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THE SOLUTION OF FRACTIONAL NONLINEAR GINZBURG–LANDAU EQUATION WITH WEAK INITIAL DATA

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Abstract. In this paper, we study the solution of the fractional nonlinear Ginzburg-Landau(FNGL) equation with weak initial data in the weighted Lebesgue spaces. The existence of a solution to this equation is proved by the contraction-mapping principle.

1. Introduction

In this paper, we study the initial value problem (IVP) of the fractional Ginzburg-Landau (FNGL) equation

$$i\frac{\partial u}{\partial t}+i\rho u+(1+i\beta)(-\triangle)^{\alpha}u+\gamma|u|^{\sigma}u=0,\quad t\geqslant 0,\quad x\in R, \tag{1.1}$$

$$u(x,0)=u_0(x), \tag{1.2}$$

where $\rho<0$, $\beta<0$, $\sigma\in[2,4]$, $\alpha(\alpha\in(\frac{1}{2},1))$ is a fixed parameter and $u(x,t)$ is a complex value function. Here Riesz potential operator $(-\triangle)^{\alpha}$ is defined through the Fourier transform

$$\widehat{f}(\xi)=\int e^{-2\pi ix\xi}f(x)dx,$$

$$(\widehat{(-\triangle)^{\alpha}f})(\xi)=(2\pi|\xi|)^{2\alpha}\widehat{f}(\xi).$$

The Ginzburg-Landau equation plays an important role in physics and mathematics, and the fractional generalization of the Ginzburg-Landau equation was suggested in [1]. In [2] Vasily E. Tarasov and George M. Zaslavsky used the fractional integrals to describe fractal media, and some simple solutions of the Ginzburg-Landau equation(GLE) for fractal media are considered and different forms of the fractional GLE are presented. In [3], A. V. Milovanov and J. Juul Rasmussen discussed the fractional modifications of the free energy functional at criticality and of the widely known GLE central to the classical Landau theory of second-type phase transitions in some detail and derived that an implication of the fractional GLE is a renormalization of the transition temperature owing to the nonlocality present.

We are mainly interested in the well-posedness result for initial data u_0 in the weighted Lebesgue spaces, $u_0\in\dot{L}_{r+\alpha,p+1}(R)$ (defined below). In order to obtain our results, it is necessary to introduce the following fractional calculus inequality, see [5-7] for a proof.

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LEMMA 1.1. *Let* $1 < p < \infty, r > 1$, *and* $h \in L^{rp}_{loc}(R)$. *Then*

$$\|(-\triangle)^{\frac{\alpha}{2}} F(f)h\|_p \leqslant C\|F'(f)\|_\infty \|(-\triangle)^{\frac{\alpha}{2}}(f)M(h^{rp})^{\frac{1}{rp}}\|_p, \tag{1.3}$$

where M *denotes the Hardy-Littlewood maximal function, i.e.,*

$$Mf(x) = \sup \frac{1}{|I|}\int_I |f(y)|dy.$$

Here the homogeneous Lebesgue space $\dot{L}_{s,q}(R)$ consists of all v such that

$$(-\triangle)^{\frac{s}{2}} v \in L^q, \quad s \in R, \quad 1 \leqslant q < \infty,$$

and the standard norm is given by

$$\|v\|_{s,q} = \|(-\triangle)^{\frac{s}{2}} v\|_{L^q}.$$

These spaces are also called the spaces of Riesz potentials, Kato and Ponce [8] consider the Navier-Stokes equations with initial data in this type of spaces.

We prove that if $\frac{1}{2} < \alpha < 1$ and $u_0 \in \dot{L}_{r+\alpha,p+1}(R)$ with r, p satisfying

$$1 < p < \infty, \quad \alpha < \frac{1}{p} < 2\alpha, \quad r = \frac{1}{p} - 2\alpha(< 0), \quad -1 \leqslant r < -\frac{1}{2},$$

then the IVP(1.1) and (1.2) is locally well-posed. The solution is global if u_0 is sufficiently small. The detail statements are given is Theorem 2.1 of the next section.

2. The work space and the main results

We'll need to use the spaces of weighted continuous functions in time, which have been introduced by Kato and Ponce [8] in solving the Navier-Stokes equations.

