MATHEMATICAL MODEL FOR VIBRATIONS OF NON-UNIFORM FLEXURAL BEAMS

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A simplified mathematical model for free vibrations of nonuniform viscoelastic flexural beams is presented. The mass intensity, the material damping intensity and the flexural stiffness of the beam are assumed varying as power functions along the beam. An analytical solution for the fourth order differential equation of beam vibration under appropriate boundary conditions is obtained by factorization. Mode shapes and damped natural frequencies of the beam are obtained for wide range of beam characteristics. The model results agree with those found in literature for uniform beams.

Key words : *damped vibration, non-uniform beam, differential equation, variable coefficients, series solution, mode shapes and natural frequencies*

1. Introduction

The dynamics of flexural beams has received an extensive research for a long period due to its wide applications in civil, mechanical and aeronautical engineering. Recently, the concept of linear systems has been generalized to nonlinear systems using perturbation techniques to solve governing equations. Caruntu [1] used the factorization method to obtain analytic solution of the differential equation of free bending vibration of nonuniform beams. Nayfeh, et al [4] used the method of multiple scales to obtain the nonlinear modes of a cantilever beam. Li [3] suggested an approach to analyze the vibrations of narrow buildings as a cantilever flexural-shear plate with variable cross section. In the mathematical treatment, the plate is modeled as flexural bar and shear bar. Taha [5] obtained the transient response of a finite viscoelastic beam resting on an elastic foundation due to stochastic dynamic load using eigen-function expansion with variation of parameter techniques. Taha [6] studied the vibration of a non-uniform shear beam resting on an elastic foundation using variation of parameters.

In the present work a simplified mathematical model for free vibrations of non-uniform viscoelastic flexural beams is presented. The beam mass intensity, the material damping intensity and the flexural stiffness are assumed varying as power functions along the beam length. The fourth order differential equation with variable coefficients of the beam vibration is obtained and factorized into two second order differential equations. The resulted equations are solved by transformation to the Bessel equations to obtain mode shapes and natural frequencies. Charts of natural frequencies for wide range of non-uniform flexural beams are conducted.

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2. Problem formulation

Assuming nonuniform simply supported beam with length L, width b and variable depth $h(x)$. The dynamic equations of the beam may be expressed as [2] :

$$\frac{\partial Q}{\partial x} + q(x,t) = m(x)\,\frac{\partial^2 y}{\partial t^2} + C(x)\,\frac{\partial y}{\partial t}\ , \tag{1}$$

$$Q = \frac{\partial M}{\partial x} \tag{2}$$

where $Q(x,t)$ is the shear force, $q(x,t)$ is the vertical excitation acting on the beam, $m(x)$ is the beam mass intensity, $C(x)$ is the material damping intensity, $y(x,t)$ is the vertical response of the beam, $M(x,t)$ is the flexural moment and t is time.

The flexural moment acting on the beam cross section is related to the vertical response as :

$$M(x,t) = -k(x)\,\frac{\partial^2 y}{\partial x^2} \tag{3}$$

where $k(x)$ is the flexural stiffness of the beam. Substitution Eqs. (2) and (3) into Eqn. (1) yields :

$$\frac{\partial^2}{\partial x^2}\left[k(x)\,\frac{\partial^2 y}{\partial x^2}\right] + m(x)\,\frac{\partial^2 y}{\partial t^2} + C(x)\,\frac{\partial y}{\partial t} = q(x,t) \tag{4}$$

2.1. The Boundary Conditions

The boundary conditions depend on the constraints at the beam ends, however for a simply supported beam whose length is L, the vertical displacement at the beam ends are given as :

$$y(0,t) = 0\ , \tag{5a}$$

$$y''(0,t) = 0\ , \tag{5b}$$

$$y(L,t) = 0\ , \tag{6a}$$

$$y''(L,t) = 0 \tag{6b}$$

where dash means derivative with respect to x.

3. Problem solution

To obtain the natural frequencies and mode shapes, one can assume :

$$q(x,t) = 0\ , \tag{7}$$

$$y(x,t) = w(x)\,\mathrm{e}^{\mathrm{i}\,\Omega\,t} \tag{8}$$

where, $w(x)$ is the mode shape and Ω is the complex damped natural frequency of the flexural beam. Substitution of Eqs. (7) and (8) into Eqn. (4), yields :

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[k(x)\,\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right] - \left[m(x)\,\Omega^2 - \mathrm{i}\,C(x)\,\Omega\right] w(x) = 0\ . \tag{9}$$

Equation (9) is a fourth order differential equation with variable coefficients whose solution depends mainly on the distribution functions representing the actual beam characteristics. Many distributions are assumed to approximate the variation of the geometric and material characteristics of the non-uniform beam as algebraic polynomials, exponential functions, trigonometric series or their combinations. Two cases are considered in the present work:

Case (1): Uniform beam

$$k(x) = k_0 \ , \tag{10}$$

$$m(x) = m_0 \ , \tag{10b}$$

$$C(x) = C_0 \tag{10c}$$

where k_0 is flexural rigidity, m_0 is mass intensity and C_0 is damping intensity of the beam.

