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Contribution to Computer Construction of Active Chain Models Via Lagrangian Form

Basic principles underlying the computer construction of the models of open kinematic chains using the recurrent relations from the rigid body kinematics are briefly presented in this paper. A new method for automatic setting of the aforementioned models via second-order Lagrange's equations is presented. This method provides a realistic basis for the application of complete dynamic models to the real-time control of robots and manipulators.

Introduction

A new class of mechanisms, which we may call active mechanisms, has appeared during the past few years. These mechanisms are mainly applied to various robots and manipulators intended for the production of different kinds of motion [1 4].

These mechanisms have a number of specific features which require new procedures to be formulated, starting with deriving mathematical models and ending with synthesizing control algorithms suitable for real-time operation.

One of the aforementioned specific features is connected with the problem of deriving dynamic models of active spatial mechanisms. What is required here is to form such an algorithm which could compose automatically dynamic equations based only on the input data on mechanisms parameters. This helps eliminate the serious problem of committing errors when forming the model "by hand."

There are, at least, two basic reasons due to which the automatic derivation of mathematical models appears necessary. The first reason is the impossibility of choosing a unique robot configuration; it is, therefore, useful to analyze various kinematic schemes in order to choose the appropriate model, depending on the particular situation. The second reason is the need for real-time control of robots and manipulators. Dynamic equations formed on the basis of such an algorithm and the creation of possibilities of its realization in real time certainly contribute directly to the synthesis of control algorithms for concrete applications.

The first algorithms meeting these requirements have appeared independently [5–7]. The first two approaches were developed in connection with the dynamic analysis of manipulators [4, 5], and the third one in connection with the synthesis of artificial anthropomorphic gait [7].

In order to give an insight into the essence of these methods, as well as various other, originating from them, we shall present their basic concepts in the lines to follow.

Kinematic Connections of Open Spatial Mechanisms

Dynamic equations of open active mechanisms can be symbolically represented by a system of differential equations in the matrix from

$$W\ddot{q} = P + V \tag{1}$$

where q is the column-matrix of generalized coordinates, P is the column-matrix of driving forces and torques, W is the inertia matrix, a square matrix, the elements of which are functions of the generalized coordinates, V is a column-matrix, which is a function of the generalized coordinates and velocities.

Explicit dependency of W and V on the state coordinates is extremely complex, so it was passed to numerical calculation of W and V for one time instant. It follows, that the essential feature here is forming of the mathematical model for one time instant. From the system obtained in this way, depending on the type of task, accelerations are calculated for known driving forces, or the necessary driving forces for known accelerations. In the case of known driving forces, accelerations (1) can be calculated, which can be considered constant in a sufficiently small time interval Δt. After that, by means of one of the numerical integration methods, new values of the system state vector are calculated. Now the system is formed again and the procedure repeated. In that way, simultaneous forming and integration of the mathematical model is being performed. Calculation of matrices W and V is performed in the process of "circling" the chain. In each iteration to the chain is added a new member, and the matrices W and V are calculated for the newly formed system. For this, it is necessary

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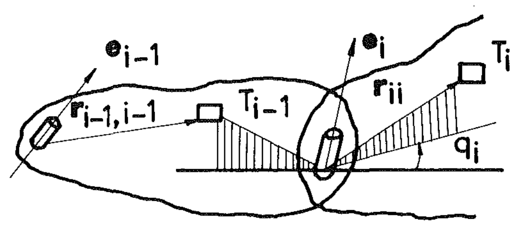

Fig. 1 Scheme of rotational joint

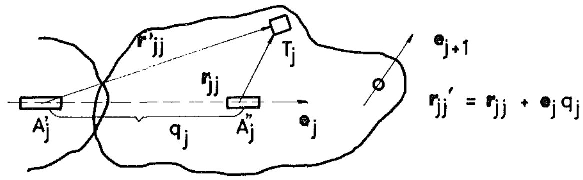

Fig. 2 Linear mechanism joint

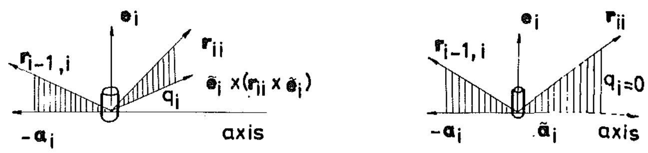

Fig. 3 Characteristic joint parameters

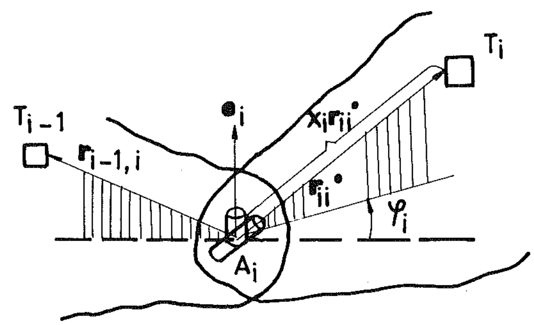

Fig. 4 Chain joint with two degrees of freedom

to know the recurrent formulas for velocities and c.o.g. accelerations as well as angular velocities and accelerations of the mechanism members.

