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Third-Order and Fourth-Order Iterative Methods Free from Second Derivative for Finding Multiple Roots of Nonlinear Equations

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Abstract. In this paper, we present two new families of third-order and fourth-order methods for finding multiple roots of nonlinear equations. Each of them requires one evaluation of the function and two of its first derivative per iteration. Several numerical examples are given to illustrate the performance of the presented methods.

Keywords: - Newtons method; Multiple roots; Iterative methods; Nonlinear equations; Order of convergence; Root-finding.

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1. Introduction

Finding the root of a nonlinear equation is a common and important problem in science and engineering. In this paper, we consider iterative methods to find a multiple root α of multiplicity m, i.e. $f^{(j)}(\alpha) = 0, j = 0, 1, \cdots, m-1$ and $f^{(m)}(\alpha) \neq 0$, of a nonlinear equation $f(x) = 0$.
The modified Newton's method for multiple roots is quadratically convergent and it is written as [20]

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} \tag{1.1}$$

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which requires the knowledge of the multiplicity m. Several methods including many multiple-root-finding methods of different orders are presented. For example, see Hansen and Patrick [7], Victory and Neta [22], Dong [6], Neta and Johnson [18], Neta [15]-[16], Chun and Neta [4], and Werner [23], etc. All of these methods require the knowledge of the multiplicity m.

The third-order Euler-Chebyshev method for finding multiple roots [21] is given by

$$x_{n+1} = x_n - \frac{m(3-m)}{2}\frac{f(x_n)}{f'(x_n)} - \frac{m^2}{2}\frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3} \tag{1.2}$$

The cubically convergent Halley's method, which is a special case of the Hansen and Patrick's method [7], is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{m+1}{2m}f'(x_n) - \frac{f(x_n)f''(x_n)}{2f'(x_n)}} \tag{1.3}$$

The third-order Osada method [19] is written as

$$x_{n+1} = x_n - \frac{1}{2}m(m+1)\frac{f(x_n)}{f'(x_n)} + \frac{1}{2}(m-1)^2\frac{f'(x_n)}{f''(x_n)} \tag{1.4}$$

Dong [5] has developed two third-order methods requiring two evaluations of f and one evaluation of f'

$$\begin{cases} y_n = x_n - \sqrt{m}u_n, \\ x_{n+1} = y_n - m(1-\frac{1}{\sqrt{m}})^{1-m}\frac{f(y_n)}{f'(x_n)}, \end{cases} \tag{1.5}$$

$$\begin{cases} y_n = x_n - u_n, \\ x_{n+1} = y_n + \frac{u_n f(y_n)}{f(y_n)-(1-\frac{1}{m})^{m-1}f(x_n)}, \end{cases} \tag{1.6}$$

where $u_n = \frac{f(x_n)}{f'(x_n)}$.

In [18], Neta and Johnson have proposed a fourth-order method requiring one-function and three-derivative evaluation per iteration. This method is based on the Jarratt method [9] given by the iteration function

$$x_{n+1} = x_n - \frac{f(x_n)}{a_1 f'(x_n) + a_2 f'(y_n) + a_3 f'(\eta_n)} \tag{1.7}$$

Table 1

m	2	2	3	4	5	6
a	1	$\frac{4}{3}$	$\frac{3}{2}$	2	$\frac{5}{2}$	3
b	free	free	free	2	$\frac{5}{2}$	3
c	free	$\frac{1-b}{3}$	$\frac{3}{5}-\frac{b}{4}$	0.06478279184	0.0217372041	0.0082119760
a_1	$-\frac{1}{2}$	$\frac{1-2b}{2}$	$\frac{25}{108}b-\frac{43}{72}$	-0.4374579865	-0.4303454005	-0.3681491853
a_2	2	$3(b-1)$	$4-\frac{25}{72}b$	7.90412890309	18.8154365391	39.6876826792
a_3	0	2	$-\frac{125}{72}$	-5.9128176652	-15.8940830499	-35.6993794378

where

$$\begin{cases} u_n = \frac{f(x_n)}{f'(x_n)}, \\ y_n = x_n - a u_n, \\ \nu_n = \frac{f(x_n)}{f'(y_n)}, \\ \eta_n = x_n - b u_n - c \nu_n. \end{cases} \tag{1.8}$$

Neta and Johnson [18] give a table of values for the parameters a, b, c, a_1, a_2, a_3 for several values of m. But, they do not give a closed formula for general case. we list this parameters for $m = 2, 3, 4, 5$ and 6 in Table 1. Neta [15] has developed another fourth-order method requiring one-function and three-derivative evaluation per iteration. This method is based on Murakami's method [14] given by

$$x_{n+1} = x_n - a_1 u_n - a_2 \nu_n - a_3 w(x_n) - \psi(x_n) \tag{1.9}$$

where u_n, y_n, ν_n and η_n are given by (1.8) and

$$w(x_n) = \frac{f(x_n)}{f'(\eta_n)},$$
$$\psi(x_n) = \frac{f(x_n)}{b_1 f'(x_n) + b_2 f'(y_n)}. \tag{1.10}$$

A table of values for the parameters $a, b, c, a_1, a_2, a_3, b_1, b_2$ for several values of m is also given by Neta [15].