Substitute Eqs. (10a), (10b) and (10c) into Eqn. (9), one obtains:

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - \frac{m_0\,\Omega^2 - \mathrm{i}\,C_0\,\Omega}{k_0}\,w(x) = 0 \ . \tag{11}$$

The general solution of the above equation is given as:

$$w(x) = A_1\,\mathrm{e}^{\theta\,x} + A_2\,\mathrm{e}^{-\theta\,x} + A_3\,\sin(\theta\,x) + A_4\,\cos(\theta\,x) \tag{12}$$

where:

$$\theta^4 = \frac{m_0\,\Omega^2 - \mathrm{i}\,C_0\,\Omega}{k_0} \ . \tag{13}$$

Case (2): Non-uniform beam

The distribution of the non-uniform characteristics may be assumed as power functions. The parameters α and n are used to approximate the actual non-uniformity of the beam, i.e.:

$$k(x) = k_0\,(1+\alpha\,x)^{n+2} \ , \tag{14a}$$

$$m(x) = m_0\,(1+\alpha\,x)^{n} \ , \tag{14b}$$

$$C(x) = C_0\,(1+\alpha\,x)^{n} \tag{14c}$$

where k_0, m_0 and C_0 are the beam characteristics at $x = 0$. Substituting Eqs. (14a), (14b) and (14c) into Eqn. (9), one obtains:

$$(1+\alpha\,x)^2\,\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + 2\,\alpha\,(n+2)\,(1+\alpha\,x)\,\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} + \alpha^2\,(n+2)\,(n+1)\,\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \theta^4\,w(x) = 0 \ . \tag{15}$$

Introduce the operator:

$$\Delta = \alpha\,(n+1)\,\frac{\mathrm{d}}{\mathrm{d}x} + (1+\alpha\,x)\,\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ . \tag{16}$$

Using the above operator, Eqn. (15) may be rewritten as:

$$(\Delta+\theta^2)\,(\Delta-\theta^2)\,w(x) = 0 \ . \tag{17}$$

The general solution of Eqn. (15) is the sum of the solutions of the two equations:

$$(1+\alpha\,x)\,\frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2}+\alpha\,(n+1)\,\frac{\mathrm{d}w_1}{\mathrm{d}x}+\theta^2 w_1(x)=0\;, \tag{18}$$

$$(1+\alpha\,x)\,\frac{\mathrm{d}^2 w_2}{\mathrm{d}x^2}+\alpha\,(n+1)\,\frac{\mathrm{d}w_2}{\mathrm{d}x}+\theta^2 w_2(x)=0\;. \tag{19}$$

Introducing new variables $\xi(x)$ and $Z_1(\xi)$, as:

$$\xi(x)=(1+\alpha\,x)^{\mu}\;, \tag{20}$$

$$Z_1(\xi)=\xi^{-\nu}\,w(x)\;. \tag{21}$$

Using the new variables (Eqs. 20 and 21), Eqn. (18) can be expressed as:

$$\frac{\mathrm{d}^2 Z_1}{\mathrm{d}\xi^2}+\frac{1}{\xi}\,\frac{\mathrm{d}Z_1}{\mathrm{d}\xi}+\left(\eta^2-\frac{\nu^2}{\xi^2}\right)Z_1(\xi)=0 \tag{22}$$

where the transformation parameters μ, ν and η are given by:

$$\mu=\frac{1}{2}\;, \tag{23}$$

$$\nu=-n\;, \tag{24}$$

$$\eta=\frac{2\,\theta}{\alpha}\;. \tag{25}$$

Equation (22) is the Bessel's differential equation of order n with parameter η. Its general solution yields, [7]:

$$w_1(x)=(1+\alpha\,x)^{-\frac{n}{2}}\,[B_1\,J_n(\eta\,\xi)+B_2\,Y_n(\eta\,\xi)]\;, \tag{26}$$

where, J_n and Y_n are the Bessel functions of first and second kind of order n.