DEFINITION 2.1. Suppose $T > 0$ and $\lambda > 0$ are real numbers. The spaces $C_{\lambda,s,q}$ and $\dot{C}_{\lambda,s,q}$ are defined as

$$C_{\lambda,s,q} = \{f \in C((0,T), \dot{L}_{s,q}),\ \|f\|_{\lambda,s,q} < \infty\}$$

where the norm is given by

$$\|f\|_{\lambda,s,q} = \sup\{t^\lambda \|f\|_{s,q},\ t \in (0,T)\}$$

$\dot{C}_{\lambda,s,q}$ is a subspace of $C_{\lambda,s,q}$:

$$\dot{C}_{\lambda,s,q} = \{f \in C_{\lambda,s,q},\ \lim_{t\to 0} t^\lambda \|f\|_{s,q} = 0\}$$

when $\lambda = 0$, $\overline{C}_{s,q}$ are used for $BC((0,T), \dot{L}_{s,q})$, where the space $BC((0,T), \dot{L}_{s,q})$ comprises all bounded and continuous functions $g : (0,T) \to \dot{L}_{s,q}$ with

$$\|g\|_{BC((0,T),\dot{L}_{s,q})} = \max_{0<t<T} \|g(t)\|_{s,q} < \infty.$$

These spaces are important in uniqueness and local existence problem, $f \in C_{\lambda,s,q}$ (resp. $f \in \dot{C}_{\lambda,s,q}$) implies that $\|f\|_{s,q} = O(t^{-\lambda})$ (resp. $o(t^{-\lambda})$).

The main result of this section is the well-posedness theorem that states

THEOREM 2.1. *Assume that* $\frac{1}{2} < \alpha < 1$ *and* $u_0 \in \dot{L}_{r+\alpha,p+1}$ *and* r,p *satisfying*

$$1 < p < \infty, \quad \alpha < \frac{1}{p} < 2\alpha, \quad r = \frac{1}{p} - 2\alpha (< 0), \quad -1 \leqslant r \leqslant -\frac{1}{2}. \tag{2.1}$$

Then there exists $T = T(u_0)$ *and a unique solution* $u(t)$ *of the IVP* (1.1) *and* (1.2) *in the time interval* $[0,T)$ *satisfying*

$$u \in Y_T = (\cap_{p \leqslant q < \infty} \overline{C}_{\frac{1}{q}-\alpha,q+1}) \cap (\cap_{p \leqslant q < \infty} \cap_{s > \frac{1}{q}-\alpha} \dot{C}_{(s-\frac{1}{q}+\alpha)/2\alpha,s,q+1}).$$

In particular,

$$u \in BC((0,T),\dot{L}_{r+\alpha,p}) \cap (\cap_{s>r+\alpha} C((0,T),\dot{L}_{s,p})).$$

Furthermore, for some neighborhood v *of* u_0*, the mapping*

$$\mathfrak{B} : v \mapsto Y_T : u_0 \to u$$

is Lipschitz.

REMARK 2.2. If $\|u_0\|_{r+\alpha,p+1}$ is small enough, then we can take $T = \infty$.

3. The estimates of the operators K and G

We write the FNGL equation (1.1) into the integral form:

$$\begin{aligned} u &= Ku_0 - G(u,t) \\ &= e^{-(\beta-i)\Lambda^{2\alpha}t}u_0 - \int_0^t e^{-(\beta-i)\Lambda^{2\alpha}(t-\tau)}(\rho u - i\gamma|u|^{\sigma}u)d\tau, \end{aligned} \tag{3.1}$$

where $K(t) = e^{-(\beta-i)\Lambda^{2\alpha}t}$ is the solution operator of the linear equation

$$i\partial_t u + (1+i\beta)\Lambda^{2\alpha}u = 0, \quad \textit{with} \quad \Lambda = (-\triangle)^{\frac{1}{2}}.$$

We shall solve (3.1) in the spaces of weighted continuous function in time introduced in the beginning of this section. To this end we need estimates for the operators K and G acting between these spaces. These are established in the two lemmas that follow.

LEMMA 3.1. (i) *For* $1 \leqslant q < \infty$ *and* $s \in R$*, the operator* K *maps continuously from* $\dot{L}_{s,q}$ *into* $\overline{C}_{s,q} = BC((0,\infty),\dot{L}_{s,q})$*.* (ii) *If* q_1, q_2, s_1, s_2 *and* α_2 *satisfy*

$$q_1 \leqslant q_2, \quad s_1 \leqslant s_2, \quad \alpha_2 = \frac{1}{2\alpha}(s_2 - s_1) + \frac{1}{2\alpha}\left(\frac{1}{q_1} - \frac{1}{q_2}\right).$$

Then K *maps continuously from* $\dot{L}_{s_1,q_1}$ *to* $\dot{C}_{\alpha_2,s_2,q_2}$.

