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# Determination of the Memory from Boundary Measurements on a Finite Time Interval

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**Abstract** — We study the problem of finding the memory term of a hyperbolic equation from the Dirichlet-to-Neumann map given on a finite time interval. We prove that this map determines uniquely some characteristics of the memory function and thereby memory functions of a special form.

## 1. STATEMENT OF THE PROBLEM AND THE MAIN RESULT

Consider the equation

$$u_{tt} - \Delta u + \int_0^t k(x, t-\tau)u(x,\tau)\,d\tau = 0 \tag{1.1}$$

in $\Omega \times [0,\infty)$, where $k(x,t) \in C^\infty(\overline{\Omega} \times [0,\infty))$ and $\Omega$ is a bounded domain in $\mathbb{R}^n$. We pose the initial-boundary value problem, by supplementing (1.1) with the conditions

$$u(x,0) = u_t(x,0) = 0, \tag{1.2}$$

$$u|_{\partial\Omega\times[0,\infty)} = g(x,t), \tag{1.3}$$

where $g \in C_0^\infty(\partial\Omega \times [0,\infty))$, the subscript $_0$ means that $g$ vanishes at $t = 0$ and for $t > T$ together with its derivatives, with $T$ some positive number, which guarantees validity of the agreement conditions. Existence and uniqueness of a solution to the problem ensues from Theorem 1.2 below.

Assume that we can choose various functions $g \in C_0^\infty(\partial\Omega \times [0,\infty))$ and measure the normal derivative $\partial u/\partial\nu$ on the set $\Gamma = \partial\Omega\times[0,T]$, $T > 0$. Observe that $\partial u/\partial\nu$ on $\Gamma$ depends only on the values of $g$ on $\Gamma$; so we can assume that

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$g = 0$ for $t > T$; i.e., $g \in C_0^\infty(\Gamma)$. The inverse problem consists in finding the memory $k(x,t)$ from the addition information $\{(g, \partial u/\partial\nu|_\Gamma) \mid g \in C_0^\infty(\Gamma)\}$.

The main result of the article is the following theorem:

**Theorem 1.1.** *Let $k_1$ and $k_2$ be two functions satisfying the above conditions. Suppose that $T > \operatorname{diam}\Omega$. If $\partial u_1/\partial\nu = \partial u_2/\partial\nu$ on $\Gamma$ for all $g \in C_0^\infty(\Gamma)$, where $u_j$ are solutions to* (1.1)–(1.3) *with $k = k_j$, then*

$$\partial_t^m k_1(x,0) = \partial_t^m k_2(x,0), \quad m = 0, 1, 2.$$

In the case $T = \infty$ we problem under consideration was studied in [1], wherein a conditional stability estimate of the logarithmic type was proven. The proof was based on the reduction of the original problem to a family of stationary problems with a parameter by means of the Fourier transform in time. In the case of a finite time interval this method is obviously inapplicable. A similar problem for the equation

$$u_{tt} - \Delta u + q(x)u = 0$$

was considered by Rakesh and Symes [2] who proved a uniqueness theorem. Stability was proven by Sun [5]. The case of time-dependent $q$ was studied in [3, 4].

For proving Theorem 1.1 we actually use the method of beam solutions proposed in [2]. The difference is that the problem in [2] reduces to the ray transform, while here we reduce the problem to the Fourier transform.

Below we need a solvability result for the direct problem.

**Theorem 1.2.** *The problem*

$$u_{tt} - \Delta u + \int_0^t k(x, t-\tau)u(x,\tau)\,d\tau = f(x,t), \quad (x,t) \in \Omega \times [0,\infty),$$
$$u(x,0) = u_t(x,0) = 0, \quad x \in \Omega,$$
$$u|_{\partial\Omega\times[0,\infty)} = g(x,t), \quad x \in \partial\Omega,\ 0 \le t < \infty,$$

*where $f \in C^\infty(\overline{\Omega} \times [0,\infty))$ is bounded and $g \in C_0^\infty(\Gamma)$, has a unique solution $u \in C^\infty(\Omega \times [0,\infty))$. Moreover, the estimate*

$$\|u\|_{H^1(\Omega\times[0,T])} \le C\Big(\|f\|_{L_2(\Omega\times[0,T])} + \|g\|_{H^1(\Gamma)}\Big),$$

*holds with some positive constant $C$ depending only on $\Omega$ and $k$.*

The assertion can be derived from the general theory of the initial-boundary value problems for hyperbolic equations as it was done, for example, in [1].