Similarly, Eqn. (19) can be transformed to the modified Bessel's equation of order n with parameter η, whose general solution is leads to, [7]:

$$w_2(x)=(1+\alpha\,x)^{-\frac{n}{2}}\,[B_3\,I_n(\eta\,\xi)+B_4\,K_n(\eta\,\xi)]\;, \tag{27}$$

where, I_n and K_n are the modified Bessel functions of the first and second kind of order n.

3.1. Mode shapes and Natural Frequencies

Applying the boundary conditions at beam ends, constants A_i and B_i, $i=1,2,3,4$ for the above two cases can be obtained. Then, the frequency equation, the mode shapes and the natural circular frequencies of the beam may be obtained.

Case (1): Uniform Beam

Mode shapes: Equation (12) may be rewritten as:

$$w(x)=C_1\,\sinh(\theta\,x)+C_2\,\cosh(\theta\,x)+C_3\,\sin(\theta\,x)+C_4\,\cos(\theta\,x)\;. \tag{28}$$

Applying the boundary conditions (5a) and (5b), yields:

$$C_2 = 0 \ , \tag{29a}$$

$$C_4 = 0 \ . \tag{29b}$$

Applying boundary conditions (6a), the nontrivial condition yields:

$$C_1 = 0 \ . \tag{29c}$$

Then, the normalized mode shapes are obtained as:

$$w(x) = \sin(\theta\, x) \ . \tag{30}$$

Natural frequencies:

Using Eqn. (30) with the boundary condition (Eqn. 6b), the frequency equation of the beam vibration may be expressed as:

$$\theta_r = \frac{r\,\pi}{L} \ , \quad r = 1, 2, 3, \dots \tag{31}$$

Substituting Eqn. (31) into Eqn. (13), the complex damped natural frequencies are obtained as:

$$\Omega_r = \frac{\mathrm{i}\, C_0}{2\, m_0} + \sqrt{\frac{r^4\, \pi^4\, k_0}{m_0\, L^4} - \frac{C_0^2}{4\, m_0^2}} \ . \tag{32}$$

The first part of Eqn. (32) represents the attenuation of the beam vibration due to damping, while the second term represents the actual natural frequency of the beam. The r-mode damped natural frequency of the beam may be expressed as:

$$\omega_{dr} = \omega_r \sqrt{1 - d^2} \tag{33}$$

where:

$$\omega_r = \frac{\pi^2\, r^2}{L^2} \sqrt{\frac{k_0}{m_0}} \ , \tag{34}$$

$$d = \frac{C_0}{C_\mathrm{c}} \ , \tag{35}$$

$$C_\mathrm{c} = \frac{2\, r^2\, \pi^2}{L^2} \sqrt{k_0\, m_0} \tag{36}$$

where, ω_r is the undamped natural frequency of r-mode, k_0 is the flexural stiffness of the beam, m_0 is the linear mass density of the beam, d is the damping ratio and C_c is the critical damping coefficient of the beam material.

Case (2): Non-uniform beams

Mode shapes:

Using Eqs. (17), (26) and (27) the general solution of Eqn. (15) is obtained as:

$$w(x) = (1 + \alpha\, x)^{-\frac{n}{2}} \left[B_1\, J_n(\eta\, \xi) + B_2\, Y_n(\eta\, \xi) + B_3\, I_n(\eta\, \xi) + B_4\, K_n(\eta\, \xi) \right] \ . \tag{37}$$

Applying boundary conditions (5a, 5b) and (6a, 6b), yields a linear system of algebraic homogeneous equations with coefficients of Bessel's functions and its derivatives at $x = 0$ and $x = L$. This system my expressed as:

$$\mathbf{A}\,\mathbf{C} = \mathbf{0} \tag{38}$$

where $\mathbf{A}$ is the coefficient matrix and $\mathbf{C}$ is the constant vector. For nontrivial solution, the frequency equation of the system may be expressed as:

$$\det(\mathbf{A}) = 0\ . \tag{39}$$

The frequency equation is a nonlinear equation in η which may be solved using appropriate iterative technique to obtain the roots η_r, then the natural frequency of the system can be determined.

Substituting the obtained values of η_r in Eqn. (38) and assuming any arbitrary value for the constant B_1 (say $B_1 = 1$), constants B_2, B_3 and B_4 may be obtained, then, the r-mode shape can be obtained.

