(B^c)}{1 - \nu(B^c)} = \frac{\pi(A \cap B, B^c)}{\pi(B, B^c)}. \tag{20}$$

Gilboa and Schmeidler (1993) suggested that this rule corresponds to a pessimistic decision maker. It indeed results from our proposal if $D' = B^c$ is taken. Thus, the received information is taken as bad news. In addition, $A \cap B$ is assigned the highest-possible rank-ordering within B. Hence, D does not contain more than B^c. The corresponding rank-ordering of events is $B^c \succcurlyeq A \cap B \succcurlyeq B \backslash A$.

The following example illustrates our updating method.

Example 9.1 Assume that a die with six numbered sides yields j if side j shows up, $j = 1,\cdots,6$. Our interest is in computing the revealed conditional likelihood of receiving 5 given the information that the prize is 3 or more. The dominating event D for $\{5\}$ is $\{6\}$ and the dominating event D' for 3 or more is the empty set. That specifies a rank-ordering of events $[6\} \succcurlyeq \{5\} \succcurlyeq \{3,4\} \succcurlyeq \{1,2\}$. Now

$$\pi(\{5\}|\, j \geq 3) = \pi(\{5\},\{6\}|\, j \geq 3,\varnothing) = \frac{\pi(\{5\},\{6\})}{\pi(\{3,4,5,6\},\varnothing)} = \frac{\nu(5,6) - \nu(6)}{\nu(3,4,5,6)}.$$

This is an example in which the event for which the revealed conditional likelihood is to be determined is neither the best nor the worst event given the conditioning event, which is a case that has not yet been considered in the literature.

We assume that $\nu(A)$ depends only on the number of elements in A and is given by Table 4. Thus, for example, $\nu(1) = \cdots = \nu(6) = 0.25$, $\nu(1,2) = \nu(5,6) = 0.40$, $\nu(1,2,5) = 0.50$, $\nu(3,4,5,6) = 0.60$, etc. A singleton event has decision weight 0.25 if it is extreme in the rank-ordering ($0.25 = 0.25-0$ if it is best, $0.25 = 1-0.75$ if it is worst), decision weight 0.15 if it is second-best ($0.15 = 0.40-0.25$) or second-worst ($0.15 = 0.75-0.60$), and decision weight 0.10 if it has a middle position ($0.10 = 0.50-0.40$ if it is third in ranking, $0.10 = 0.60-0.50$ if it is fourth in ranking). This capacity ν is symmetric and satisfies bounded subadditivity.

Formula (19) gives

Table 4.

$\|A\|$	0	1	2	3	4	5	6
ν	0	0.25	0.40	0.50	0.60	0.75	1

$$\pi(\{5\}|\ j \geq 3) = \frac{\nu(5)}{\nu(3,4,5,6)} = \frac{0.25}{0.60} = 0.42,$$

and the Dempster-Shafer update rule (20) gives

$$\pi(\{5\}|\ j \geq 3) = \frac{\pi(\{5\},\{1,2\})}{\pi(\{3,4,5,6\},\{1,2\})} = \frac{0.10}{0.60} = 0.17.$$

Our update rule (18) gives

$$\pi(\{5\}|\ j \geq 3) = \pi(\{5\},\{6\}|\ j \geq 3,\emptyset) = \frac{\pi(\{5\},\{6\})}{\pi(\{3,4,5,6\},\emptyset)} = \frac{0.15}{0.60} = 0.25.$$

The dominating event for $\{5\}$ is $\{6\}$ and our update rule (18) assigns the weight $\nu\{5,6\} - \nu\{6\} = 0.40 - 0.25 = 0.15$ to event $\{5\}$. Formula (19) assumes that no event dominates $\{5\}$ and thus it overweights $\{5\}$ by assigning a decision weight $\nu\{5\} = 0.25$. The Dempster-Shafer Formula (20) treats $\{1,2\}$ as the dominating event for $\{5\}$ and thereby underweights $\{5\}$ by assigning a decision weight $\nu\{1,2,5\} - \nu\{1,2\} = 0.50 - 0.40 = 0.10$ to $\{5\}$.