The proof of the Lemma 3.1 is similar to that in [4], which solved the quasi-geostrophic type equations.

Now we give estimates for the operator G:

$$G(g) = \int_0^t K(t-\tau)g(\tau)d\tau.$$

LEMMA 3.2. *If q_1, q_2, s_1, s_2, α_1 and α_2 satisfy*

$$q_1 \leqslant q_2, \quad s_1 \leqslant s_2 < s_1 + 2\alpha - \left(\frac{1}{q_1} - \frac{1}{q_2}\right),$$

$$\alpha_1 < 1, \quad \alpha_2 = \alpha_1 - 1 + \frac{1}{2\alpha}\left[s_2 - s_1 + \frac{1}{q_1} - \frac{1}{q_2}\right].$$

Then G maps continuously from $\dot{C}_{\alpha_1,s_1,q_1}$ to $\dot{C}_{\alpha_2,s_2,q_2}$.

Proof. Let $g \in \dot{C}_{\alpha_1,s_1,q_1}$, clearly

$$\|G(g)\|_{\alpha_2,s_2,q_2} = \sup_{t\in[0,T)} t^{\alpha_2} \int_0^t \|(-\triangle)^{\frac{s_0}{2}} K(t-\tau)(-\triangle)^{\frac{s_1}{2}} g(\tau)\|_{L^{q_2}} d\tau,$$

where $s_0 = s_2 - s_1$, using Young's inequality, we have

$$\|G(g)\|_{\alpha_2,s_2,q_2} \leqslant \sup_{t\in[0,T)} t^{\alpha_2} \int_0^t \|(-\triangle)^{\frac{s_0}{2}} K(t-\tau)\|_{L^q} \|(-\triangle)^{\frac{s_1}{2}} g(\tau)\|_{L^{q_1}} d\tau,$$

with $\frac{1}{q} = 1 - (\frac{1}{q_1} - \frac{1}{q_2})$. If $s_0 \geqslant 0$, the operator $(-\triangle)^{\frac{s_0}{2}} K(t)$ has the property

$$\|(-\triangle)^{\frac{s_0}{2}} K(t)\|_{L^q(R)} \leqslant c t^{\frac{1}{2\alpha}(-s_0-1+\frac{1}{q})}, \tag{3.2}$$

where $q \in [1,\infty)$ and c is a constant, the proof of this property is similar to that for the heat operator [8,9].

So we obtain

$$\begin{aligned}
\|G(g)\|_{\alpha_2,s_2,q_2} &\leqslant c\|g\|_{\alpha_1,s_1,q_1} \sup_{t\in[0,T)} t^{\alpha_2} \int_0^t (t-\tau)^{-\frac{1}{2\alpha}(s_0+1-\frac{1}{q})} \tau^{-\alpha_1} d\tau \\
&\leqslant c\|g\|_{\alpha_1,s_1,q_1} \sup_{t\in[0,T)} t^{\alpha_2-\alpha_1+1-\frac{1}{2\alpha}(s_0+1-\frac{1}{q})} \\
&\quad \times B\left(1 - \frac{1}{2\alpha}\left(s_0 + 1 - \frac{1}{q}\right), 1-\alpha_1\right),
\end{aligned}$$

where c is a constant and $B(a,b)$ is the Beta function

$$B(a,b) = \int_0^1 (1-x)^{a-1} x^{b-1} dx.$$

By noticing that $B(a,b)$ is a finite when $a>0$, $b>0$ and that

$$s_0 = s_2 - s_1, \quad 1 - \frac{1}{q} = \frac{1}{q_1} - \frac{1}{q_2}.$$

We obtain

$$\|G(g)\|_{\alpha_2,s_2,q_2} \leqslant c\|g\|_{\alpha_1,s_1,q_1},$$

if the indices satisfy

$$0 \leqslant s_2 - s_1 < 2\alpha - \left(\frac{1}{q_1} - \frac{1}{q_2}\right),$$

$$\alpha_1 < 1, \quad \alpha_2 = \alpha_1 - 1 + \frac{1}{2\alpha}\left[s_2 - s_1 + \frac{1}{q_1} - \frac{1}{q_2}\right]. \quad \square$$

4. The proof of Theorem 2.1

We prove Theorem 2.1 by the method of integral equations and contraction-mapping arguments.