Natural frequencies:

Using Eqs. (13) and (25), the r-mode complex damped frequency is obtained as:

$$\Omega_r = \frac{\mathrm{i}\,C_0}{2\,m_0} + \sqrt{\frac{\alpha^4\,\eta_r^4\,k_0}{16\,m_0} - \frac{C_0^2}{4\,m_0^2}}\ . \tag{40}$$

Similarly, the first part of Eqn. (40) represents the attenuation of the beam vibration due to the damping, while the second term represents the actual natural frequency of the r-mode. The r-mode damped natural frequency of the beam may be expressed as:

$$\omega_{dr} = \omega_r\,\sqrt{1 - d^2} \tag{41}$$

where:

$$\omega_r = \frac{\alpha^2\,\eta_r^2}{4}\sqrt{\frac{k_0}{m_0}}\ , \tag{42}$$

$$d = \frac{C_0}{C_{\mathrm{c}}}\ , \tag{43}$$

$$C_{\mathrm{c}} = \frac{\alpha^2\,\eta_r^2}{4}\sqrt{k_0\,m_0} \tag{44}$$

where, ω_r is the undamped natural frequency of r-mode, d is the damping ratio and C_{c} is the critical damping coefficient of the beam.

4. Numerical results

Many cases occur in actual applications where the beam is made of homogeneous material with constant cross section (prismatic). In such cases, the present model can predict the exact natural undamped frequency [2]. To obtain the damped vibration behavior of the uniform homogeneous beam the damping coefficient C must be known. However, using the results of present model, it is easy to measure the damping coefficient of the system experimentally by implementation of the techniques of single degree of freedom system. Also, it is obvious from Eqn. (33) that the damping is only significant in the case of highly damped system.

For the case of non-uniform beams, the proposed model can deal with any variations in the beam cross section by identifying suitable values for n and α that approximate the beam configurations. However, more frequent cases in real applications of beams made of homogeneous material with uniform width and linearly variable depth (beams with linearly tapered depth). In this case the beam depth is given by:

$$h(x) = h_0\,(1 + \alpha\,x) \ . \tag{45}$$

In this case ($n = 1$), the nonlinear frequency equation (39) is solved numerically to obtain the roots η_r, and then the mode shapes and natural frequency are obtained.

The significant parameters of the non-uniform beam are the slenderness ratio λ and the non-uniformity factor α. The slenderness ratio λ is defined as:

$$\lambda = \frac{L}{h_0} \ . \tag{46}$$

Figure (1) shows the first four mode shapes of a beam with linearly increasing depth ($\alpha = 0.2$, $\lambda = 10$). The amplitude of vibration decreases as x increases as the rigidity of the beam increases.

Figure (2) depicts the effect of nonuniformity factor α on the vibration amplitude of the mode shapes ($\lambda = 10$, $r = 1, 2$). It is found that as α increases, the vibration amplitude decreases as the beam depth increases, i.e., the beam becomes more rigid.

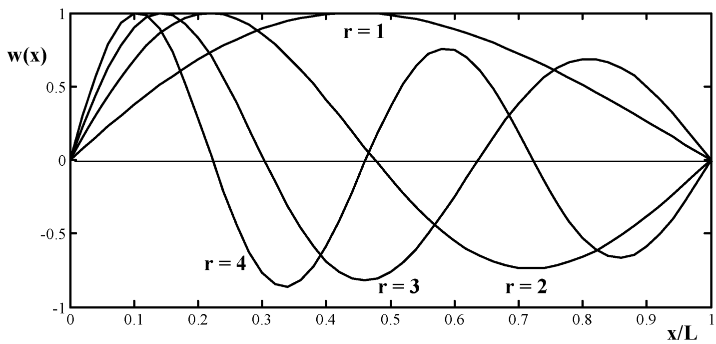

Fig.1: Mode shapes for beam with linearly variable depth ($\alpha = 0.2$, $n = 1$, $\lambda = 10$)

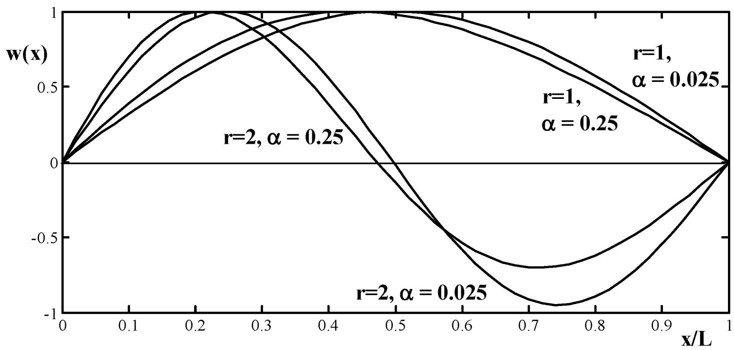

Fig.2: Effect of nonlinearity factor α on mode shapes ($r = 1, 2$, $n = 1$, $\lambda = 10$)

The effect of α on the first four dimensionless natural frequencies ω^* of the nonuniform beam is shown in Figure (3). The dimensionless natural frequency ω^* is defined as:

$$\omega^* = \frac{\omega_r L^2}{\nu h_0} \,, \qquad \nu = \sqrt{\frac{E}{\varrho}} \tag{47}$$

where ν is the longitudinal wave velocity in the beam material, E is the modulus of elasticity of the beam material and ϱ is the volumetric density of the beam material.