The case of an one-degree-of-freedom joint is being considered, with one rotation about axis $\mathbf{e}_i$ (Fig. 1). T_i designates the c.o.g. of the ith member, and the generalized coordinate, corresponding to this degree of freedom is defined as the angle q_i, the angle of joint turning, angle between the projections of the vectors $\mathbf{r}_{i-1,i}$ and $\mathbf{r}_{ii}$ onto the plane orthogonal to $\mathbf{e}_i$. If by $\mathbf{v}_i$, $\mathbf{w}_i$ are designated the velocity and acceleration of the body c.o.g., and by ω_i, ϵ_i the angular velocity and accelerations of the ith mechanism member, then the known relations of the rigid body mechanics can be applied

$$\omega_i = \omega_{i-1} + \dot{q}_i \mathbf{e}_i; \quad \mathbf{v}_i = \mathbf{v}_{i-1} - \omega_{i-1} \times \mathbf{r}_{i-1,i} + \omega_i \times \mathbf{r}_{ii}$$

$$\epsilon_i = \epsilon_{i-1} + \ddot{q}_i \mathbf{e}_i + \dot{q}_i \omega_{i-1} \times \mathbf{e}_i; \quad \mathbf{w}_i = \mathbf{w}_{i-1} - \epsilon_{i-1} \times \mathbf{r}_{i-1,i} \\ -\omega_{i-1} \times (\omega_{i-1} \times \mathbf{r}_{i-1}) + \epsilon_i \times \mathbf{r}_{ii} + \omega_1 \times (\omega_i \times \mathbf{r}_{ii}) \quad (2)$$

For the joints with one linear degree of freedom along axis $\mathbf{e}_i$, the recurrent relations (2) become (Fig. 2)

$$\omega_j = \omega_{j-1}; \quad \mathbf{v}_j = \mathbf{v}_{j-1} - \omega_{j-1} \times \mathbf{r}_{j-1,j} + \omega_j \times \mathbf{r}_{jj}' + \dot{q}_j \mathbf{e}_j$$

$$\epsilon_j = \epsilon_{j-1}; \ \mathbf{w}_j = \mathbf{w}_{j-1} - \epsilon_{j-1} \times \mathbf{r}_{j-1,j} - \omega_{j-1} \times (\omega_{j-1} \times \mathbf{r}_{j-1,j}) \\ + \epsilon_j \times \mathbf{r}_{jj}' + \omega_j \times (\omega_j \times \mathbf{r}_{jj}') + (\ddot{q}_j \mathbf{e}_j + 2\omega_{j-1} \times \mathbf{e}_j \dot{q}_j) \quad (3)$$

A_j' = fixed point in relation to the $(j-1)$th member

A_j'' = fixed point in relation to the jth member

After that, one coordinate system with its axes along the main axes of inertia is connected to every mechanism member and the following designations are introduced:

$\mathbf{a}_i$ is a vector in the fixed coordinate system
$\tilde{\mathbf{a}}_i$ is the same vector in the ith member connected system.
$\underset{\sim}{\mathbf{a}}_i$ is the same vector in the $(i-1)$th member system.

Both the vector and matrix calculus will be utilized in the lines to come. Hence let us introduce the following designations. For instance, $\mathbf{a}$ designates a vector, and a designates a column-matrix, corresponding to that vector. In the vector calculus will be utilized $\mathbf{a}$, and in the matrix one a. It will be evident from the relevant text which values are in question.

The transient matrices are defined from the ith member system to the fixed, and from the $(i-1)$th member system to the ith member system, i.e.,

$$\mathbf{a}_i = A_i \, \tilde{\mathbf{a}}_i; \quad \tilde{\mathbf{a}}_i = A_{i,i-1} \mathbf{a}_i \quad (4)$$

The transient matrix A_i is calculated in the course of adding the member i to the chain. Let us suppose the case of a rotational joint. As A_{i-1} is known, so $\mathbf{r}_{i-1,i} = A_{i-1}\tilde{\mathbf{r}}_{i-1,i}$ and $\mathbf{e}_i = A_{i-1}\underset{\sim}{\mathbf{e}}_i$ are also known.

