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Singular Integral Operators Method in Unsteady Spillway Flows

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Abstract A classical open channel hydraulics problem is the determination of the free-surface profile of an unsteady flow over a spillway flow. Thus, by using the Singular Integral Operators Method (S.I.O.M.) then the above problem can be solved by applying numerical evaluation. When a flow rate Q is known, then the velocities and the elevations are computed on the free surface of the spillway flow. For the numerical evaluation of the singular integral equations are used both constant and linear elements. An application is finally given to the determination of the free-surface profile of a special spillway and comparing the numerical results with corresponding results by the Boundary Integral Equation Method (B.I.E.M.).

Keywords Spillway, Free-Surface Flows, Unsteady Flows, Open Channel Hydraulics, Singular Integral Operators Method (S.I.O.M.), Potential Flows, Constant & Linear Elements

1. Introduction

Unsteady free-surface flows, which belong to a wider field of classical hydraulics and fluid mechanics, were quite difficult over the past years to be solved very accurately and efficiently, because several design and measurement purposes occurred during their solution. The two main reasons of the difficulty for solving such hydraulics problems, is firstly the nonlinear character of the boundary conditions and secondly the fact that the boundary of the free-surface flow is not known from the beginning.

In general the spillway problems are usually more difficult to be solved than the normal free-surface open channel hydraulics problems. Some basic parameters of the spillway flows, like the discharge, the free surfaces and the speeds are very important for the design of the hydraulics structures. During the past years the above mentioned parameters were usually obtained through experiments. Beyond the above, the increasing development of computer techniques in hydraulics and fluid mechanics problems over the recent years, made efficient the possibility of obtaining such data by using numerical methods, as well.

As a beginning R.V. Southwell and G. Vaisey[1] used finite differences for the determination of the free waterfall. Some years later was used the finite difference method with a satisfactory success by J.S. Mc Nown, E.Y. Hsu & C.S. Yih[2] and by J.J. Cassidy[3] for the calculation of the flow over a spillway.

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Furthermore, the application of Finite Elements for the study of hydraulics problems was introduced by J.A. Mc Corquodale and C.Y.Li[4] who investigated sluice gate flows. The finite element method was also applied by S.T.K. Chan, B.E. Larock and L.R. Hermann[5] for the solution of the surface fluid flows and M. Ikegawa and K. Washizu[6] for the investigation of a flow over a spillway crest.

Beyond the above, the finite element method was improved by L.T. Isaacs[7],[8] for solving potential flow problems and sluice gate flows. On the other hand, by using Finite Elements B.E. Larock[9] studied spillway flows and H.J. Diersch, A. Schirmer and K.F. Bush[10] several generalized open channel hydraulics problems. Furthermore, E. Varoglou and W.D.L. Finn[11] and P.L. Betts[12] applied the finite element method for the solution of free surface gravity flows.

On the other hand, the Boundary Element Method (B.E.M.) was further used for the solution of open channel flows hydraulics problems and especially those involved to the determination of a free surface under non-linear boundary conditions, by J.A.Ligget[13] and A.H.-D. Cheng, J.A. Liggett and P.L.-F Liu[14].

The complex variable function theory was also used for the solution of free surface potential flow problems. The above method was applied when the effort of gravity is neglected and the geometry of the solid boundary consists of straight segments. T.S. Strelkoff[15] used a computational method based on the complex variable function theory for the numerical evaluation of the sharp-crested weir flows. By using a corresponding method T.S. Strelkoff and M.S. Moayeri[16] studied the waterfall from a flat channel with horizontal and vertical walls. Also, Y. Guo et al.[17] proposed a numerical method for the determination of the spillway flow with free drop and initially unknown discharge.