Both (18) and (19) treat $\{3,4,5,6\}$ as a good-news event, i.e., take an empty dominating event and assign decision weight 0.60 to $\{3,4,5,6\}$. Formula (20) takes $\{3,4,5,6\}$ as a bad-news event, which differs from our interpretation but, by symmetry, assigns the same decision weight 0.60 as our method does. □

The central aspect of Bayes theorem is to derive the probability of B given A from the probability of A given B. That is, in our case, $\pi(A,D|B,D')$ is to be related to $\pi(B,\bar{D}|A,\bar{D}')$, where the dominating events are discussed next. There cannot be expected to be a simple relation between the two conditional likelihoods if $D \neq \bar{D}$, i.e., if in one case $A \cap B$ has a different dominating event than in the other. However, as soon as $D = \bar{D}$, we obtain the following extension of the Bayesian calculation:

$$\pi(A,D|B,D')\pi(B,D') = \pi(A \cap B,D) = \pi(A \cap B,\bar{D}) = \pi(B,\bar{D}|A,\bar{D}')\pi(A,\bar{D}')$$

whenever $\pi(B,D')$ and $\pi(A,\bar{D}')$ are nonzero. This analysis shows that the inverse relation for revealed conditional likelihood also holds for (19), because here all dominating events in the conditionalization are chosen empty, but the inverse relation will not hold for the Dempster-Shafer update rule (20) because in $\pi(A|B)$ the dominating event for $A \cap B$ is B^c, in $\pi(B|A)$ it is A^c.

We briefly mention a second case in which Formula (17) yields a tractable result requiring only a partial specification of the rank-ordering of events. It concerns the case in which $A \cap B$ and $B \backslash A$ are connected. Again, we only need to specify two dominating events, D for $A \cap B$ and D' for $B \backslash A$, and (17) results in

$$\pi(A|B) = \frac{\pi(A \cap B,D)}{\pi(A \cap B,D) + \pi(B\backslash A,D')}. \tag{21}$$

The case in which $D=\varnothing$ and $D'=(B\backslash A)^c$ has received much attention in the literature (Jaffray, 1992; Denneberg, 1994). In this case, $A \cap B$ is a good-news event but the other part of the conditioning event, $B\backslash A$, is a bad-news event, and B is not connected. B^c is connected and contains the intermediate states. The belonging rank-ordering of events is $A \cap B \succcurlyeq B^c \succcurlyeq B\backslash A$. If the capacity is convex ("pessimism"), for instance if it is a Dempster-Shafer belief function, then the decision weight of $A \cap B$ is minimal if the event is good news and the decision weight of $B\backslash A$ is maximal if it is bad news. Hence these choices of D and D' minimize $\pi(A|B)$, thus maximize pessimism. The formula also results if one identifies the capacity with the set of dominating probability distributions and applies conditionalization to each dominating probability measure separately.

An alternative method for updating nonadditive measures has been proposed by Lehrer (1996). He considers the more general notion of conditional expectation. His conditional expectation of a function h (e.g., h(s) may be U(f(s)) for an act f), given a sub-sigma algebra, is a function measurable with respect to that sub-sigma algebra that satisfies some requirements (e.g., it should have the same expectation as h and nowhere exceed maximal and minimal values of h). Given those requirements, it should minimize the quadratic distance with respect to h.

The various notions of revealed conditional likelihood that have been discussed can be tested empirically. Some work along this line has begun (Cohen, Gilboa, Jaffray, and Schmeidler, in preparation). Specifically, it will be interesting to examine the role of dominating events in the revision of beliefs.

10. Revealed likelihood and beliefs

In SEU, revealed likelihood can be interpreted as a measure of belief. This interpretation is appealing because probabilities that measure revealed likelihoods are independent of tastes. In nonadditive models, revealed likelihoods are measured by decision weights. Decision weights are, however, not independent of the rank-ordering of consequences. Therefore, if decision weights are interpreted as measures of belief then independence of beliefs from tastes cannot be entirely maintained. It is quite possible that capacities and decision weights reflect not only beliefs but also decision attitudes (e.g., ambiguity aversion).