Now the following vectors can be calculated:

$$\mathbf{a}_i = \frac{-\mathbf{e}_i \times (\mathbf{r}_{i-1,i} \times \mathbf{e}_i)}{|-\mathbf{e}_i \times (\mathbf{r}_{i-1,i} \times \mathbf{e}_i)|} \quad \text{and} \quad \tilde{\mathbf{a}}_i = \frac{\tilde{\mathbf{e}}_i \times (\mathbf{r}_{ii} \times \tilde{\mathbf{e}}_i)}{|\tilde{\mathbf{e}}_i \times (\mathbf{r}_{ii} \times \tilde{\mathbf{e}}_i)|} \quad (5)$$

which are perpendicular to $\mathbf{e}_i$ and $\tilde{\mathbf{e}}_i$, respectively.

Vector $\mathbf{a}_i$ is the unit vector of axis a, and the second equation (5) is valid for the case $q_i = 0$ (Fig. 3). Introducing the vectors $\mathbf{l}_i = \mathbf{e}_i \times \mathbf{a}_i$ and $\tilde{\mathbf{l}}_i = \tilde{\mathbf{e}}_i \times \tilde{\mathbf{a}}_i$, triplets of linearly independent vectors $\{\mathbf{e}_i, \mathbf{a}_i, \mathbf{l}_i\}$ and $\{\tilde{\mathbf{e}}_i, \tilde{\mathbf{a}}_i, \tilde{\mathbf{l}}_i\}$, respectively, are obtained. As it is

$$\mathbf{e}_i = A_i{}^0 \tilde{\mathbf{e}}_i, \quad \mathbf{a}_i = A_i{}^0 \tilde{\mathbf{a}}_i, \quad \mathbf{l}_i = A_i{}^0 \tilde{\mathbf{l}}_i \quad (6)$$

where the upper index 0 designates, that the transient matrix, corresponding to $q_i = 0$, is in question

$$A_i{}^0 = [e_i a_i l_i][\tilde{e}_i \tilde{a}_i \tilde{l}_i]^{-1} \quad (7)$$

The columns of the transient matrix represent the unit vectors of the connected coordinate system, expressed in the fixed system. In order to obtain the transient matrix, corresponding to a certain angle q_i, turning should be performed, i.e., rotation, according to Rodrig's formula, of each unit vector by the angle q_i about axis $\mathbf{e}_i$

$$Q_{ij} = q_{ij} \cos q_i + (1 - \cos q_i) \cdot (e_i q_{ij}) e_i + e_i \times q_{ij} \sin q_i \quad (8)$$

where Q_{ij} is the jth column of matrix $A_i{}^0$ (unit vector after turning). Thus the transient matrix is formed

$$A_i{}^0 = [Q_{i1} Q_{i2} Q_{i3}] \quad (9)$$

For a linear joint only "joining" of the member is performed. By analogous procedure the transient matrix $A_{i,i-1}$ is determined. Now we can proceed to the calculation of matrices W and V, determining the mathematical model in the given time instant. In this paper we shall present a new method of forming mathematical models of the open chain dynamics, based on previously presented postulates.

Models Based on Lagrange's Equations

The kinematic chain on which we shall demonstrate this method, possesses joints with two d.o.f., one rotation about axis $\mathbf{e}_i$ and one translation of member i with respect to the $(i-1)$th member, along the axis $\mathbf{r}_{ii}{}^0$ (Fig. 4).

In each joint is acting a driving torque $M_i \mathbf{e}_i$ and driving force $F_i \mathbf{r}_{ii}{}^0$. Generalized coordinates are defined as the turning angle φ_i and intensity of vector $A_i T_i$.

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A_i = fixed point in relation to the $(i-1)$th mechanism member.

The vector of generalized coordinates is then

$$q = [\varphi_1, x_1, \ldots, \varphi_n, x_n]^T \quad (10)$$

The recurrent relation for the c.o.g. velocity and angular velocity is

$$\omega_i = \omega_{i-1} + \dot{\varphi}_i \mathbf{e}_i; \quad \mathbf{v}_i = \mathbf{v}_{i-1} - \omega_{i-1} \times \mathbf{r}_{i-1,i} + \omega_i \times \mathbf{r}_{ii}{}^0 x_i + \dot{x}_i \mathbf{r}_{ii}{}^0 \quad (11)$$