In [11], Li et al. have proposed a fourth-order method requiring one-function and two-derivative evaluation per iteration. This method is based on the Jarratt method [1] given by the iteration function

$$\begin{cases} y_n = x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \frac{\frac{1}{2}m(m-2)(\frac{m}{m+2})^{-m} f'(y_n) - \frac{m^2}{2} f'(x_n)}{f'(x_n) - (\frac{m}{m+2})^{-m} f'(y_n)} \frac{f(x_n)}{f'(x_n)}. \end{cases} \tag{1.11}$$

In [12], a fourth-order method is proposed,

$$\begin{cases} y_n = x_n - rac2mm + 2\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - a_3\frac{f(x_n)}{f'(y_n)} - \frac{f(x_n)}{b_1 f'(x_n) + b_2 f'(y_n)} \end{cases} \tag{1.12}$$

where

$$\begin{aligned} a_3 &= -\frac{1}{2}\frac{(\frac{m}{m+2})^m m(m^4+4m^3-16m-16)}{m^3-4m+8}, \\ b_1 &= -\frac{(m^3-4m+8)^2}{m(m^4+4m^3-4m^2-16m+16)(m^2+2m-4)}, \\ b_2 &= \frac{m^2(m^3-4m+8)}{(\frac{m}{m+2})^m(m^4+4m^3-4m^2-16m+16)(m^2+2m-4)}. \end{aligned}$$

This method require one-function and two-derivative evaluation per iteration.
Heydari et al. [13] have developed two fourth-order methods requiring two-function and two-derivative evaluation per iteration. This method is based on Chun fourth-order method (for simple roots) [3] given by the iteration function

$$\begin{cases} y_n = x_n - \theta_i\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n + \beta_i\frac{f(x_n)}{f'(x_n)} + \lambda_i\frac{f(y_n)}{f'(x_n)} + \delta_i\frac{f(y_n)f'(y_n)}{f'(x_n)^2}, \ i = 1, 2, \end{cases} \tag{1.13}$$

where

$$\begin{cases} \theta_1 = 1, \\ \beta_1 = -m^3 + 3\,m^2 - 3\,m, \\ \lambda_1 = -2m\,(m-1)\,\left(\frac{m-1}{m}\right)^{-m}, \\ \delta_1 = m\,(m-1)^2\left(\frac{m-1}{m}\right)^{-2\,m}, \end{cases} \tag{1.14}$$

and

$$\begin{cases} \theta_2 = \frac{2m}{m+1}, \\ \beta_2 = \frac{1}{4}\,m^2 - m - \frac{1}{4}. \\ \lambda_2 = -\frac{1}{4}\,(m-1)\,(m+1)^2\left(\frac{m-1}{m+1}\right)^{-m}, \\ \delta_2 = \frac{1}{4}m\,(m-1)^2\,\left(\frac{m-1}{m+1}\right)^{-2\,m}. \end{cases} \tag{1.15}$$

The above-mentioned methods have been proven to be competitive to Newton's method in their performance and efficiency. There are, however, not yet many methods known in the literature that can handle the case of multiple roots, see [17]. Motivated and inspired by the recent activities in this direction, in this paper we present two new families of third-order and fourth-order methods for finding multiple roots of nonlinear equations. Each of them requires one evaluation of the function and two of its first derivative per iteration.

2. Development of methods and convergence analysis

Now, we consider the following iteration scheme:

$$\begin{cases} y_n = \phi_i(x_n, \theta), \\ x_{n+1} = x_n - H(\xi_n)\frac{f(x_n)}{f'(x_n)}, \ i = 1, 2 \end{cases} \tag{2.1}$$

where $\xi_n = \frac{f'(y_n)}{f'(x_n)}$, $H(t)$ represents a real-valued function and $\phi_i(x_n, \theta),\ i = 1, 2$ are the second-order iteration functions known in the literature(for $\theta = 1$), which are given as follows.

$$\phi_1(x, \theta) = x - \theta \frac{f(x)f'(x)}{f^2(x) + f'^2(x)} \tag{2.2}$$

$$\phi_2(x, \theta) = x - \theta \frac{f(x)}{f'(x)} \tag{2.3}$$

(2.3) is Newton's iteration function and (2.2) the iteration function derived in [8].

2.1. New third-order schemes free of second derivatives.

For simplicity, we define

$$A_j = \frac{f^{(m+j)}(\alpha)}{f^{(m)}(\alpha)},\ j = 1, 2, \cdots,\ \mu = \frac{m - \theta}{m} \tag{2.4}$$

we consider the following iteration functions

$$\begin{cases} y_n = \phi_1(x_n, \theta), \\ x_{n+1} = x_n - H(\xi_n)\frac{f(x_n)}{f'(x_n)}, \end{cases} \tag{2.5}$$

We can state the following convergence theorems for the two-step method defined by (2.5).

Theorem 2.1. *Let $\alpha \in I$ be a multiple root of multiplicity m of sufficiently differentiable function $f : I \longrightarrow R$ for an open interval I and $H_1(t)$ be a real-valued function as follows*

$$H_1(t) = a_1 + b_1 t \tag{2.6}$$

If x_0 is sufficiently close to α, then the method defined by (2.5) has third-order convergence, when

$$a_1 = -\frac{m\left(-\theta^2\,(m+1)+\theta\,m+m^2\right)}{\theta\,(\theta\,(m+1)-2\,m)}, \tag{2.7}$$

$$b_1 = \frac{(m-\theta)^2\,m\left(\frac{m-\theta}{m}\right)^{-m}}{\theta\,(\theta\,(m+1)-2\,m)}, \tag{2.8}$$

and satisfy the error equation

$$e_{n+1} = [\chi_1(m,\theta)A_1^2 + \psi_1(m,\theta)A_2]e_n^3 + O(e_n^4), \tag{2.9}$$

where $e_n = x_n - \alpha$ and A_1, A_2 are defined in (2.4) and

$$\chi_1(m,\theta) = \frac{1}{2}\,\frac{-2\,m^3-6\,m^2+\left(m^3+9\,m^2+6\,m\right)\theta}{(-2\,m+\theta(m+1))\,(m+1)^2\,(m-\theta)\,m^2} + \frac{1}{2}\,\frac{\left(-2\,m^2-4\,m-2\right)\theta^2}{(-2\,m+\theta(m+1))\,(m+1)^2\,(m-\theta)\,m^2} \tag{2.10}$$

$$\psi_1(m,\theta) = \frac{2\,m^2+\left(-m^2-4\,m\right)\theta+(m+2)\,\theta^2}{(-2\,m+\theta(m+1))\,(m+1)\,m^2\,(m+2)} \tag{2.11}$$

for any $\theta \in R$ and $\theta \neq m, \frac{2m}{m+1}$.