Furthermore, over the last years E.G. Ladopoulos[18] -[23] and E.G. Ladopoulos and V.A. Zisis[24],[25] introduced and investigated linear and non-linear singular integral equations methods for the solution of fluid mechanics problems. In the present research the above methods will be extended to the solution of potential and unsteady flows problems over spillways. Consequently, the Singular Integral Operators Method (S.I.O.M.)[23],[26]-[33] is applied to the determination of the free-surface profile of a spillway. For the numerical evaluation of the singular integral equations both constant and linear elements are used. An application is finally given to the determination of the free-surface profile of a spillway and comparing the outprints with corresponding results by the Boundary Integral Equation Method (B.I.E.M.).

2. Potential Flow Problems Formulations

Let an homogeneous, incompressible and inviscid fluid, which flows over a spillway. As the flow is irrotational, then for the stream function f with $\mathbf{f} = \nabla f$, one has:[23]

$$\nabla \times \mathbf{f} = 0 \tag{2.1}$$

Beyond the above, because of the conservation of mass for an incompressible fluid, then it is valid:

$$\nabla \bullet \mathbf{f} = 0 \tag{2.2}$$

By combining (2.1) and (2.2.) we obtain the equation of Laplace which is the governing equation in the domain Ω:

$$\nabla^2 f = 0 \tag{2.3}$$

The boundary conditions corresponding to the above hydraulics problem are:

a. Essential conditions of the type: $f=0$ on the lower boundary and on the spillway wall (2.4)
and $f = Q$ on the free surface
where Q denotes the flow rate per unit width.

b. Natural conditions of the type as follows:

$$v = \frac{\vartheta f}{\vartheta n} \tag{2.5}$$

in which v is the velocity and n the unit normal from the free surface.

Also, on the free surface the dynamic boundary condition is valid:

$$\frac{v^2}{2g} + y = H \tag{2.6}$$

where g is the acceleration of gravity, y the free surface elevation and H the design load. (see: Fig. 1).

Consequently, because of (2.6), then the natural condition (2.5) may be written as:

$$\frac{\vartheta f}{\vartheta n} = \sqrt{2g(H-y)} \tag{2.7}$$

Moreover, in the current research of the flow over a spillway, the flow rate Q is known, while the design load H is required as part of the solution.

It is obvious that the spillway flow extend to ±∞, but for the purposes of the numerical evaluation the inflow and outflow streams are cut at right angles to the primary velocity. Thus on the cut portions the following boundary condition is valid:

$$\frac{\vartheta f}{\vartheta n} = 0 \tag{2.8}$$

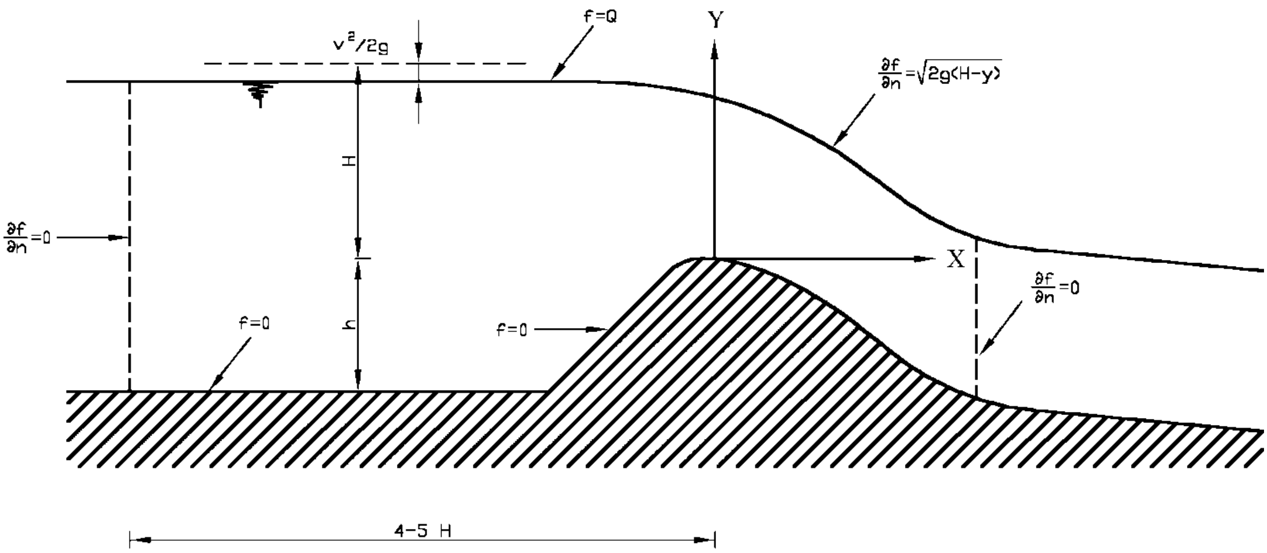

Figure 1. Free-surface profile of a Spillway

By condition (2.8) follows that there is no velocity normal to the main flow. Although this condition is approximate, it is applied enough far from the spillway crest and thus any small error does not affect the interesting part of the flow.

Thus, when a flow rate Q is known, then the position of the free-surface boundary is assumed and the problem is solved by using the above described boundary conditions. Furthermore, by equation (2.6) the hydraulic load H is calculated on the free surface. Hence, if H is the same for all free-surface points, then the problem is solved. Otherwise, the assumed free surface is changed, so that the hydraulic load H becomes constant at all points.

3. Singular Integral Operators Method for Hydraulics

Let a weighting function f^*, in order to have continuous first derivatives. Thus, the function f^* produces the weighted residual statement as following:[23],[26] -[33]

$$\int_{\Omega} (\nabla^2 f) f^* d\Omega = \int_{\Gamma_2} \left(\frac{\vartheta f}{\vartheta n} - \bar{v} \right) f^* d\Gamma - \int_{\Gamma_1} (f - \bar{f}) \frac{\vartheta f^*}{\vartheta n} d\Gamma \quad (3.1)$$

where by (-) are meant average values and Γ_1, Γ_2 are the boundaries where the essential and the natural conditions are affected, respectively.

Furthermore, by integrating by parts the left hand side of eqn (3.1) one has:

$$-\int_{\Omega} \left(\frac{\vartheta f}{\vartheta x_k} \frac{\vartheta f^*}{\vartheta x_k} \right) d\Omega = \int_{\Gamma_2} \frac{\vartheta f}{\vartheta n} f^* d\Gamma - \int_{\Gamma_2} \bar{v} f^* d\Gamma - \int_{\Gamma_1} f \frac{\vartheta f^*}{\vartheta n} d\Gamma + \int_{\Gamma_1} \bar{f} \frac{\vartheta f^*}{\vartheta n} d\Gamma \quad (3.2)$$

A second integration in the left hand side of eqn (3.2) gives:

$$\int_{\Omega} f (\nabla^2 f^*) d\Omega = \int_{\Gamma_2} f \frac{\vartheta f^*}{\vartheta n} d\Gamma - \int_{\Gamma_2} \bar{v} f^* d\Gamma - \int_{\Gamma_1} \frac{\vartheta f}{\vartheta n} f^* d\Gamma + \int_{\Gamma_1} \bar{f} \frac{\vartheta f^*}{\vartheta n} d\Gamma \quad (3.3)$$

For a solution satisfying the Laplace equation, the governing equation takes the form:

$$\nabla^2 f^* + \Delta_i = 0 \quad (3.4)$$

where Δ_i denotes the Dirac delta function

Consequently, the solution of eqn (3.4) is called the fundamental solution and has the property such that:

$$\int_{\Omega} f (\nabla^2 f^* + \Delta_i) d\Omega = \int_{\Omega} f \nabla^2 f^* d\Omega + f_i \quad (3.5)$$

where f_i is the value of the unknown function at the point "i" where a concentrated load is acting.