From (10) and (11) it follows:

$$\omega_i = \sum_{j=1}^{i} \beta_j \dot{\varphi}_j \quad (12)$$

where

$$\beta_j = A_{j-1} \underline{\mathbf{e}}_j \quad (13)$$

$$\mathbf{v}_i = \sum_{j=1}^{i} \alpha_j{}^i \dot{\varphi}_j + \sum_{j=1}^{i} \gamma_j \dot{x}_j \quad (14)$$

where

$$\gamma_j = A_j \tilde{\mathbf{r}}_{jj} \quad (15)$$

and

$$\alpha_j{}^i = \begin{cases} -\sum_{k=j+1}^{i} \mathbf{R}_{kj} + \sum_{k=1}^{i} \mathbf{S}_{kj}; & j < i \\ \mathbf{S}_{ii}; & j = i \end{cases} \quad (16)$$

$$\mathbf{R}_{kj} = A_{j-1} \underline{\mathbf{e}}_j \times A_{k-1} \mathbf{r}_{k-1,k} = \beta_j \times \boldsymbol{\delta}_k$$
$$\mathbf{S}_{kj} = A_{j-1} \underline{\mathbf{e}}_j \times A_k \tilde{\mathbf{r}}_{kk}{}^0 x_k = \beta_j \times \gamma_k x_k \quad (17)$$

where

$$\boldsymbol{\delta}_k = A_{k-1} \tilde{\mathbf{r}}_{k-1,k} \quad (17a)$$

In matrix form (12) and (14) become

$$\omega_i = N^{(i)} \dot{q}; \quad (18)$$

$$v_i = M^{(i)} \dot{q} \quad (19)$$

where

$$N^{(i)} = [\beta_1 0 \ldots \beta_i 000 \ldots 0] \quad (20)$$

$$M^{(i)} = [\alpha_1{}^i \gamma_1 \ldots \alpha_i{}^i \gamma_i 0 \ldots 0] \quad (21)$$

For calculating the kinetic energy, the expression for angular velocity is necessary in the connected coordinate system

$$\tilde{\omega}_i = A_i{}^{-1} \omega_i = A_i{}^{-1} N^{(i)} \dot{q} = \tilde{N}^{(i)} \dot{q} \quad (22)$$

where

$$\tilde{N}^{(i)} = [B_1{}^i 0 \ldots B_i{}^i 00 \ldots] \quad B_j{}^{(i)} = A_i{}^{-1} \beta_j = A_i{}^{-1} A_{j-1} \underline{\mathbf{e}}_j \quad (23)$$

Kinetic energy of the ith chain member is

$$T_i = \frac{1}{2} (m_i v_i{}^T v + \tilde{\omega}_i{}^T J_i \tilde{\omega}_i) \quad (24)$$

or

$$T_i = \frac{1}{2} \dot{q}^T (m_i M^{(i)T} M^{(i)} + \tilde{N}^{(i)T} J_i \tilde{N}^{(i)}) \dot{q} = \frac{1}{2} \dot{q}^T Z_i \dot{q} \quad (25)$$

where

$$Z_i = m_i M^{(i)T} M^{(i)} + \tilde{N}^{(i)T} J_i \tilde{N}^{(i)}$$

Total kinetic energy of the open chain is

$$T = \sum_{i=1}^{n} T_i = \frac{1}{2} \dot{q}^T \left(\sum_{i=1}^{n} Z_i \right) \dot{q} = \frac{1}{2} \dot{q}^T W \dot{q},$$

where

$$W = \sum_{i=1}^{n} Z_i \quad (26)$$

As known, the system of Lagrange's equations can be written in matrix form

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} = Q \quad (27)$$

where $Q = [Q_1{}^M Q_1{}^F, \ldots, Q_n{}^M Q_n{}^F]^T$ is the column-matrix of generalized force.

By substituting quadratic form (26) into (27), one obtains

$$W \ddot{q} + \dot{W} \dot{q} - \frac{\partial T}{\partial q} = Q \quad (28)$$

If the generalized forces are calculated in the form $Q = P + Y$, $P = [M_1, F_1, \ldots, M_n, F_n]^T$ and if we introduce

$$V = Y + \frac{\partial T}{\partial q} - \dot{W} \dot{q} \quad (29)$$

then (28) reduces to

$$W \ddot{q} = P + V \quad (30)$$

which is, evidently, identical to equation (1).

The column-matrix Y will be defined in the text to follow.