Proof. Let $\alpha \in R$ be a multiple root of multiplicity m of a sufficiently smooth function $f(x)$, $e_n = x_n - \alpha$ and $\hat{e}_n = y_n - \alpha$, where y_n is defined in (2.5). Using the Taylor expansion of $f(x_n), f'(x_n)$ and $f'(y_n)$ about α, we have

$$f(x_n) = \frac{f^{(m)}(\alpha)}{m!}e_n^m[1 + C_1e_n + C_2e_n^2 + C_3e_n^3 + C_4e_n^4 + O(e_n^5)], \tag{2.12}$$

$$f'(x_n) = \frac{f^{(m)}(\alpha)}{(m-1)!}e_n^{m-1}[1 + D_1e_n + D_2e_n^2 + D_3e_n^3 + D_4e_n^4 + O(e_n^5)], \tag{2.13}$$

$$f'(y_n) = \frac{f^{(m)}(\alpha)}{(m-1)!}\hat{e}_n^{m-1}[1 + D_1\hat{e}_n + D_2\hat{e}_n^2 + D_3\hat{e}_n^3 + D_4\hat{e}_n^4 + O(\hat{e}_n^5)], \tag{2.14}$$

where $C_j = \frac{m!}{(m+j)!}A_j$ and $D_j = \frac{(m-1)!}{(m+j-1)!}A_j$. From (2.12) and (2.13), we can get

$$\frac{f(x_n)}{f'(x_n)} = \frac{e_n}{m}[1 + (C_1 - D_1)e_n + (C_2 - D_2 + D_1^2 - C_1D_1)e_n^2 + (C_3 - D_3 + (D_1 - C_1)D_2 + (D_2 - C_2 + C_1D_1 - D_1^2)D_1)e_n^3 + O(e_n^4)]. \tag{2.15}$$

$$\frac{f(x_n)f'(x_n)}{f^2(x_n)+f'^2(x_n)} = \frac{1}{m}e_n + \left(\frac{D_1+C_1}{m} - 2\,\frac{D_1}{m}\right)e_n^2$$
$$+\Big(\frac{D_2+C_1D_1+C_2}{m} - \frac{{D_1}^2+2\,D_2+m^{-2}}{m}$$
$$-2\,\frac{(-D_1+C_1)\,D_1}{m}){e_n}^3 + O\left(e_n^4\right) \qquad (2.16)$$

So, from (2.16) we have

$$\hat{e}_n = e_n - \theta\frac{f(x_n)f'(x_n)}{f^2(x_n)+f'^2(x_n)} = d_0e_n + d_1e_n^2 + d_2e_n^3 + d_3e_n^4 + O(e_n^5) \quad (2.17)$$

where

$$d_0 = \mu$$
$$d_1 = \frac{\theta\,(D_1-C_1)}{m}$$
$$d_2 = \frac{\theta\,\left(D_2m^2+m^2C_1D_1-m^2C_2-{D_1}^2m^2+1\right)}{m^3}$$
$$d_3 = -\frac{\theta\,\left(-D_3m^2+m^2C_3-m^2C_1D_2-m^2C_2D_1\right)}{m^3}$$
$$-\frac{\theta\,\left(2\,D_1D_2m^2-3\,C_1-{D_1}^3m^2+3\,D_1+C_1{D_1}^2m^2\right)}{m^3}$$

By substituting (2.17) into (2.14) , we can get

$$f'(y_n) = \frac{f^{(m)}(\alpha)}{(m-1)!}e_n^{m-1}\Lambda[1+D_1\hat{e}_n+D_2\hat{e}_n^2+D_3\hat{e}_n^3+D_4\hat{e}_n^4+O(\hat{e}_n^5)], \quad (2.18)$$

where

$$\Lambda = (d_0+d_1e_n^1+d_2e_n^2+d_3e_n^3+O(e_n^4))^{m-1}$$
$$= d_0^{m-1}+(m-1)d_0^{m-2}d_1e_n+\{\binom{m-1}{2}d_1^2d_0^{m-3}+(m-1)d_2d_0^{m-2}\}e_n^2$$
$$+\{2\binom{m-1}{2}d_1d_2d_0^{m-3}+(m-1)d_3d_0^{m-2}$$
$$+\binom{m-1}{3}d_1^3d_0^{m-4}\}e_n^3+O(e_n^4). \qquad (2.19)$$

Dividing (2.14) by (2.13), we have

$$\xi_n = \frac{f'(y_n)}{f'(x_n)}$$
$$= \mu^{m-1} - \frac{\theta\,\left(\mu^{m-2}D_1\,(\mu-m+1)+\mu^{m-2}C_1\,(m-1)\right)e_n}{m} + O\left(e_n^2\right). \qquad (2.20)$$

Now from (2.5), (2.6), (2.15) and (2.20) we have

$$e_{n+1} = e_n - H_1(\xi_n)\frac{f(x_n)}{f'(x_n)} = K_1 e_n + K_2 e_n^2 + K_3 e_n^3 + O(e_n^4), \quad (2.21)$$

where

$$K_1 = -\frac{-m + a_1 + b_1\mu^{m-1}}{m}, \quad (2.22)$$

$$K_2 = -\frac{\left(b_1\mu^{m-2}\left(\theta\, m - \theta - \mu\, m - \mu\,\theta - \mu\, m\theta\right) - a_1 m\right) A_1}{m^3\,(m+1)} \quad (2.23)$$

Before we list K_3, we choose a_1 and b_1 to annihilate the coefficients K_1 and K_2, so we have

$$a_1 = -\frac{m\left(-\theta^2\,(m+1) + \theta\, m + m^2\right)}{\theta\,\left(\theta\,\left(m+1\right) - 2\, m\right)}, \quad (2.24)$$

$$b_1 = \frac{(m-\theta)^2\, m\left(\frac{m-\theta}{m}\right)^{-m}}{\theta\,\left(\theta\,\left(m+1\right) - 2\, m\right)}, \quad (2.25)$$