Figure (4) shows the dimensionless fundamental natural frequency for a homogeneous beam with linearly increasing depth for a very wide range of beam characteristics. However, the fundamental natural frequency is important for design purposes. Using curve fitting techniques, the fundamental natural frequency of any beam of that type can be calculated (to accuracy of 0.1 %) as:

$$\omega_1^* = 2.831529 + 0.004979\,\lambda + 0.718315\,\alpha + 0.531724\,\lambda\,\alpha \,. \tag{47}$$

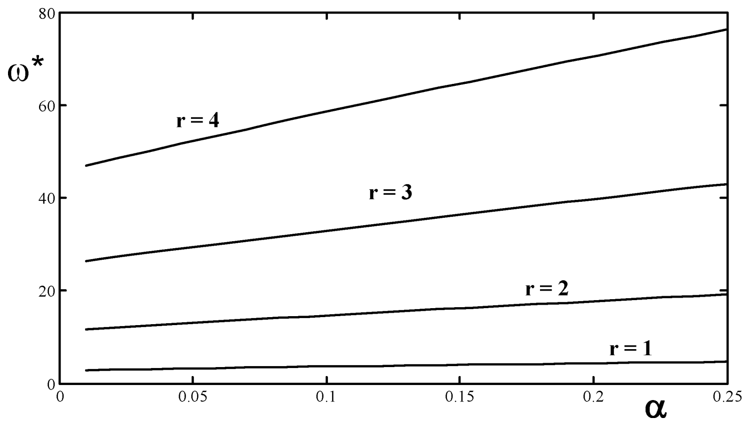

Fig.3: Variation of the First Four Eigen Values with α ($n = 1$, $\lambda = 12$)

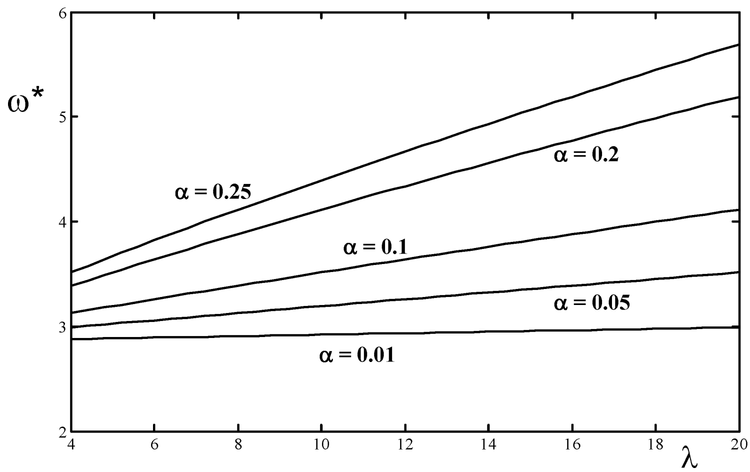

Fig.4: Variation of fundamental natural frequency with slenderness ratio λ ($n = 1$, $r = 1$)

5. Conclusions

A simplified model predicting the vibration behavior of viscoelastic uniform and non-uniform flexural beam is presented. Two distributions for beam characteristics are suggested, the first is constant distribution to represent prismatic beam and the second as a power function to represent the non-uniform beam. The dynamic equation of the beam is solved by introducing new variables to transform the equation to the Bessel differential equations. The obtained solutions are used to find the mode shapes and the natural frequencies. Charts that depicts the variation of the natural frequencies and mode shapes of the first four frequencies (ω_r, $r = 1, 2, 3, 4$) of homogenous beam with constant width and linearly increasing depth are given of a wide range of practical beam characteristics for design purposes. An accurate expression to calculate the fundamental natural frequency of homogeneous beams of constant width and linearly tapered depth is given. The fundamental natural frequency decreases as the slenderness ratio λ increases and as the nonuniformity factor α decreases, i.e., as the beam becomes more flexible.

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Received in editor's office: May 1, 2007
Approved for publishing: August 22, 2008