Matrix W is a calculated in a recurrent way in the course of iteration, i.e., according to (26) it will be

$$W^{(i)} = W^{(i-1)} + Z_i \quad (31)$$

where

$$Z_i = m_i M^{(i)T} M^{(i)} + \tilde{N}^{(i)T} J_i \tilde{N}^{(i)}$$

Matrix $\dot{W}$ is also calculated in a recurrent way

$$\dot{W}^{(i)} = \dot{W}^{(i-1)} + \dot{Z}_i \quad (32)$$

where

$$\dot{Z}_i = m_i (\dot{M}^{(i)T} M^{(i)} + M^{(i)T} \dot{M}^{(i)} + \dot{\tilde{N}}^{(i)T} J_i \tilde{N}^{(i)} + \tilde{N}^{(i)T} J_i \dot{\tilde{N}}^{(i)} \quad (33)$$

Matrices $\dot{M}^{(i)}$, $\dot{\tilde{N}}^{(i)}$ are

$$\dot{M}^{(i)} = [\dot{\alpha}_1{}^i \dot{\gamma}_1 \ldots \dot{\alpha}_i{}^i \dot{\gamma}_i 0 \ldots 0]$$
$$\dot{\tilde{N}}^{(i)} = [\dot{B}_1{}^i 0 \ldots \dot{B}_i{}^i 00 \ldots 0]$$
$$\dot{L}^{(i)} = [\dot{\delta}_1 0 \ldots \dot{\delta}_i 0 \ldots 0]$$

where, based on (13), (16), (17), (17a), and (23)

$$\dot{\mathbf{B}}_j{}^i = -A_i{}^{-1} \dot{A}_i A_i{}^{-1} \beta_j + A_j{}^{-1} \dot{\beta}_j; \quad \dot{\beta}_j = \dot{A}_{j-1} \mathbf{e}_j \quad (34)$$

$$\dot{\gamma}_j = \dot{A}_j \cdot \tilde{\mathbf{r}}_{jj}{}^0 \quad (35)$$

$$\dot{\boldsymbol{\delta}}_j = \dot{A}_{j-1} \tilde{\mathbf{r}}_{j-1,j} \quad (36)$$

$$\dot{\alpha}_j{}^i = \begin{cases} -\sum_{k=j+1}^{i} \dot{\mathbf{R}}_{kj} + \sum_{k=j}^{i} \dot{\mathbf{S}}_{kj}; & j < i \\ \dot{\mathbf{S}}_{ii}; & j = i \end{cases} \quad (37)$$

$$\dot{\mathbf{R}}_{kj} = \dot{\beta}_j \times \boldsymbol{\delta}_k + \beta_j \times \dot{\boldsymbol{\delta}}_k; \quad \dot{\mathbf{S}}_{kj} = \dot{\beta}_j \times \gamma_k x_k + \beta_j \times \dot{\gamma}_k x_k + \beta_j \times \gamma_k \dot{x}_k \quad (38)$$

Column-matrix $\partial T / \partial q$, needed for calculating V in (29) can be written as

$$\frac{\partial T}{\partial q} = \sum_{i=1}^{n} \frac{\partial T_i}{\partial q} \quad (39)$$

In the ith iteration it will be

$$\frac{\partial T_i}{\partial q} = \left[\frac{\partial T_i}{\partial \varphi_1} \frac{\partial T_i}{\partial x_i} \ldots \frac{\partial T_i}{\partial \varphi_i} \frac{\partial T_i}{\partial x_i} 0 \ldots 0 \right]^T \quad (40)$$

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where, due to (25)

$$\frac{\partial T}{\partial \varphi_j} = \frac{1}{2}\dot{q}^T \frac{\partial Z_i}{\partial \varphi_j}\dot{q} \tag{41}$$

$$\frac{\partial Z_i}{\partial \varphi_j} = m_i\left(\frac{\partial M^{(i)}}{\partial \varphi_j}\right)^T M^{(i)} + m_i M^{(i)T}\frac{\partial M^{(i)}}{\partial \varphi_s} + \left(\frac{\partial \tilde{N}^{(i)}}{\partial \varphi_s}\right)^T J_i \tilde{N}^{(i)} + \tilde{N}^{(i)T} J_i \frac{\partial \tilde{N}^{(i)}}{\partial \varphi_s} \tag{42}$$