## 2. AUXILIARY ASSERTIONS

Henceforth we denote the convolution $\int_0^t k(x,t-\tau)u(x,\tau)\,d\tau$ by $k * u$.

In the proof of Theorem 1 we need the following two lemmas:

**Lemma 2.1.** *Under the conditions of Theorem 1.1,*

$$\int_\Omega (k_1 - k_2) * u_1 * u_2 \, dx = 0, \quad 0 < t < T, \tag{2.1}$$

*where $u_j$ are arbitrary solutions to* (1.1)–(1.2) *with $k = k_j$, $j = 1, 2$,* (*not necessarily coinciding on* $\Gamma$).

In what follows without loss of generality we assume that the origin coincides with the center of the minimal closed ball containing $\overline{\Omega}$. Let $d$ be the diameter of this ball and $r$, its radius.

**Lemma 2.2.** *Problem* (1.1), (1.2) *has solutions of the form*

$$u(x,t) = \theta_\varepsilon(x \cdot \omega + t - r) e^{i\sigma(x\cdot\omega + t)} + R(x,t), \tag{2.2}$$

*where $\omega \in \mathbb{R}^n$, $|\omega| = 1$, $\sigma > 0$ is arbitrary, $\theta_\varepsilon \in C^\infty(\mathbb{R})$ is such that $\theta'_\varepsilon \geq 0$, $\theta_\varepsilon = 0$ for $t \leq 0$, $\theta_\varepsilon = 1$ for $t \geq \varepsilon$ and $\int_0^\varepsilon \theta_e = \varepsilon/2$. Moreover, $R(x,t)$ satisfies the condition $R|_\Gamma = 0$ and the estimate*

$$\|R\|_{L_2(\Omega\times[0,T])} \leq \frac{C}{\sigma^2}, \quad \sigma \to \infty, \tag{2.3}$$

*where $C$ depends only on $\Omega$ and $k$.*

**Proof of Lemma 2.1.** Take the convolution of the equation for $u_1$ with $u_2$ and the convolution of the equation for $u_2$ with $u_1$ and subtract the resulting equalities:

$$u_2 * (\partial_t^2 u_1) - u_1 * (\partial_t^2 u_2) + u_1 * \Delta u_2 - u_2 * \Delta u_1 + (k_1 - k_2) * u_1 * u_2 = 0.$$

Since $\partial_t^2(u_1 * u_2) = u_1 * (\partial_t^2 u_2) = u_2 * (\partial_t^2 u_1)$ in view of the initial conditions, we have (like $\Delta$ the operations div and grad are taken over $x$)

$$\int_\Omega (k_1 - k_2) * u_1 * u_2 \, dx = \int_\Omega \big(u_2 * \Delta u_1 - u_1 * \Delta u_2\big)\, dx$$
$$= \int_\Omega \operatorname{div}\big(u_2 * \operatorname{grad} u_1 - u_1 * \operatorname{grad} u_2\big)\, dx = \int_{\partial\Omega} \left(u_2 * \frac{\partial u_1}{\partial \nu} - u_1 * \frac{\partial u_2}{\partial \nu}\right) dS.$$

Let $v$ be a solution to (1.1) and (1.2) with $k = k_1$ and the boundary condition $v|_\Gamma = u_2|_\Gamma$. Letting $u_2$ in the above equality equal $v$, we obtain

$$0 = \int_\Omega (k_1 - k_1) * u_1 * v \, dx = \int_{\partial\Omega} \left(v * \frac{\partial u_1}{\partial \nu} - u_1 * \frac{\partial v}{\partial \nu}\right) dS$$
$$= \int_{\partial\Omega} \left(u_2 * \frac{\partial u_1}{\partial \nu} - u_1 * \frac{\partial u_2}{\partial \nu}\right) dS.$$