By substituting (2.24) and (2.25) into K_3, we get

$$K_3 = \chi_1(m,\theta)A_1^2 + \psi_1(m,\theta)A_2, \quad (2.26)$$

where

$$\chi_1(m,\theta) = \frac{1}{2}\,\frac{-2\,m^3 - 6\,m^2 + \left(m^3 + 9\,m^2 + 6\,m\right)\theta}{\left(-2\,m + \theta(m+1)\right)(m+1)^2\,(m-\theta)\,m^2} + \frac{1}{2}\,\frac{\left(-2\,m^2 - 4\,m - 2\right)\theta^2}{\left(-2\,m + \theta(m+1)\right)(m+1)^2\,(m-\theta)\,m^2} \quad (2.27)$$

$$\psi_1(m,\theta) = \frac{2\,m^2 + \left(-m^2 - 4\,m\right)\theta + (m+2)\,\theta^2}{\left(-2\,m + \theta(m+1)\right)(m+1)\,m^2\,(m+2)} \quad (2.28)$$

and $\theta \neq m, \frac{2m}{m+1}$. Therefore, we have

$$e_{n+1} = [\chi_1(m,\theta)A_1^2 + \psi_1(m,\theta)A_2]e_n^3 + O(e_n^4), \quad (2.29)$$

which indicates that the order of convergence of the methods defined by (2.5) is at least three. This completes the proof.

Theorem 2.2. *Let $\alpha \in I$ be a multiple root of multiplicity m of sufficiently differentiable function $f : I \longrightarrow R$ for an open interval I and $H_2(t)$ be a real-valued function as follows*

$$H_2(t) = \frac{1}{a_2 + b_2 t}. \quad (2.30)$$

If x_0 is sufficiently close to α, then the method defined by (2.5) has third-order convergence, when

$$a_2 = \frac{m^2 - 3\,\theta\, m + (m+1)\,\theta^2}{m\theta\,(\theta\,(m+1) - 2\,m)}, \tag{2.31}$$

$$b_2 = -\frac{(m-\theta)^2 \left(\frac{m-\theta}{m}\right)^{-m}}{m\theta\,(\theta\,(m+1) - 2\,m)}, \tag{2.32}$$

and satisfy the error equation

$$e_{n+1} = [\chi_2(m,\theta)A_1^2 + \psi_2(m,\theta)A_2]e_n^3 + O(e_n^4), \tag{2.33}$$

where $e_n = x_n - \alpha$ and A_1, A_2 are defined in (2.4) and

$$\chi_2(m,\theta) = \frac{1}{2}\,\frac{-2\,m^2 - 2\,m + \left(m^2 + 7\,m\right)\theta + (-2\,m - 2)\,\theta^2}{m\,(\theta\, m - 2\,m + \theta)\,(m+1)^2\,(m-\theta)}, \tag{2.34}$$

$$\psi_2(m,\theta) = \frac{2\,m^2 + \left(-m^2 - 4\,m\right)\theta + (m+2)\,\theta^2}{(-2\,m + \theta(m+1))\,(m+1)\,m^2\,(m+2)} \tag{2.35}$$

for any $\theta \in R$ and $\theta \neq m, \frac{2m}{m+1}$.

Proof. The proof method is similar to the Theorem 2.1's, it's easy so omit.

Theorem 2.3. *Let $\alpha \in I$ be a multiple root of multiplicity m of sufficiently differentiable function $f : I \longrightarrow R$ for an open interval I and $H_3(t)$ be a real-valued function as follows*

$$H_3(t) = 1 + \frac{a_3 t}{1 + b_3 t}. \tag{2.36}$$

If x_0 is sufficiently close to α, then the method defined by (2.5) has third-order convergence, when

$$a_3 = \frac{\theta\,(m-1)\left(\frac{m-\theta}{m}\right)^{-m}\left(m^2\theta - 2\,m^2 + 2\,m - \theta\right)}{m^3}, \tag{2.37}$$

$$b_3 = -\frac{\left(\frac{m-\theta}{m}\right)^{-m}\left(m^3 + m^2\theta - \theta^2 m^2 - 2\,\theta\, m + \theta^2\right)}{m^3} \tag{2.38}$$

and satisfy the error equation

$$e_{n+1} = [\chi_3(m,\theta)A_1^2 + \psi_3(m,\theta)A_2]e_n^3 + O(e_n^4), \tag{2.39}$$

where $e_n = x_n - \alpha$ and A_1, A_2 are defined in (2.4) and

$$\chi_3(m,\theta) = \frac{1}{2}\,\frac{\eta(m,\theta)}{(m+1)^2\,(m-\theta)\,m^3\,(\theta\, m^2 - 2\,m^2 + 2\,m - \theta)}, \tag{2.40}$$

$$\psi_3(m,\theta) = \frac{2\,m^2 + \left(-m^2 - 4\,m\right)\theta + (m+2)\,\theta^2}{(-2\,m + \theta(m+1))\,(m+1)\,m^2\,(m+2)} \tag{2.41}$$

$$\eta(m,\theta) = -2\,m^5 + 6\,m^3 + \left(m^5 - m^3 - 14\,m^2 + 6\,m^4\right)\theta + \left(-8\,m^3 - 2\,m^4 + 4\,m^2 + 10\,m\right)\theta^2 + \left(2\,m^3 + 2\,m^2 - 2\,m - 2\right)\theta^3 \tag{2.42}$$

for any $\theta \in R$ *and* $\theta \neq m, \frac{2m}{m+1}$.

Proof. The proof is similar to that of Theorem 2.1's, so it's omitted.

2.2. **New fourth-order schemes free of second derivatives.** Now we consider the following iteration functions

$$\begin{cases} y_n = \phi_2(x_n,\theta), \\ x_{n+1} = x_n - H(\xi_n)\frac{f(x_n)}{f'(x_n)}, \end{cases} \tag{2.43}$$

We can state the following convergence theorems for the two-step method defined by (2.43).