Then, if eqn (3.4) is satisfied by the fundamental solution, one has:

$$\int_{\Omega} f (\nabla^2 f^*) d\Omega = -f_i \quad (3.6)$$

By using further (3.6), then eqn (3.3) can be written as:

$$f_i + \int_{\Gamma_2} f \frac{\vartheta f^*}{\vartheta n} d\Gamma + \int_{\Gamma_1} \bar{f} \frac{\vartheta f^*}{\vartheta n} d\Gamma = \int_{\Gamma_2} \bar{v} f^* d\Gamma + \int_{\Gamma_1} \frac{\vartheta f}{\vartheta n} f^* d\Gamma \quad (3.7)$$

Beyond the above, by taking the point "i" on the boundary, then the term f_i in eqn (3.7) has to be multiplied by 1/2 for a smooth boundary. On the other hand, if the boundary is not smooth at the point "i", then the number 1/2 has to be replaced by a constant which can be determined from constant potential considerations.

Hence, eqn (3.7) takes the form:

$$c_i f_i + \int_{\Gamma} f \frac{\vartheta f^*}{\vartheta n} d\Gamma = \int_{\Gamma} \frac{\vartheta f}{\vartheta n} f^* d\Gamma \quad (3.8)$$

in which $\Gamma = \Gamma_1 + \Gamma_2$ under the conditions $f = \bar{f}$ on Γ_1 and $\frac{\vartheta f}{\vartheta n} = v = \bar{v}$ on Γ_2. Furthermore, the constant c_i can be determined by the relation:

$$c_i = \frac{\Theta}{2\pi} \quad (3.9)$$

where Θ denotes the internal angle of the corner in rad.

(a) Constant Elements

In order eqn (3.8) to be numerically evaluated by using constant elements, then this equation may be written as:

$$c_i f_i + \sum_{j=1}^{n} f_j \int_{\Gamma_j} \frac{\vartheta f^*}{\vartheta n} d\Gamma = \sum_{j=1}^{n} \frac{\vartheta f_j}{\vartheta n} \int_{\Gamma_j} f^* d\Gamma \quad (3.10)$$

Moreover, eqn (3.10) can be further written as:

$$c_i f_i + \sum_{j=1}^{n} f_j A_{ij}^* = \sum_{j=1}^{n} \frac{\vartheta f_j}{\vartheta n} B_{ij} \quad (3.11)$$

in which: $A_{ij} = A_{ij}^*$, when $i \neq j$

$$A_{ij} = A_{ij}^* + c_i, \text{ when } i = j \quad (3.12)$$

Thus, eqn (3.11) takes the form:

$$\sum_{j=1}^{n} A_{ij} f_j = \sum_{j=1}^{n} B_{ij} \frac{\vartheta f_j}{\vartheta n} \quad (3.13)$$

or in matrix form will be written as:

$$\mathbf{A\, f = B\, v} \quad (3.14)$$

Furthermore, by reordering the above equation so that all the unknows are on the left hand side, then one has:

$$\mathbf{C\, X = D} \quad (3.15)$$

where $\mathbf{X}$ is the vector of unknows f and v.

Hence, if the values of f and v on the whole boundary are known, then f can be calculated at any interior point by the following formula:

$$f_i = \sum_{j=1}^{n} \frac{\vartheta f_j}{\vartheta n} B_{ij} - \sum_{j=1}^{n} f_j A^*_{ij} \quad (3.16)$$

(b) Linear Elements

On the other hand, for the numerical evaluation of eqn (3.8) by using linear elements, then this equation may be written as:

$$c_i f_i + \sum_{j=1}^{n} \int_{\Gamma_j} f \frac{\vartheta f^*}{\vartheta n} d\Gamma = \sum_{j=1}^{n} \int_{\Gamma_j} \frac{\vartheta f}{\vartheta n} f^* d\Gamma \quad (3.17)$$

In the present case, in contrary to eqn (3.10), the variables f_j and $\dfrac{\vartheta f_j}{\vartheta n}$ cannot be taken out of the integral as they vary linearly within the element.