Here we use the fact that $\partial v/\partial\nu = \partial u_2/\partial\nu$ on $\Gamma$ by the assumption of Theorem 1, for $v = u_2$ on $\Gamma$. Combining the so-obtained equalities, we complete the proof of the lemma. $\square$

**Proof of Lemma 2.2.** Inserting (2.2) in (1.1) and (1.2), we obtain the following equation in $R(x,t)$:

$$R_{tt} - \Delta R + k * R = -k * \big(\theta_\varepsilon(x\cdot\omega + t - r)e^{i\sigma(x\cdot\omega+t)}\big). \tag{2.4}$$

Moreover,

$$R(x,0) = R_t(x,0) = 0, \tag{2.5}$$

since $\theta_\varepsilon(x\cdot\omega - r) = \theta'_\varepsilon(x\cdot\omega - r) = 0$. Supplement (2.4) and (2.5) with the boundary condition

$$R|_\Gamma = 0. \tag{2.6}$$

Problem (2.4)–(2.6) has a unique solution by Theorem 1.2. We are left with proving estimate (2.3). To this end, we apply the integration operator $\mathcal{I}R = \int_0^t R(x,\tau)\,d\tau$ to equation (2.4). In view of the obvious relations, we find that

$$(\mathcal{I}R)_{tt} - \Delta(\mathcal{I}R) + k*(\mathcal{I}R) = -\mathcal{I}\Big(k * \big(\theta_\varepsilon e^{i\sigma(x\cdot\omega+t)}\big)\Big).$$

Moreover, the function $\mathcal{I}R$ meets the zero initial and boundary conditions. Hence, by Theorem 2,

$$\|R\|_{L_2(\Omega\times[0,T])} \le \|\mathcal{I}R\|_{H^1(\Omega\times[0,T])} \le C\|\mathcal{I}\Big(k * \big(\theta_\varepsilon e^{i\sigma(x\cdot\omega+t)}\big)\Big)\|_{L_2(\Omega\times[0,T])}.$$

Consider the function

$$\begin{aligned} k * \big(\theta_\varepsilon e^{i\sigma(x\cdot\omega+t)}\big) &= \int_0^t k(x,\tau)\theta_\varepsilon(x\cdot\omega + t - \tau - r)e^{i\sigma(x\cdot\omega+t-\tau)}\,d\tau \\ &= e^{i\sigma(x\cdot\omega+t)}\int_0^t e^{-i\sigma\tau}(\theta_\varepsilon k)\,d\tau \end{aligned}$$

$$= \frac{e^{i\sigma(x\cdot\omega+t)}}{-i\sigma}\left[\theta_\varepsilon(x\cdot\omega+t-r)k(x,0) - \int_0^t e^{-i\sigma\tau}(\theta_\varepsilon k)_\tau\,d\tau\right], \quad x\in\Omega,\ t\ge 0.$$

Denote the function in the square brackets by $f(x,t)$. Now,

$$\begin{aligned} \mathcal{I}(k*\theta_\varepsilon e^{i\sigma(x\cdot\omega+t)}) &= \frac{e^{i\sigma x\cdot\omega}}{-i\sigma}\int_0^t e^{i\sigma\tau}f(x,\tau)\,d\tau \\ &= \frac{e^{i\sigma x\cdot\omega}}{\sigma^2}\left[e^{i\sigma t}f(x,t) - \int_0^t e^{i\sigma\tau}f_\tau(x,\tau)\,d\tau\right], \quad x\in\Omega,\ t\ge 0. \end{aligned}$$

Since $f$ is bounded uniformly in $x$, $t$ and $\sigma$ together with its derivative with respect to $t$ by a constant depending only on $k$,

$$\|\mathcal{I}(k*\theta_\varepsilon e^{i\sigma(x\cdot\omega+t)})\|_{L_2(\Omega\times[0,T])} \le \frac{C}{\sigma^2},$$

where $C$ depends on $\Omega$ and $k$. The lemma is proven. □

## 3. PROOF OF THEOREM 1.1

By Lemma 2.2, problem (1.1), (1.2) with $k = k_j$ has a solution of the form

$$u_j(x,t) = \theta_\varepsilon(x\cdot\omega_j + t - r)e^{i\sigma(x\cdot\omega_j+t)} + R_j(x,t).$$