Theorem 2.4. *Let* $\alpha \in I$ *be a multiple root of multiplicity* m *of sufficiently differentiable function* $f : I \longrightarrow R$ *for an open interval* I *and* $H_4(t)$ *be a real-valued function as follows*

$$H_4(t) = a_4 + b_4 t + c_4 \frac{t^2}{2}. \tag{2.44}$$

If x_0 *is sufficiently close to* α*, then the method defined by (2.43) has fourth-order convergence, when*

$$\theta = \frac{2m}{m+2} \tag{2.45}$$

$$a_4 = \frac{1}{8} m\left(m^3 + 6\,m^2 + 8\,m + 8\right), \tag{2.46}$$

$$b_4 = -\frac{1}{4}\,m^3\,(m+3)\left(\frac{m}{m+2}\right)^{-m}, \tag{2.47}$$

$$c_4 = \frac{1}{4}\,m^4\left(\frac{m}{m+2}\right)^{-2\,m} \tag{2.48}$$

and satisfy the error equation

$$e_{n+1} = K_4 e_n^4 + O(e_n^5), \tag{2.49}$$

where $e_n = x_n - \alpha$ *and the error constant* K_4 *is given by*

$$K_4 = \frac{1}{3}\,\frac{m^4 + 2\,m^3 + 2\,m^2 - 2\,m + 12}{(m+1)^3\,m^5} A_1^3 - \frac{A_1 A_2}{m\,(m+2)\,(m+1)^2} + \frac{m A_3}{(m+3)\,(m+2)^3\,(m+1)}. \tag{2.50}$$

Proof. From (2.15) and (2.43) we have

$$\tilde{e}_n = e_n - \theta \frac{f(x_n)}{f'(x_n)} = p_0 e_n + p_1 e_n^2 + p_2 e_n^3 + p_3 e_n^4 + O(e_n^5), \tag{2.51}$$

where

$$p_0 = \mu, \tag{2.52}$$

$$p_1 = -\frac{\theta(C_1 - D_1)}{m}, \tag{2.53}$$

$$p_2 = -\frac{\theta(C_2 - D_2 + D_1^2 - C_1 D_1)}{m}, \tag{2.54}$$

$$\begin{aligned} p_3 &= -\frac{\theta(C_3 - D_3 + (D_1 - C_1)D_2}{m} \\ &- \frac{\theta(D_2 - C_2 + C_1 D_1 - D_1^2)D_1)}{m}. \end{aligned} \tag{2.55}$$

and

$$f'(y_n) = \frac{f^{(m)}(\alpha)}{(m-1)!} \tilde{e}_n^{m-1} [1 + D_1 \tilde{e}_n + D_2 \tilde{e}_n^2 + D_3 \tilde{e}_n^3 + D_4 \tilde{e}_n^4 + O(\tilde{e}_n^5)], \tag{2.56}$$

By substituting (2.51) into (2.56) , we can get

$$f'(y_n) = \frac{f^{(m)}(\alpha)}{(m-1)!} e_n^{m-1} \Delta [1 + D_1 \tilde{e}_n + D_2 \tilde{e}_n^2 + D_3 \tilde{e}_n^3 + D_4 \tilde{e}_n^4 + O(\tilde{e}_n^5)], \tag{2.57}$$

where

$$\begin{aligned} \Delta &= (p_0 + p_1 e_n^1 + p_2 e_n^2 + p_3 e_n^3 + O(e_n^4))^{m-1} \\ &= p_0^{m-1} + (m-1) p_0^{m-2} p_1 e_n + \{ \binom{m-1}{2} p_1^2 p_0^{m-3} \\ &+ (m-1) p_2 p_0^{m-2} \} e_n^2 + \{ 2 \binom{m-1}{2} p_1 p_2 p_0^{m-3} + (m-1) p_3 p_0^{m-2} \\ &+ \binom{m-1}{3} p_1^3 p_0^{m-4} \} e_n^3 + O(e_n^4). \end{aligned} \tag{2.58}$$

Now from (2.15), (2.20), (2.43) and (2.44) we have

$$e_{n+1} = e_n - H_4(\xi_n) \frac{f(x_n)}{f'(x_n)} = K_1 e_n + K_2 e_n^2 + K_3 e_n^3 + K_4 e_n^4 + O(e_n^5), \tag{2.59}$$

where

$$K_1 = -\frac{1}{2m} \left(2a_4 + 2b_4 \mu^{m-1} + c_4 \mu^{2m-2} - 2m \right), \tag{2.60}$$

$$K_2 = \{\frac{1}{(m+1)\,m^2}a_4 + \frac{\mu^{m-1}\theta\,m + \mu^{m-1}\theta - \mu^{m-2}\theta\,m + \mu^{m-2}\theta + \mu^{m-1}m}{(m+1)\,m^3}b_4 + \frac{1}{2}\,\frac{2\,\mu^{2\,m-2}\theta\,m + 2\,\mu^{2\,m-2}\theta - 2\,\mu^{2\,m-3}\theta\,m + 2\,\mu^{2\,m-3}\theta + \mu^{2\,m-2}m}{(m+1)\,m^3}c_4\}A_1 \tag{2.61}$$

Before we list K_3, we choose a_4 and b_4 to annihilate the coefficients K_1 and K_2, so we have

$$a_4 = -\frac{1}{2\,(\mu - m + m\mu + 1)\,\theta}\{\theta\,\left((m-1)\,\mu^{2\,m-2} - (m+1)\,\mu^{2\,m-1}\right)c_4 + (-2\,\mu\,m^2 + \left(2\,m^2 - 2\,\mu\,m^2 - 2\,m - 2\,m\mu\right)\theta\} \tag{2.62}$$

$$b_4 = -\frac{1}{(\mu - m + m\mu + 1)\,\theta}\{m^2\mu^{2-m} + (\theta((m+1)\mu^m + (1-m)\mu^{m-1})c_4\}. \tag{2.63}$$