We defined

$$X = \overline{C}_{r+\alpha,p+1} \cap \dot{C}_{-\frac{r}{2\alpha},\alpha,p+1},$$

with norm for $u \in X$ given by

$$\|u\|_X = \|u - Ku_0\|_{0,r+\alpha,p+1} + \|u\|_{-\frac{r}{2\alpha},\alpha,p+1},$$

and the complete metric space X_R to be [-R,R] in X . Consider the operator $\mathscr{A}(u,u_0)$: $X_R \times V \mapsto X$.

$$\mathscr{A}(u,u_0)(t) = Ku_0 - G(\rho u - i\gamma|u|^\sigma u)(t), \quad 0 < t < T,$$

where V is some neighborhood of u_0 in $\dot{L}_{r+\alpha,p+1}$ and T will be choosen.

Using lemma 3.1 by substituting $s = r+\alpha$, $q = p+1$ in (i) and

$$q_1 = q_2 = p, \quad s_1 = r+\alpha, \quad s_2 = \alpha, \quad \alpha_2 = -\frac{r}{2\alpha}$$

in (ii), we find that $K\widetilde{u}_0 \in X_R$ for $\widetilde{u}_0 \in V$ if T is taken small enough and V is choosen properly.

To estimate G, we use lemma 3.2 with

$$q_1 = \frac{p+1}{2}, \quad q_2 = p+1, \quad s_1 = \alpha, \quad s_2 = l+r+\alpha+\frac{1}{p(p+1)}, \quad \alpha_1 = -\frac{r}{\alpha}, \quad \alpha_2 = \frac{l}{2\alpha}.$$

To obtain for some constant c

$$\|G(\rho u - i\gamma|u|^\sigma u)\|_{\frac{l}{2\alpha},l+r+\alpha+\frac{1}{p(p+1)},p+1} \leqslant c\|u\|_{-\frac{r}{\alpha},\alpha,\frac{p+1}{2}} + c\||u|^\sigma u\|_{-\frac{r}{\alpha},\alpha,\frac{p+1}{2}}. \quad (4.1)$$

Using Lemma 1 and imbedding theorem, we have

$$\begin{aligned}\||u|^{\sigma}u\|_{-\frac{r}{\alpha},\alpha.\frac{p+1}{2}} &\leqslant c\||u|^{\sigma}\|_{\infty}\|u\|_{-\frac{r}{\alpha},\alpha,\frac{p+1}{2}}\\ &\leqslant c\||u|^{\sigma}\|_{-\frac{r}{2\alpha},\alpha,p+1}\|u\|_{-\frac{r}{2\alpha},\alpha,p+1}.\end{aligned} \tag{4.2}$$

Then we have

$$\|G(\rho u-i\gamma|u|^{\sigma}u)\|_{\frac{l}{2\alpha},l+r+\alpha+\frac{1}{p(p+1)},p+1} \leqslant c\|u\|_{-\frac{r}{2\alpha},\alpha,p+1}+c\||u|^{\sigma}\|_{-\frac{r}{2\alpha},\alpha,p+1}\|u\|_{-\frac{r}{2\alpha},\alpha,p+1}, \tag{4.3}$$

for $l\in[-r-\frac{1}{p(p+1)},-2r)$.

Using Lemma 1 again, we have

$$\||u|^{\sigma}\|_{-\frac{r}{2\alpha},\alpha,p+1}\leqslant c\|u\|^{\sigma}_{-\frac{r}{2\alpha},\alpha,p+1}\leqslant cR^{\sigma}.$$

Then we get

$$\|G(\rho u-i\gamma|u|^{\sigma}u)\|_{\frac{l}{2\alpha},l+r+\alpha+\frac{1}{p(p+1)},p+1}\leqslant cR(R^{\sigma}+1).$$

We should notice that the restriction (2.1) on r,p are necessary in order to apply lemma 3.1 and lemma 3.2.

Furthermore

$$\|\mathscr{A}(u,u_0)-\mathscr{A}(\widetilde{u},u_0)\|_X=\|G(\rho u-ir|u|^{\sigma}u)-G(\rho\widetilde{u}-ir|\widetilde{u}|^{\sigma}\widetilde{u})\|_X.$$

By (4.1) and (4.2), we obtain

$$\begin{aligned}&\|\mathscr{A}(u,u_0)-\mathscr{A}(\widetilde{u},u_0)\|_X\\ \leqslant& c\|u-\widetilde{u}\|_X+c\||u|^{\sigma}\|_X\|u-\widetilde{u}\|_X+c\|(|u|-|\widetilde{u}|)(\sum_{i+j=\sigma-1}|u|^i|\widetilde{u}^j|)\|_X\|\widetilde{u}\|_X\\ \leqslant& \|u-\widetilde{u}\|_X(c+c\||u|\|^{\sigma}_X+ch^{\sigma-1}\|\widetilde{u}\|_X),\end{aligned}$$

where $h=\max\{1,R\}$.