Take an arbitrary $\xi \in \mathbb{R}^n$. Choose $\omega_1, \omega_2 \in \mathbb{R}^n$, $|\omega_j| = 1$, so that $-\xi = \sigma(\omega_1+\omega_2)$ which is obviously possible for every sufficiently large positive $\sigma$. Here the vectors $\omega_j$ depend naturally on $\sigma$. Note that the sum $\omega_1 + \omega_2$ and the scalar products $\xi \cdot \omega_j$ vanish as $\sigma \to \infty$.

Insert the solutions $u_j$ with the so-chosen $\omega_j$ in identity (2.1), denoting $k = k_1 - k_2$:

$$0 = \int_\Omega k * u_1 * u_2\, dx = \sum_{j=1}^{4} I_j, \quad 0 \le t \le T.$$

Here

$$I_1(t) = \int_\Omega k * (e^{-i\xi x}e^{i\sigma t}\alpha(x,t))\, dx,$$

$$\alpha(x,t) = \int_0^t \theta_\varepsilon(x\cdot\omega_1 + t - \tau - r)\theta_\varepsilon(x\cdot\omega_2 + \tau - r)\, d\tau;$$

$$I_2(t) = \int_\Omega k * \Big(\theta_\varepsilon(x\cdot\omega_1 + t - r)e^{i\sigma(x\cdot\omega_1+t)}\Big) * R_2\, dx;$$

$$I_3(t) = \int_\Omega k * \Big(\theta_\varepsilon(x\cdot\omega_2 + t - r)e^{i\sigma(x\cdot\omega_2+t)}\Big) * R_1\, dx;$$

$$I_4(t) = \int_\Omega k * R_1 * R_2\, dx.$$

Observe that $\alpha(x,t)$ is a smooth function which is equal to zero for $t \le 2r + \frac{\xi\cdot x}{\sigma}$ and coincides with the linear function $y = t - (2r + \frac{\xi\cdot x}{\sigma} + \varepsilon)$ as $t \ge 2r + \frac{\xi\cdot x}{\sigma} + 2\varepsilon$.

Examine the asymptotic behavior of each integral as $\sigma \to \infty$. We start with $I_1$:

$$\begin{aligned} I_1(t) &= \int_\Omega \int_0^t k(x,\tau)e^{-i\xi x}e^{i\sigma(t-\tau)}\alpha(x,t-\tau)\, d\tau\, dx \\ &= e^{i\sigma t}\Bigg[\int_0^t e^{-i\sigma\tau}\hat{k}(\xi,\tau)\alpha(0,t-\tau)\, d\tau \qquad (3.1) \\ &+ \int_\Omega e^{-i\xi x}\int_0^t e^{-i\sigma\tau}k(x,\tau)\big(\alpha(x,t-\tau) - \alpha(0,t-\tau)\big)\, d\tau\, dx\Bigg], \end{aligned}$$

where $\hat{k}(\xi,t) = \int_\Omega e^{-i\xi x}k(x,t)\, dx$ is the Fourier transform. Integrating by parts in the first integral in the square brackets, we obtain

$$\int_0^t e^{-i\sigma\tau}(\hat{k}\alpha)\, d\tau = \frac{1}{-i\sigma}\Big[-(\hat{k}\alpha)|_{\tau=0} - \int_0^t e^{-i\sigma\tau}(\hat{k}\alpha)_\tau\, d\tau\Big]$$

$$= \frac{1}{-i\sigma}\left[-(\hat{k}\alpha)|_{\tau=0} - \frac{1}{-i\sigma}\left(-(\hat{k}\alpha)_\tau|_{\tau=0} - \int_0^t e^{-i\sigma\tau}(\hat{k}\alpha)_\tau\,d\tau\right)\right]$$

$$= \sum_{j=0}^{N} \frac{\partial_\tau^j(\hat{k}\alpha)|_{\tau=0}}{(i\sigma)^{j+1}} + \frac{1}{(i\sigma)^{N+1}}\int_0^t e^{-i\sigma\tau}\partial_\tau^{N+1}(\hat{k}\alpha)\,d\tau$$

for every natural $N$.