By substituting (2.62) and (2.63) into K_3, we get

$$K_3 = \frac{\varphi_1(\theta, m, c_4)}{2\,(m+1)^2\,m^5\mu\,(-m + \mu + m\mu + 1)}A_1^2 + \frac{\varphi_2(\theta, m)}{m^2\,(m+1)\,(m+2)\,(-m + \mu + m\mu + 1)}A_2, \tag{2.64}$$

$$\varphi_1(\theta, m, c_4) = -\frac{\theta^2\,(\mu^m)^2\,(\mu + 1 + m\mu - m)^3}{\mu^3}c_4 + m^2\left(2\,\mu\,\theta - 2\,\mu\,m^2\theta + 2\,m\mu - 2\,\mu\,m^2 + 4\,\mu^2m^2 + 4\,\mu^2m + m^2\theta - 3\,m\theta + 2\,\theta\right. \tag{2.65}$$

$$\varphi_2(\theta, m) = \mu\,(-2\,m + \theta\,m + 2\,\theta)\,. \tag{2.66}$$

Now we choose θ and c_4 to annihilate the coefficients $\varphi_1(\theta, m, c_4)$ and $\varphi_2(\theta, m)$ in K_3, so we can get

$$\theta = \frac{2m}{m+2} \tag{2.67}$$

and

$$c_4 = \frac{1}{4}\,m^4\left(\frac{m}{m+2}\right)^{-2\,m} \tag{2.68}$$

By substituting (2.67) and (2.68) into (**??**) and (2.63), we get

$$a_4 = \frac{1}{8}m\left(m^3 + 6\,m^2 + 8\,m + 8\right) \tag{2.69}$$

$$b_4 = -\frac{1}{4}\,m^3\,(3+m)\left(\frac{m}{m+2}\right)^{-m} \tag{2.70}$$

Substituting (2.67)-(2.70) into (2.59), we can get the error equation

$$e_{n+1} = K_4 e_n^4 + O(e_n^5), \tag{2.71}$$

where

$$K_4 = \frac{1}{3}\,\frac{m^4+2\,m^3+2\,m^2-2\,m+12}{(m+1)^3\,m^5}A_1^3 - \frac{A_1A_2}{m\,(m+2)\,(m+1)^2} + \frac{mA_3}{(m+3)\,(m+2)^3\,(m+1)}. \tag{2.72}$$

which indicates that the order of convergence of the methods defined by (2.43) is at least four. This completes the proof.

Theorem 2.5. *Let $\alpha \in I$ be a multiple root of multiplicity m of sufficiently differentiable function $f : I \longrightarrow R$ for an open interval I and $H_5(t)$ be a real-valued function as follows*

$$H_5(t) = \frac{1}{a_5 + b_5 t + c_5 t^2}. \tag{2.73}$$

If x_0 is sufficiently close to α, then the method defined by (2.43) has fourth-order convergence, when

$$\theta = \frac{2m}{m+2} \tag{2.74}$$

$$a_5 = \frac{1}{16}\,\frac{m^4+2\,m^3-8\,m^2-16\,m+16}{m}, \tag{2.75}$$

$$b_5 = -\frac{1}{8}\left(\frac{m}{m+2}\right)^{-m} m\left(m^2-6\right), \tag{2.76}$$

$$c_5 = \frac{1}{16}\,m^2\,(m-2)\left(\frac{m}{m+2}\right)^{-2\,m} \tag{2.77}$$

and satisfy the error equation

$$e_{n+1} = K_4 e_n^4 + O(e_n^5), \tag{2.78}$$

where $e_n = x_n - \alpha$ and the error constant K_4 is given by

$$K_4 = \frac{1}{3}\,\frac{m^4+2\,m^3+5\,m^2-14\,m+12}{(m+1)^3\,m^5}A_1^3 - \frac{A_1A_2}{m\,(m+2)\,(m+1)^2} + \frac{mA_3}{(m+3)\,(m+2)^3\,(m+1)}. \tag{2.79}$$

Proof. The proof is similar to that of Theorem 2.4's, so it's omitted.

Theorem 2.6. *Let $\alpha \in I$ be a multiple root of multiplicity m of sufficiently differentiable function $f : I \longrightarrow R$ for an open interval I and*

$H_6(t)$ be a real-valued function as follows

$$H_6(t) = \frac{a_6 + b_6 t}{1 + c_6 t}. \tag{2.80}$$

If x_0 is sufficiently close to α, then the method defined by (2.43) has fourth-order convergence, when

$$\theta = \frac{2m}{m+2} \tag{2.81}$$
$$a_6 = -\frac{1}{2}m^2, \tag{2.82}$$
$$b_6 = \frac{1}{2}m\left(\frac{m}{m+2}\right)^{-m}(m-2), \tag{2.83}$$
$$c_6 = -\left(\frac{m}{m+2}\right)^{-m} \tag{2.84}$$

and satisfy the error equation

$$e_{n+1} = K_4 e_n^4 + O(e_n^5), \tag{2.85}$$

where $e_n = x_n - \alpha$ and the error constant K_4 is given by

$$K_4 = \frac{1}{3}\frac{m^3 + 2\,m^2 + 2\,m - 2}{(m+1)^3\,m^4}A_1^3 - \frac{A_1 A_2}{m\,(m+2)\,(m+1)^2} + \frac{mA_3}{(m+3)\,(m+2)^3\,(m+1)}. \tag{2.86}$$

Proof. The proof is similar to that of Theorem 2.4's, so it's omitted.