Thus, by using linear elements then eqn (3.17) can be also written as:

$$c_i f_i + \sum_{j=1}^{n} f_j A_{ij}^{*} = \sum_{j=1}^{n} \frac{\vartheta f_j}{\vartheta n} B_{ij} \qquad (3.18)$$

By using a corresponding method, as for eqn (3.13), then the above equation takes the form:

$$\sum_{j=1}^{n} A_{ij} f_j = \sum_{j=1}^{n} B_{ij} \frac{\vartheta f_j}{\vartheta n} \qquad (3.19)$$

and in matrix form:

$$\mathbf{A\,f} = \mathbf{B\,v} \qquad (3.20)$$

Hence, by using either the constant elements or the linear elements then the velocities

$v = \partial f/\partial n$ are computed on the free surface of the spillway flow.

Then, the free surface elevations y, are further computed by the formula:

$$y = \frac{Q}{v} \qquad (3.21)$$

and thus the free-surface profile is fully determined.

4. Two-dimensional Free-Surface Profile of a Spillway

The previous outlined theory will be applied to the determination of the free-surface profile over a spillway of height h = 7.55 m, designed for a flow rate of 3.72 m^3/sec/m of width. The above problem has been previously solved by A.H.D. Cheng, J.A. Liggett and P.L.F. Lin[14] by using the Boundary Integral Equation Method (B.I.E.M.). Thus a comparison will be made between the results of the Singular Integral Operators Method (S.I.O.M.), and the Boundary Elements.

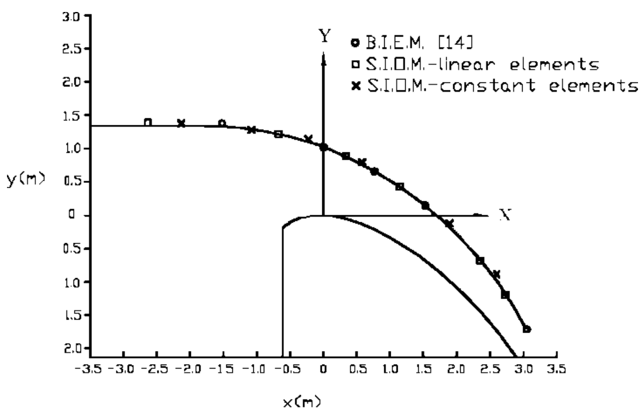

Figure 2. Surface Profile for a Spillway with Q = 3.72 m^3/sec/m width

This problem was solved by using both constant and linear elements. Thus, as it can be seen from Figure 2 the results of the linear elements by using the S.I.O.M., are in good agreement with the coresponding results by the Boundary Element Method. Beyond the above, there is a small disagreement between the results of the constant elements of the S.I.O.M. and the Boundary Elements. This is basically explained by the fact that the constant elements are not well fitted in the zone of uncertainty of the flow over the spillway.

5. Conclusions

The free-surface profile of the unsteady flow over a spillway was determined by using the S.I.O.M. (Singular Integral Operators Method). This is one of the main problems of classical hydraulics and especially in the theory of open channel unsteady flows. Beyond the above, some basic parameters of the unsteady spillway flows, like the discharge, the free surfaces and speeds are too important for the design of the spillways. During the past years the above mentioned hydraulic parameters were obtained through experiments, but with the continuous development of the numerical recipies in hydraulics problems, became efficient the possibility of obtaining such parameters through computational methods.

On the other hand, the governing equation for solving potential flow problems is the equation of Laplace. Consequently, by using the Laplacean and choosing some proper boundary conditions, then the unsteady flows over a spillway are calculated by using a computational method based on the singular integral equations. For the numerical solution of the singular integral equations were used both constant and linear elements. An application was given to the determination of the free-surface profile of a special spillway and the results were compared with corresponding numerical results by the Boundary Integral Equation Method (B.I.E.M.).

The proposed method by using the Laplacean together with Singular Integral Operators Method for solving potential problems can be applied in several other hydraulics problems of open channel flows. Hence, in future special attention must be given to the research and application of the integral equation methods to the solution of several important hydraulics problems of open channel flows.

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