Capacities and decision weights may be different than likelihood elicited through introspection. Some may regard that beliefs are best captured by an extraneous notion (introspection, verbal report) of likelihood that precedes preferences. In this view, beliefs depend only on the degree and extent of information that a decision maker possesses. Even under SEU, revealed likelihood through bets may not represent beliefs (Kadane and Winkler, 1988; Karni, 1996). Shafer's (1976) belief functions provide an example of beliefs that precede preferences. Ellsberg (1961, p. 659) wrote: "[One] can always assign relative likelihoods to states of nature. But how does he *act* in the presence of his uncertainty? The answer to that may depend on another sort of judgement..." Future studies may be able to disentangle beliefs and decision attitudes in the analysis of decision weights (Jaffray, 1989; Tversky and Fox, 1995; Wu and Gonzalez, 1997; Epstein and Zhang, 1998; Tversky and Wakker, 1998).

11. Summary and conclusion

In decision under uncertainty, there is often a difficulty in assigning probabilities to events. Ellsberg's examples demonstrated these difficulties convincingly. In recent years, Choquet-expected utility (CEU) has been introduced to describe the observed violations of expected utility as in the Ellsberg examples. In the context of CEU, we propose that decision weights be interpreted as a measure of revealed likelihood. Under this interpretation, the revealed likelihood of an event depends on the dominating event.

Several applications of our measure of revealed likelihood are illustrated. The definition of null events and supports is clarified, new interpretations are given for convexity, concavity, bounded subadditivity, and probabilistic sophistication. We define revealed conditional likelihood in the context of CEU and dynamic consistency, and show several implications for existing rules of updating if new information is gathered.

In CEU, capacities resemble probabilities and therefore are often treated as measures of belief. We have raised two objections against this customary interpretation of capacities. First, this interpretation, arbitrarily, considers events only in the role of good-news events. Events may as well play the role of bad-news events, in which case the dual capacity should be considered. Indeed, a number of papers have pointed out that the dual capacity is just as valid a measure of belief as the capacity (Gilboa, 1989a) or, similarly but in qualitative terms, that bets against events provide as valid an ordering of likelihood as bets on events (Nehring, 1994). We have argued that, more generally, events may also play the role of intermediate events and that in many respects (such as the study of bounded subadditivity) decision weights are relevant, rather than capacities or their duals. Second, capacities, their duals, and decision weights, all may comprise not only a belief component but may also be affected by decision attitudes. To avoid commitment to a pure belief-interpretation, we used the term revealed likelihood rather than likelihood throughout the paper.

We realize that our interpretations are subject to counter viewpoints and that better arguments for (or against) defining revealed likelihood in CEU may yet emerge. We do hope that the interpretation of capacities as degrees of belief, and bets on events as elicitations of likelihood ordering, will no longer be accepted without any qualification in CEU.

Appendix

Proofs

Proof of Observation 4.2 First cumulative dominance is derived for CEU. To do so, we use the following formula, where we write $\nu^{\uparrow}$ for ν.

$$\mathrm{CEU}(f) = \int_{\mathrm{IR}+} \nu^{\uparrow}[\mathrm{Uof} \geq t]dt + \int_{\mathrm{IR}-} (\nu^{\uparrow}[\mathrm{Uof} \geq t] - 1)dt \tag{22}$$

It is well-known, and can be derived by partial integration, that (22) provides an alternative manner for writing the upper Choquet integral of Uof with respect to the capacity $\nu^{\uparrow}$, i.e., for calculating CEU(f). To prove cumulative dominance, assume that $[f \geq x] \succcurlyeq^{\uparrow} [g \geq x]$ for all x. Because $\nu^{\uparrow}$ represents $\succcurlyeq^{\uparrow}$, for all t the integrand in (22) is at least as large as the integrand when g is substituted for f. Hence, the CEU value of f exceeds that of g and $f \succcurlyeq g$ follows. Cumulative dominance has been demonstrated.