Now, turn to the second integral in (3.1), denoting $\beta(x,t) := \alpha(x,t)-\alpha(0,t)$. By analogy with the above, we can show that

$$\int_0^t e^{-i\sigma\tau}k(x,\tau)\beta(x,t-\tau)\,d\tau = \sum_{j=0}^{N}\frac{\partial_\tau^j(k\beta)|_{\tau=0}}{(i\sigma)^{j+1}} + \frac{1}{(i\sigma)^{N+1}}\int_0^t e^{-i\sigma\tau}\partial_\tau^{N+1}(k\beta)\,d\tau.$$

Observing that $\beta(x,t) = -\frac{x\cdot\xi}{\sigma}$ for $t > d+2\varepsilon+\frac{|x\cdot\xi|}{\sigma}$ (we can make the right-hand side of the last inequality arbitrarily close to $d$) and $|\partial_\tau^j\beta(x,t)| \le \frac{C_j|\xi|}{\sigma}$ for all $t$, with $C_j$ depending only on $j$ and $\Omega$, we obtain

$$\int_0^t e^{-i\sigma\tau}k(x,\tau)\beta(x,t-\tau)\,d\tau$$
$$= -\sum_{j=0}^{N}\frac{\partial_\tau^j k(x,0)(x\cdot\xi)}{\sigma(i\sigma)^{j+1}} + \frac{1}{(i\sigma)^{N+1}}\int_0^t e^{-i\sigma\tau}\partial_\tau^{N+1}(k\beta)\,d\tau$$
$$= -\sum_{j=0}^{N}\frac{\partial_\tau^j k(x,0)(x\cdot\xi)}{\sigma(i\sigma)^{j+1}} + O(\sigma^{-N-2}), \quad \sigma\to\infty,$$

for $t > d+2\varepsilon+\frac{|x\cdot\xi|}{\sigma}$, $O$ is understood in the sense of uniform boundedness over $x$ with a constant depending only on $N$, $\Omega$, and $k$. Thus,

$$I_1(t) = e^{-i\sigma t}\left[\sum_{j=0}^{N}\frac{\partial_\tau^j(\hat{k}(\xi,\tau)\alpha(0,t-\tau))|_{\tau=0}}{(i\sigma)^{j+1}}\right.$$
$$\left.-\sum_{j=0}^{N}\int_\Omega e^{-i\xi x}\frac{\partial_\tau^j k(x,0)(x\cdot\xi)}{\sigma(i\sigma)^{j+1}}\,dx + O(\sigma^{-N-1})\right], \quad \sigma\to\infty,$$

for $t > d+2\varepsilon+\frac{r|\xi|}{\sigma}$.

Now, turn to $I_2$. By analogy with the above, we obtain

$$k * \left(\theta_\varepsilon(x\cdot\omega_1+t-r)e^{i\sigma(x\cdot\omega_1+t)}\right)$$
$$= e^{i\sigma(x\cdot\omega_1+t)}\int_0^t e^{-i\sigma\tau}(\theta_\varepsilon(x\cdot\omega_1+t-\tau-r)k(x,\tau))\,d\tau$$
$$= e^{i\sigma(x\cdot\omega_1+t)}\left[\sum_{j=0}^{N}\frac{\partial_\tau^j(\theta_\varepsilon k)|_{\tau=0}}{(i\sigma)^{j+1}} + \frac{1}{(i\sigma)^{N+1}}\int_0^t e^{-i\sigma\tau}\partial_\tau^{N+1}(\theta_\varepsilon k)\,d\tau\right]$$