Theorem 2.7. *Let $\alpha \in I$ be a multiple root of multiplicity m of sufficiently differentiable function $f : I \longrightarrow R$ for an open interval I and $H_7(t)$ be a real-valued function as follows*

$$H_7(t) = \frac{a_7 + t + b_7 t^2}{c_7 + t}. \tag{2.87}$$

If x_0 is sufficiently close to α, then the method defined by (2.43) has fourth-order convergence, when

$$\theta = \frac{2m}{m+2} \tag{2.88}$$

$$a_7 = \frac{1}{4}\frac{\left(m^5+2\,m^4+2\,m^3-4\,m^2-16\right)\left(\frac{m}{m+2}\right)^m}{m^2\,(m+3)}, \tag{2.89}$$

$$b_7 = -\frac{1}{4}\frac{\left(\frac{m}{m+2}\right)^{-m} m\left(m^2-2\,m+2\right)}{m+3} \tag{2.90}$$

$$c_7 = -\frac{\left(\frac{m}{m+2}\right)^m \left(m^4+3\,m^3+2\,m^2-4\,m+4\right)}{m^3\,(m+3)} \tag{2.91}$$

and satisfy the error equation

$$e_{n+1} = K_4 e_n^4 + O(e_n^5), \tag{2.92}$$

where $e_n = x_n - \alpha$ and the error constant K_4 is given by

$$K_4 = \frac{1}{3}\frac{m^7+4\,m^6+8\,m^5+4\,m^4-4\,m^3-20\,m^2+28\,m-24}{(m+1)^3\,m^5\,(2\,m+m^3+2\,m^2-2)}A_1^3 - \frac{A_1A_2}{m\,(m+2)\,(m+1)^2} + \frac{mA_3}{(m+3)\,(m+2)^3\,(m+1)}. \tag{2.93}$$

Proof. The proof is similar to that of Theorem 2.4's, so it's omitted.

Remark 2.8. From Theorem (2.6) we can see the method defined by (2.43) with $H_6(t)$ is the equivalent to the method (1.11).

Remark 2.9. Any method of the family (2.43) uses three evaluations per iteration, and has fourth-order convergence with conditions of Theorem 2.4-2.7, which accord with the conjecture of Kung-Traub that a multipoint iteration without memory based on n evaluations achieves optimal convergence order 2^{n-1} for $n = 3$.

Remark 2.10. Per iteration the presented method requires one evaluation of the function, two of its first derivative. We consider the definition of efficiency index [24] as $p^{\frac{1}{w}}$, where p is the order of the method and w is the number of function evaluations per iteration required by the method. If we assume that all the evaluations have the same cost as function one, we have that the presented method has the efficiency index $I = \sqrt[3]{4} \simeq 1.587$, which is better than $I = \sqrt[2]{2} \simeq 1.414$ of Newtons method, $I = \sqrt[3]{3} \simeq 1.442$ of third-order methods (1.2), (1.3), (1.4), (1.5) and (1.6) and $I = \sqrt[4]{4} \simeq 1.414$ of the fourth-order methods proposed in [9, 13, 18].

3. Numerical Examples

In this section, some numerical test of some various multiple-root-finding methods as well as our new methods and Newton's method are presented. All computations were done using the MAPLE 13 with 128 digit floating point arithmetics(Digits: = 128).We use the following functions, which have also been considered in [2, 4, 17].

$f_1(x) = (x^3+4x^2-10)^3,\ m = 3,\ x_* = 1.3652300134140968457608068290,$

$f_2(x) = (\sin^2 x - x^2+1)^2,\ m = 2,\ x_* = 1.4044916482153412260350868178,$

$f_3(x) = (x^2-e^x-3x+2)^5,\ m = 5,\ x_* = 0.25753028543986076045536730494,$

$f_4(x) = (\cos x - x)^3,\ m = 3,\ x_* = 0.7390851332151606416553120876 7,$

$f_5(x) = (xe^{x^2} - \sin^2 x + 3\cos x + 5)^4,\ m = 4,$

$x_* = -1.2076478271309189270094167584,$

$f_6(x) = (\sin x - \frac{x}{2})^2,\ m = 2,\ x_* = 1.8954942670339809471440357381,$

We present some numerical test results for various cubically convergent iterative schemes in Table 1. Compared were the Newton method (1.1)(NM), the method of Halley-like method (1.3) (HM),Osada's method (1.4) (OM), Euler-Chebyshev method (1.2) (ECM), Dong's method (1.6) (DM) and the methods (2.5) with $H_1(t)$ and $\theta = -1$ (PM1), (2.5) with $H_2(t)$ and $\theta = -1$ (PM2) and (2.5) with $H_2(t)$ and $\theta = -1$ (PM3), respectively, introduced in the present contribution.

Table 1 shows the number of iterations (IT) required such that $|f(x_n)| < 10^{-32}$, the number of function evaluations(NFE) and the values of $|f(x_n)|$. The test results in Table 1 show that for most of the functions we tested, the methods introduced in the present presentation have at least equal performance compared to the other third-order methods, and can also compete with Newtons method.

We also present some numerical test results for various quartically convergent iterative schemes in Table 2. Compared were Newton's method (1.1)(NM), Neta method (1.7) (NEM)(with $b = 0$), Li et al's method (1.12)(LM), Heydari et al's method (1.13) and (1.14) (HEM1), Heydari et al's method (1.13) and (1.15) (HEM2) and the methods (2.43) with $H_4(t)$ (PM4), (2.43) with $H_5(t)$ (PM5), (2.43) with $H_6(t)$ (PM6) and (2.43) with $H_7(t)$ (PM7), respectively, introduced in the present contribution. We used the same test functions as in the test for the above cubically convergent methods.

Table 2 shows the number of iterations required such that $|f(x_n)| < 10^{-64}$ and the number of function evaluations (NFEs) after required iterations in parentheses.