Matrices $\partial \tilde{N}^{(i)}/\partial \varphi_s$, $\partial M^{(i)}/\partial \varphi_s$ are

$$\frac{\partial \tilde{N}^{(i)}}{\partial \varphi_s} = \left[\frac{\partial B_i{}^i}{\partial \varphi_s} 0 \ldots \frac{\partial B_i{}^i}{\partial \varphi_s} 00 \ldots 0\right] \tag{43}$$

$$\frac{\partial M^{(i)}}{\partial \varphi_s} = \left[\frac{\partial \alpha_i{}^i}{\partial \varphi_s}\frac{\partial \alpha_1{}^i}{\partial \varphi_s} \ldots \frac{\partial \alpha_i{}^i}{\partial \varphi_s}\frac{\partial \gamma_i{}^i}{\partial \varphi_s} 0 \ldots 0\right] \tag{44}$$

where

$$\frac{\partial \mathbf{B}_j{}^i}{\partial \varphi_s} = -A_i{}^{-1}\frac{\partial A_i}{\partial \varphi_s} A_i{}^{-1}\beta_j + A_i{}^{-1}\frac{\partial \beta_j}{\partial \varphi_s};$$

$$\frac{\partial \beta_j}{\partial \varphi_s} = \frac{\partial A_{i-1}}{\partial \varphi_s}\underline{\mathbf{e}}_j \tag{45}$$

$$\frac{\partial \gamma_j}{\partial \varphi_s} = \frac{\partial A_j}{\partial \varphi_s}\tilde{\mathbf{r}}_{jj} \tag{46}$$

$$\frac{\partial \delta_j}{\partial \varphi_s} = \frac{\partial A_{j-1}}{\partial \varphi_s}\tilde{\mathbf{r}}_{j-1,j} \tag{47}$$

$$\alpha_j{}^i = \begin{cases} -\sum_{k=j+1}^{i}\frac{\partial \mathbf{R}_{kj}}{\partial \varphi_s} + \sum_{k=j}^{i}\frac{\partial \mathbf{S}_{kj}}{\partial \varphi_s}; & j < i \\ \frac{\partial \mathbf{S}_{ij}}{\partial \varphi_s}; & j = 1 \end{cases} \tag{48}$$

$$\frac{\partial \mathbf{R}_{kj}}{\partial \varphi_s} = \frac{\partial \beta_j}{\partial \varphi_s} \times \delta_k + \beta_j \times \frac{\partial \delta_k}{\partial \varphi_s};$$

$$\mathbf{S}_{kj} = \frac{\partial \beta_j}{\partial \varphi_s} \times \gamma_k x_k + \beta_j \times \frac{\partial \gamma_k}{\partial \varphi_s} x_k \tag{49}$$

Parameters for the variable x are prepared in an analogous way, taking care that in the case equations (45)–(49) have the following form:

$$\frac{\partial \mathbf{B}_j{}^i}{\partial x_s} = 0, \quad \frac{\partial \gamma_j}{\partial x_s} = 0, \quad \frac{\partial \delta_j}{\partial x_s} = 0 \tag{50}$$

$$\frac{\partial \alpha_j{}^i}{\partial x_s} = \begin{cases} \sum_{k=j}^{i}\frac{\partial \mathbf{S}_{kj}}{\partial x_s}; & j < i \\ \frac{\partial \mathbf{S}_{ii}}{\partial x_s}; & j = i \end{cases};$$

$$\frac{S_{kj}}{\partial x_s} = \begin{bmatrix} 0; & k \neq s \\ \beta_j \times \gamma_k; & k = s \end{bmatrix} \tag{51}$$

It follows, that apart from the time derivations of the transient matrices it is necessary to calculate also the partial derivatives of the same with respect to coordinates. Using the procedure for recurrent relations of the transient matrices, a procedure can be derived for recurrent calculating of the partial derivatives $\partial A_i/\partial \varphi_s$, starting from A_{i-1} and $\partial A_{i-1}/\partial \varphi_s$. It is evident, that one of the key points in the algorithm is the calculation of matrices $N^{(i)}$, $\tilde{N}^{(i)}$, $M^{(i)}$, $\tilde{M}^{(i)}$, $\dot{N}^{(i)}$, $\dot{\tilde{N}}^{(i)}$, $\dot{M}^{(i)}$, $L^{(i)}$, $\dot{L}^{(i)}$ and matrices $\partial N^{(i)}/\partial \varphi_s$, $\partial \tilde{N}^{(i)}/\partial \varphi_s$, $\partial M^{(i)}/\partial \varphi_s$, $s = 1, \ldots, i$. The same holds for the coordinates x_s.

For the sake of algorithm efficiency, the foregoing values are calculated in a recurrent way, which means that, in each iteration, they are changed and supplemented with respect to the preceding iteration. The column-vector y, necessary for forming the equation (29), is realized in the following way.