Our above estimates show that if we choose T to be small and R appropriately, then $\mathscr{A}$ maps X_R into itself and is a contraction. Consequently there exists a unique fixed point $u\in X_R: u=\mathfrak{B}(u_0)$ satisfying $u=\mathscr{A}(u,u_0)$. It is easy to see from these estimates that the uniqueness can be extended to all R' by further reducing the time interval and thus to the whole X.

To prove the Lipschitz continuity of $\mathfrak{B}$ on V. Let $u=\mathfrak{B}(u_0)$ and $\varsigma=\mathfrak{B}(\varsigma_0)$ for $u_0,\varsigma_0\in V$. Then

$$\begin{aligned}\|u-\varsigma\|_X&=\|\mathscr{A}(u,u_0)-\mathscr{A}(\widetilde{\varsigma},\varsigma_0)\|_X\\ &\leqslant\|\mathscr{A}(u,u_0)-\mathscr{A}(\widetilde{\varsigma},u_0)\|_X-\|\mathscr{A}(\varsigma,u_0)-\mathscr{A}(\widetilde{\varsigma},\varsigma_0)\|_X\\ &\leqslant c\|u-\varsigma\|_X+\|K(u_0-\varsigma_0)\|.\end{aligned}$$

Since $\mathscr{A}$ is a contraction, $c<1$. Therefore the asserted property is obtained by applying lemma 3.1 to the second term of the last inequality.

To show that u is in the asserted class Y_T, we notice that

$$u = \mathscr{A}(u,u_0) = Ku_0 - G(\rho u - ir|u|^\sigma u).$$

We apply lemma 3.1 twice to Ku_0 to show that

$$Ku_0 \in \overline{C}_{\frac{1}{q}-\alpha,q+1}, \quad Ku_0 \in \dot{C}_{(s-\frac{1}{q}+\alpha)/(2\alpha),s,q+1}$$

for any $p \leqslant q < \infty$ and $s > \frac{1}{q} - \alpha$. To show the second part

$$G(\rho u - ir|u|^\sigma u) \in \overline{C}_{\frac{1}{q}-\alpha,q+1}, \quad p \leqslant q < \infty. \tag{4.4}$$

We use lemma 3.2 with

$$\begin{aligned} q_1 &= \frac{p+1}{2}, \quad q_2 = q+1, \\ s_1 &= \alpha - 1, \quad s_2 = \frac{1}{q+1} - \alpha + \frac{2}{p(p+1)}, \\ \alpha_1 &= -\frac{2r+1}{2\alpha}, \quad \alpha_2 = 0, \end{aligned}$$

$$\begin{aligned} \|G(\rho u - ir|u|^\sigma u)\|_{0,\frac{1}{q+1}-\alpha+\frac{2}{p(p+1)},q+1} &\leqslant c\|\rho u - ir|u|^\sigma u\|_{-\frac{2r+1}{2\alpha},\alpha-1,\frac{p+1}{2}} \\ &\leqslant c\|\rho u - ir|u|^\sigma u\|_{-\frac{r}{\alpha},\alpha,\frac{p+1}{2}} \\ &\leqslant c\|u\|_X + c\|u\|_X^{\sigma+1}. \end{aligned}$$

Once again we apply lemma 3.2 with

$$\begin{aligned} q_1 &= \frac{p+1}{2}, \quad q_2 = q+1, \\ s_1 &= \alpha - 1, \quad s_2 = s + \frac{2}{p(p+1)} - \frac{1}{q(q+1)}, \\ \alpha_1 &= -\frac{r}{\alpha} - \frac{1}{2\alpha}, \quad \alpha_2 = \frac{s-\frac{1}{q}+\alpha}{2\alpha} \end{aligned}$$

to show that

$$G(\rho u - ir|u|^\sigma u) \in \dot{C}_{(s-\frac{1}{q}+\alpha)/(2\alpha),s+\frac{2}{p(p+1)}-\frac{1}{q(q+1)},q+1} \quad \text{for} \quad s > \frac{1}{q} - \alpha, \tag{4.5}$$

but s should also satisfy

$$s < 3\alpha - 1 - \frac{2}{p} + \frac{1}{q},$$

as requried by lemma 3.2.

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