Thereby,

$$I_2(t) = e^{i\sigma t}\sum_{j=0}^{N}\int_\Omega\int_0^t e^{i\sigma(x\cdot\omega_1 - s)}\frac{\partial_\tau^j(k\theta_\varepsilon)|_{\tau=0,t\to t-s}}{(i\sigma)^{j+1}}R_2(x,s)\,ds\,dx$$
$$+\frac{e^{i\sigma t}}{(i\sigma)^{N+1}}\int_\Omega\int_0^t e^{i\sigma(x\cdot\omega_1 - s)}\left(\int_0^{t-s} e^{-i\sigma\tau}\partial_\tau^{N+1}(k\theta_\varepsilon)|_{t\to t-s}\,d\tau\right)R_2(x,s)\,ds\,dx.$$

By the estimate of Lemma 2 and Hölder's inequality,

$$|I_2(t)| \le \sum_{j=0}^{N}\frac{C_j\|\partial_\tau^j k(\,\cdot\,,0)\|_{L_2}}{\sigma^{j+3}} + \frac{C_{N+1}\|k\|_{H^{N+1}}}{\sigma^{N+3}}, \quad 0 \le t \le T,$$

where the constants $C_j$ depend only on $\Omega$ and $T$.

Similarly, we estimate $I_3$:

$$|I_3(t)| \le \sum_{j=0}^{N}\frac{C_j\|\partial_\tau^j k(\,\cdot\,,0)\|_{L_2}}{\sigma^{j+3}} + \frac{C_{N+1}\|k\|_{H^{N+1}}}{\sigma^{N+3}}, \quad 0 \le t \le T,$$

We easily estimate the last integral, using Lemma 2 and Hölder's inequality:

$$|I_4(t)| \le \frac{C}{\sigma^4}, \quad 0 \le t \le T,$$

where the constant $C$ depends on $k_1$, $k_2$, $\Omega$, and $T$.

Recalling that the sum of the integrals $I_1$–$I_4$ equals zero for $0 \le t \le T$, we write down

$$\left|\sum_{j=0}^{3}\frac{\partial_\tau^j\big(\hat{k}(\xi,\tau)\alpha(0,t-\tau)\big)|_{\tau=0}}{\sigma^{j+1}}\right|$$
$$\le \sum_{j=0}^{3}\frac{C_j\|\partial_\tau^j k(\,\cdot\,,0)\|_{L_2}}{\sigma^{j+2}} + \sum_{j=0}^{3}\frac{C_j\|\partial_\tau^j k(\,\cdot\,,0)\|_{L_2}}{\sigma^{j+3}} + \frac{C}{\sigma^4}.$$

Suppose that $t$ is such that $d + 2\varepsilon + \frac{r|\xi|}{\sigma} \le t \le T$ (which is possible for a small $\varepsilon$ and a large $\sigma$). Multiplying the resulting inequality by $\sigma$ and letting $\sigma$ tend to infinity, we see that $\hat{k}(\xi,0) = 0$ and hence $k(x,0) = 0$ in view of the arbitrariness of $\xi$. Now, multiplying by $\sigma^2$, using the equality $k(x,0) = 0$, and letting $\sigma$ tend to infinity, we obtain $\partial_\tau k(x,0) = 0$. Similarly, $\partial_\tau^2 k(x,0) = 0$. The theorem is proven.

**Remark 3.1.** We illustrate how the above results can be applied. From the physical viewpoint it is natural to suppose that $k(x,t)$ is a monotone positive function tending to zero as $t \to \infty$. Thus, we can consider models in which $k(x,t) = \alpha(x)(1+t)^{-\beta(x)}$ or $k(x,t) = \alpha(x)e^{-\beta(x)t}$, where $\alpha(x)$ and $\beta(x)$ are smooth positive functions. Once $k(x,0)$ and $k_t(x,0)$ can be found, in both cases we can determine $\alpha(x)$ and $\alpha(x)\beta(x)$ and hence $\beta(x)$. The knowledge of the derivative $k_{tt}(x, 0)$ yields $\alpha(x)\beta(x)(1 + \beta(x))$ in the first case and $\alpha(x)\beta^2(x)$ in the second. Thus, we can decide which model suits best.

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