The generalized force, corresponding to the coordinate q, can be calculated using the following expression:

$$Q_i{}^M = M_i + [m_i\mathbf{g}, \mathbf{e}_i, \mathbf{r}_{ii}] + \sum_{k=1}^{n-1}[m_{i+k}\mathbf{g}, \mathbf{e}_i, \mathbf{r}_k] \tag{52}$$

where

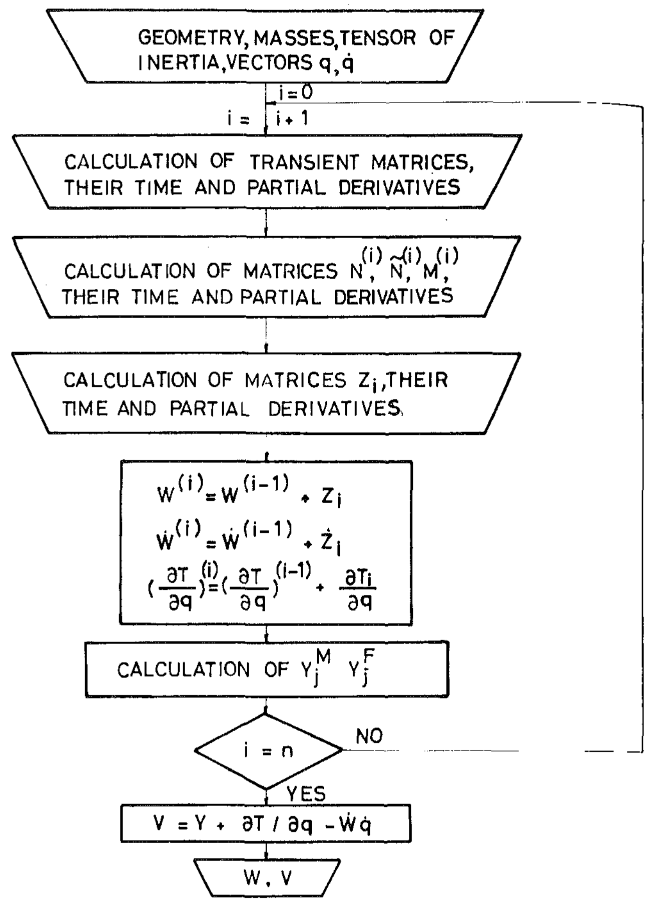

Fig. 5 Block-diagram of Lagrange's equations method

$$\mathbf{r}_k = \sum_{l=0}^{k}\mathbf{r}_{i+l,i+l} - \sum_{l=0}^{k-1}\mathbf{r}_{i+l,i+l+1}. \tag{53}$$

$[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ designates the triple scalar product.

The generalized force, corresponding to x_i, can be represented by

$$Q_i{}^F = F_i + \mathbf{e}_i\sum_{k=0}^{n-i} m_{i+k}\mathbf{g} \tag{54}$$

Expressions (52) and (54) can be calculated according to recurrent relations.

Now the column-matrix

$$Y = [Y_1{}^M Y_1{}^F, \ldots, Y_n{}^M Y_n{}^F]^T,$$

where

$$Y_i{}^M = [m_i\mathbf{g}, \mathbf{e}_i, \mathbf{r}_{ii}] + \sum_{k=1}^{n-1}[m_{i+k}\mathbf{g}, \mathbf{e}_i, \mathbf{r}_k]$$

$$Y_i{}^F = \mathbf{e}_i\sum_{k=0}^{n-i} m_{i+k}\mathbf{g}$$

can be calculated independently from the driving forces and torques. The described algorithm follows easily from the block-diagram in Fig. 5.

Example

In order to show the efficiency of the proposed computer algorithm for automatic forming of dynamic equations of open spatial mechanisms, an example of a manipulator with three degrees of freedom is presented here.

Initial position of the considered mechanism is shown in Fig. 6, where its basic geometry is given.

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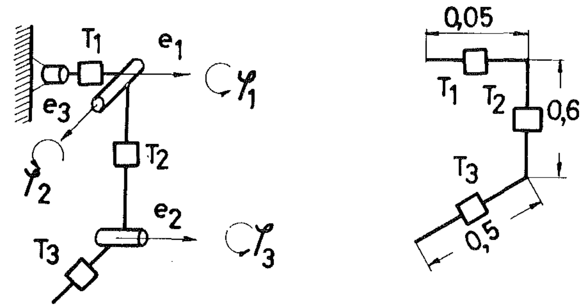