The derivation of dual cumulative dominance now follows from Observation 4.1. We prefer, however, giving an independent derivation, so as to further illustrate the duality of good- and bad-news events. We use the following formula, where $\nu^{\downarrow}$ denotes the dual of ν.

$$\mathrm{CEU}(f) = \int_{\mathrm{IR}+} (1 - \nu^{\downarrow}[\mathrm{Uof} \leq t])dt - \int_{\mathrm{IR}-} \nu^{\downarrow}[\mathrm{Uof} \leq t]\, dt \tag{23}$$

It is also well-known, and can be derived by partial integration, that (23) provides an alternative manner for writing the lower integral of Uof with respect to the capacity $\nu^{\downarrow}$; this also yields CEU(f) (Gilboa, 1989a). To prove dual cumulative dominance, assume that $[f \leq x] \preccurlyeq^{\downarrow} [g \leq x]$ for all x. Because $\nu^{\downarrow}$ represents $\succcurlyeq^{\downarrow}$, $\nu^{\downarrow}[\mathrm{Uof} \leq t]$ in (23) is less than or equal to $\nu^{\downarrow}[\mathrm{Uog} \leq t]$, for all t. Therefore the CEU value of f exceeds that of g and $f \succcurlyeq g$ follows. Dual cumulative dominance has been demonstrated.

To further clarify the duality between good-news and bad-news events in (23) and (24), note that (23) is equal to

$$\mathrm{CEU}(f) = \int_{\mathrm{IR}+} (1 - \nu^{\downarrow}[\mathrm{Uof} < t])dt - \int_{\mathrm{IR}-} \nu^{\downarrow}[\mathrm{Uof} < t]\, dt. \tag{24}$$

The nondecreasing integrands in (23) and (24) can have at most countably many discontinuities and therefore differ at most at countably many t. Those t provide a Lebesgue 0 set and do not contribute to the integrals. After substitution of the definition of $\nu^{\downarrow}$ in (24), (22) results. □

Proof of Theorem 8.1 First assume that the conditions (i), (ii), and (iii) hold. We prove the existence of P and ϕ as described in the theorem. We write $A \succcurlyeq B$ for events A, B whenever $\nu(A) \geq \nu(B)$. No confusion with the preference relation $\succcurlyeq$ on acts will arise. First we derive the properties of a qualitative probability ordering for $\succcurlyeq$.

Obviously $\succcurlyeq$ is a weak order; the notation $\succ$ is as usual. We have $S \succcurlyeq A \succcurlyeq \varnothing$ for all events A and $S \succ \varnothing$. Finally, assume that event D is disjoint from events A,B. Then the following six statements are equivalent: (1) $A \succcurlyeq B$; (2) $\nu(A) \geq \nu(B)$; (3) $\pi(A,\varnothing) \geq \pi(B,\varnothing)$; (4) $\pi(A,D) \geq \pi(B,D)$; (5) $\nu(A \cup D) \geq \nu(B \cup D)$; (6) $A \cup D \succcurlyeq B \cup D$. The equivalence of (3) and (4) holds because of invariance of likelihood orderings. The equivalence of (1) and (6) is the well-known additivity condition of qualitative probability theory. Thus, $\succcurlyeq$ is a qualitative probability ordering (Villegas, 1964).

Solvability of ν implies that no atoms exist and Condition (ii) implies monotone continuity of Villegas (1964). Therefore, by Villegas' Theorem 4.3, there exists a unique countably additive atomless probability measure P on the sigma algebra of events that represents the qualitative probability ordering $\succcurlyeq$. Hence there exists a strictly increasing transformation ϕ such that $\nu = \phi o P$. By solvability of ν, the range of ϕ is [0,1], hence ϕ must be continuous.

For the reversed implication, solvability of ν is implied because P is atomless and ϕ is continuous, Condition (ii) is implied by sigma-additivity of P and continuity of ϕ, and invariance of likelihood orderings follows from additivity of P.

Our proof has been based on the qualitative probability result of Villegas (1964), which implies countably additivity measures. Finite additivity can be obtained by using different qualitative probability results, such as Savage's (1954). Then, however, the fineness condition is more complicated (Gilboa, 1985). □

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Notes

1. This condition was called to our attention by Alain Chateauneuf (1991, personal communication).

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