Table 1. Comparison of various third-order convergent iterative methods and Newton's method ('div' means divergent)

	IT	NFE	$\|f(x_n)\|$	IT	NFE	$\|f(x_n)\|$
	$f_1(x), x_0 = -0.4$			$f_1(x), x_0 = 3.0$		
NM	167	334	6.86e-63	6	12	6.11e-56
HM	92	276	5.87e-92	4	12	1.08e-81
OM	11	33	2.43e-38	4	12	1.30e-46
ECM	7	21	5.81e-97	4	12	1.79e-57
DM	28	84	1.40e-98	4	12	1.28e-84
PM1	4	12	1.36e-46	4	12	3.43e-48
PM2	24	72	6.21e-46	4	12	4.86e-61
PM3	10	30	1.68e-52	4	12	1.37e-55
	$f_2(x), x_0 = 1.2$			$f_2(x), x_0 = 2.0$		
NM	5	10	6.61e-47	6	12	5.12e-64
HM	3	9	1.49e-42	4	12	7.43e-77
OM	4	12	5.78e-46	4	12	3.54e-51
ECM	4	12	4.55e-92	4	12	1.53e-63
DM	3	9	1.08e-45	4	12	5.54e-85
PM1	3	9	1.21e-33	4	12	1.13e-58
PM2	3	9	3.36e-50	4	12	4.33e-69
PM3	3	9	1.38e-45	4	12	9.94e-67
	$f_3(x), x_0 = -1.0$			$f_3(x), x_0 = 2.5$		
NM	4	8	1.19e-59	5	10	1.12e-71
HM	2	6	5.24e-35	3	9	3.44e-50
OM	3	9	2.18e-85	4	12	5.59e-101
ECM	3	9	4.89e-89	3	9	9.27e-35
DM	2	6	2.32e-35	3	9	1.33e-59
PM1	3	9	5.28e-88	3	9	8.77e-41
PM2	3	9	2.81e-86	3	9	1.18e-62
PM3	3	9	2.99e-90	3	9	7.41e-81
	$f_4(x), x_0 = 0.2$			$f_4(x), x_0 = 2.5$		
NM	5	10	1.59e-76	5	10	2.79e-51
HM	3	9	1.85e-54	4	12	3.17e-71
OM	4	12	9.41e-97	div		
ECM	3	9	7.67e-42	4	12	2.25e-48
DM	3	9	1.41e-59	4	12	6.86e-90
PM1	3	9	1.93e-39	4	12	1.90e-40
PM2	3	9	4.01e-51	4	12	4.37e-62
PM3	3	9	1.02e-46	4	12	6.05e-55
	$f_5(x), x_0 = -2.0$			$f_5(x), x_0 = -0.5$		
NM	7	14	5.61e-37	9	18	4.62e-53
HM	4	12	1.61e-61	3	9	3.50e-46
OM	5	15	5.10e-45	div		
ECM	5	15	3.21e-64	div		
DM	4	12	2.45e-52	3	9	1.50e-40
PM1	5	15	1.99e-84	div		
PM2	4	12	1.58e-86	3	9	1.95e-36
PM3	4	12	2.45e-74	3	9	1.81e-61
	$f_6(x), x_0 = 1.6$			$f_6(x), x_0 = 2.5$		
NM	5	10	1.26e-43	5	10	1.70e-40
HM	3	9	1.12e-35	4	12	4.51e-88
OM	7	21	6.41e-84	4	12	4.41e-66
ECM	4	12	3.18e-72	4	12	1.61e-77
DM	3	9	6.79e-41	3	9	2.46e-33
PM1	4	12	3.42e-76	4	12	8.66e-68
PM2	3	9	6.56e-38	4	12	4.17e-74
PM3	3	9	2.92e-35	4	12	8.04e-73

Table 2. The number of iterations and NFEs.

$f(x)$	x_0	NM	NEM	HEM1	HEM2	LM	PM4	PM5	PM6	PM7
f_1	$x_0 = 1.0$	6(12)	4(16)	3(12)	3(12)	3(9)	3(9)	3(9)	3(9)	3(9)
	$x_0 = 3.0$	7(14)	4(16)	4(16)	4(16)	4(12)	4(12)	4(12)	4(12)	4(12)
f_2	$x_0 = 1.2$	6(12)	4(16)	3(12)	3(12)	3(9)	3(9)	3(9)	3(9)	3(9)
	$x_0 = 3.5$	7(14)	4(16)	4(16)	4(16)	4(12)	4(12)	4(12)	4(12)	4(12)
f_3	$x_0 = -1.0$	5(10)	3(12)	3(12)	2(8)	2(6)	2(6)	3(9)	2(6)	2(6)
	$x_0 = 4.5$	7(14)	4(16)	4(16)	4(16)	4(12)	4(12)	4(12)	4(12)	4(12)
f_4	$x_0 = 1.7$	5(10)	3(12)	3(12)	3(12)	3(9)	3(9)	3(9)	3(9)	3(9)
	$x_0 = 2.5$	6(12)	23(92)	4(16)	3(12)	4(12)	4(12)	4(12)	4(12)	4(12)
f_5	$x_0 = -3.5$	17(34)	10(40)	10(40)	9(36)	9(27)	9(27)	10(30)	9(27)	9(27)
	$x_0 = -2.5$	11(22)	6(24)	6(24)	6(24)	6(18)	6(18)	6(18)	6(18)	6(18)
f_6	$x_0 = 1.7$	6(12)	3(12)	3(12)	3(12)	3(9)	3(9)	3(9)	3(9)	3(9)
	$x_0 = 2.0$	5(10)	3(12)	3(12)	3(12)	3(9)	3(9)	3(9)	3(9)	3(9)

The results presented in Table 2 show that for the considered test functions and considered initial guesses the proposed fourth-order methods converge more rapidly than Newton's method and require the less number of function evaluations, so that they can compete with Newton's method. Furthermore, for most of the functions we tested, the new methods have at least equal performance when compared to the other well-known classical methods of the same order.

4. Conclusion

In this work, we have suggested two new family of third-order and fourth-order methods for finding multiple roots of nonlinear equations. The presented methods are compared in their performance with various cubically and quartically convergent iteration methods, and it is observed that they have at least equal performance. The result presented in this work can be continuously applied to developing the other cubically and quartically convergent iterative schemes.

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