Fig. 6 Considered configuration of manipulator

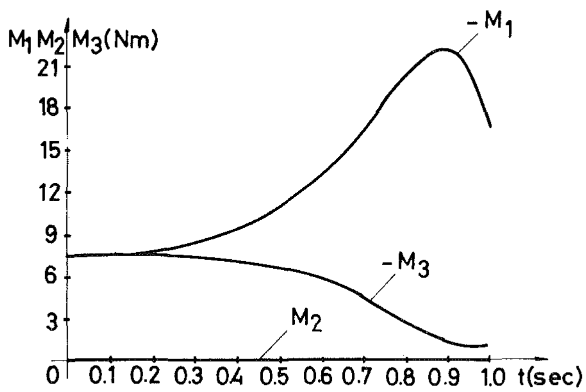

Fig. 7 Driving torques versus time

The lengths of mechanism members are given in meters. Initial state values of the system are

$$\varphi_1 = \pi \qquad \varphi_2 = -\pi/2 \qquad \varphi_3 = -\pi/2$$
$$\dot\varphi_1 = 0 \qquad \dot\varphi_2 = 0 \qquad \dot\varphi_3 = 0$$

Masses and tensor of inertia in MKS system are given with $m_1 = 0$, 1 kg, $m_2 = m_3 = 2$ kg.

For this mechanism, the synthesis of one functional movement has been performed, namely, the movement of the mechanism tip (manipulator gripper) vertically upward, with constant acceleration of 2 m/sec^2, of one second duration. This is being realized by means of driving torques, which, together with the trajectories, are presented in Figs. 7 and 8 as the results of simulation. The calculation is performed on a PDP-11/45 computer using the FORTRAN PLUS.

Digital simulation of automatic forming of differential equations using this method has shown that the time to form the equations is 0.3 sec.

This fact is of great importance for the on-line control of robots and manipulators. In the analysis of computer time, it is necessary to take into account the time of integration for one step. This integration gives new dynamic states of the system, necessary for motion performance. The integration time depends on the method of integration, i.e., on the number of calling of block for forming the equations. For example,

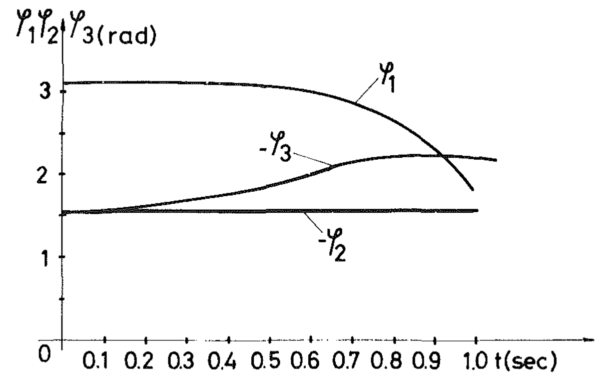

Fig. 8 Trajectories of joint angles versus time

in the case of Euler's method of integration, the integration time may, practically, be neglected, while for other, more precise methods of numerical integration, this time is up to triple the time to form the equations. The integration problem cannot be considered as separated from the manipulator positioning problem. However, this problem will not be dealt with here.

Conclusion

A new method enabling the computer construction of differential equations of motion of arbitrarily complex open-chain spatial mechanisms using the second-order Lagrange's equations is presented in this paper.

The problem of setting the equations automatically offers enormous possibilities of a systematic and fast choice of the appropriate configuration for the set manipulation task. It should be said here that the real-time computation of dynamics of such systems is not of some special significance in those tasks which are performed in predetermined conditions. However, in some classes of tasks, in which uncertainty and changing working conditions may arise, the real-time calculation of dynamic parameters of the system may have practical significance.

References

1 Vukobratović, M., and Stepanenko, Yu., "Mathematical Models of General Anthropomorphic Systems," *Math. Biosciences,* Vol. 17, 1973, pp. 130–155.

2 Vukobratović, M., "Legged Locomotion Robots and Anthropomorphic Mechanisms," Monograph, Mihailo Pupin Inst., Beograd, P.O.B. 906, 1975.

3 Vukobratović, M., Hristić, D., and Stokić D., "A New Control Concept of Anthropomorphic Manipulators," *IFToMM Journal Mechanism and Machine Theory,* Vol. 12, No. 5, 1977, pp. 515–530.

4 Stepanenko, Yu., and Vukobratović, M., "Dynamics of Open-Chain Articulated Active Mechanisms," *Math. Biosciences,* Vol. 28, No. 1/2, 1976, pp. 137–170.

5 Stepanenko, Yu., "A Method of Analyzing Lever Space Mechanisms," *Mekhanika mashin* (in Russian), Vol. 23, Moscow, 1970, pp. 80–89.

6 Juričić, D., and Vukobratović, M., "Mathematical Modeling of Bipedal Walking System," ASME Paper No. 72-WA/BHF-13.

7 Popov E. P., et al., "Design of Robot Control Using Dynamic Models of Manipulator Devices," *Proceedings, VI IFAC Symposium on Automatic Control in Space,* USSR, 1974.

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